Unfolding Orthogonal Terrains

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Abstract

It is shown that every orthogonal terrain, i.e., an orthogonal (right-angled) polyhedron based on a rectangle that meets every vertical line in a segment, has a grid unfolding: its surface may be unfolded to a single non-overlapping piece by cutting along grid edges defined by coordinate planes through every vertex.

1 Introduction

This paper is concerned with *unfolding* the surface of a polyhedron to a single, connected planar piece that avoids overlap. We will concentrate on *orthogonal polyhedra*: those whose faces meet at angles that are multiples of 90°, and whose edges are parallel to Cartesian xyz-axes. Figure 1 shows an *edge unfolding* of an orthogonal polyhedron, an unfolding produced by cutting along edges of the polyhedron. Note that we permit boundary overlap, but no interior points of the planar piece overlap. Thus the shape could be cut out of paper and folded up to form the surface of the polyhedron.

The study of unfolding orthogonal polyhedra was initiated in [BDD$^+$98], and there are now many results, which we will not survey (see [O'R07] and [DO07]). It will suffice here to note that an easy example (a small box in the center of a larger box's top face) demonstrates that not every orthogonal polyhedron may be edge-unfolded. Consequently, loosenings of the unfolding criteria have been explored. A *grid unfolding* adds edges (*grid edges*) to the surface by intersecting the polyhedron with planes parallel to Cartesian coordinate planes through every vertex, as in Figure 1(c), permitting cutting along these grid edges. Even this freedom has not proven sufficient to obtain broadly applicable algorithms, so grid refinements have been studied. A $k_1 \times k_2$ *refinement* of a surface [DO05] partitions each face into a $k_1 \times k_2$ grid of faces (with the convention that a 1×1 refinement is an unrefined grid unfolding). Athough there have now been several grid refinement algorithms developed that unfold special classes of orthogonal polyhedra (surveyed in [O'R07]), it remains unknown whether every orthogonal

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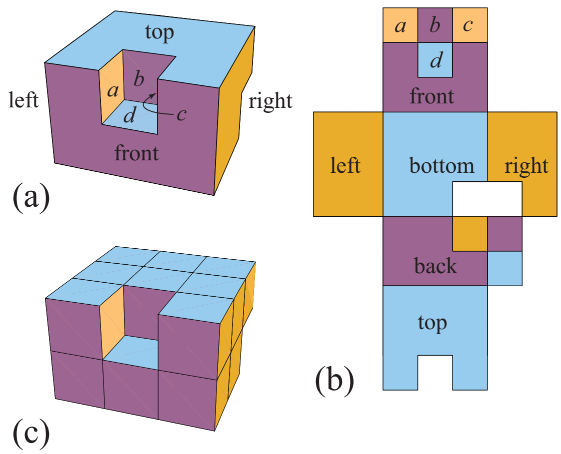

Figure 1: (a) An orthogonal polyhedron. (b) An edge unfolding of the polyhedron in (a). (c) Grid edges added to (a) by intersecting with coordinate planes through every vertex.

polyhedron has a (1×1) grid unfolding. This paper shows that a special class of orthogonal polyhedra does have a grid unfolding.

This class we call *orthogonal terrains*. Let P be the surface of an orthogonal polyhedron, and $\mathcal{P}$ the closed, solid whose boundary is P. An orthogonal terrain satisfies two properties: (1) there is a distinguished rectangular face of P called the *base* B; and (2) every vertical line L (parallel to the z-axis) that intersects $\mathcal{P}$ meets it in a single segment, $L \cap \mathcal{P} = s$, with s a finite-length line segment with one endpoint on B: $s \in B$. $P \setminus B$ is a "monotone surface," and models a terrain of elevations. In fact, any Digital Elevation Model (DEM), i.e., any rectangular array of heights, can be viewed as an orthogonal terrain (when closed with sides and a base). Figure 2 shows an example we will use throughout the paper (Figure 1(a) is not a terrain because its base is not a rectangle).

A slightly broader class of shapes, the "Manahattan towers," were studied in [DFO05]. These differ from terrains only in permitting the base B to be an arbitrary orthogonal polygon. This apparently small generalization considerably complicates the situation, and that paper achieved only a 5×4 grid unfolding. Insisting that B be a rectangle permits a completely different, and relatively simple algorithm to achieve a 1×1 grid unfolding.

2 Terrain Unfolding Algorithm

We now proceed to describe that algorithm, relying on illustrations to avoid excessive formality. Unlike most unfolding algorithms, this one can be specified as a continuous motion that avoids self-intersection throughout (as opposed to only guaranteeing nonoverlap at the planar conclusion). The first two steps

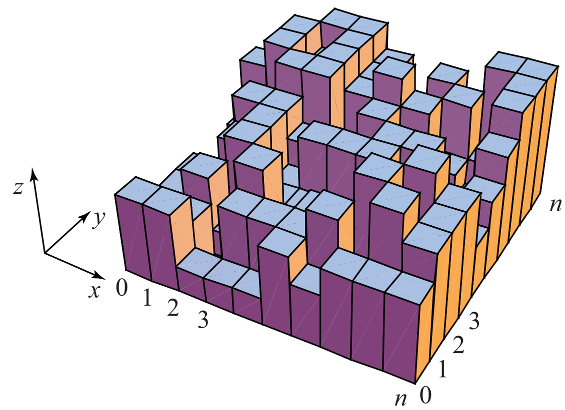

Figure 2: A orthogonal terrain with grid edges added, in this case via a plane at every integer coordinate. The base B underneath is a $10{\times}10$ square.

are straightforward. First, the right ($+x$), left ($-x$), and back ($+y$) vertical faces are unfolded to the xy-plane while remaining attached to the base B. See Figure 3. Second, B and its attachments are rotated around the x-axis, and

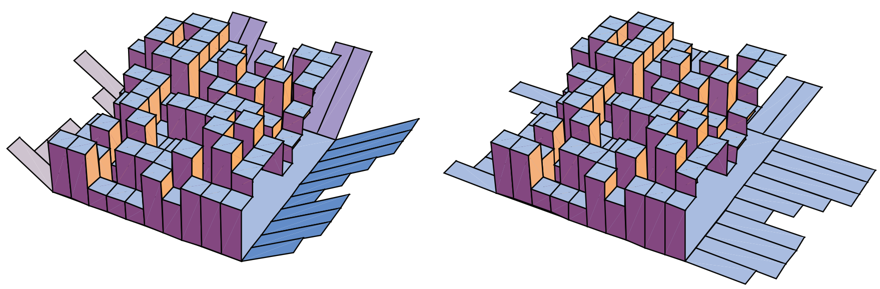

Figure 3: Unfolding the right, left, and back sides of P.

then the front vertical faces unfolded horizontally as in Figure 4. Here the line of rotation is $x = 0 \cap z = h$, where h is the height of the tallest front face ($h = 3$ in the figure; six front faces are tied for tallest).

All this is straightforward. The third step of the algorithmm is the heart of it. Define an *x_i-strip* as the sequence of faces between $y = i$ and $y = i + 1$ ($i = 0, \ldots, n{-}1$) on the "top" of P: the horizontal xy-faces, and the vertical yz right and left faces connecting them in a sinuous path. Each x_i-strip will be unfolded as a unit, into a (long) rectangle stretching in the x-direction. For example, the first x_0-strip (covering $y = [0, 1]$) in Figure 2 unfolds to a $16{\times}1$ rectangle: $n = 10$ unit square top xy-faces, connected by right/left pairs of $1{\times}2$ and $1{\times}1$ vertical faces. See Figure 6.

Consider any adjacent x-strips, x_{i-1} and x_i. In the original P, they are connected by a number of vertical xz-faces, some rising at $y{=}i$ to connect to

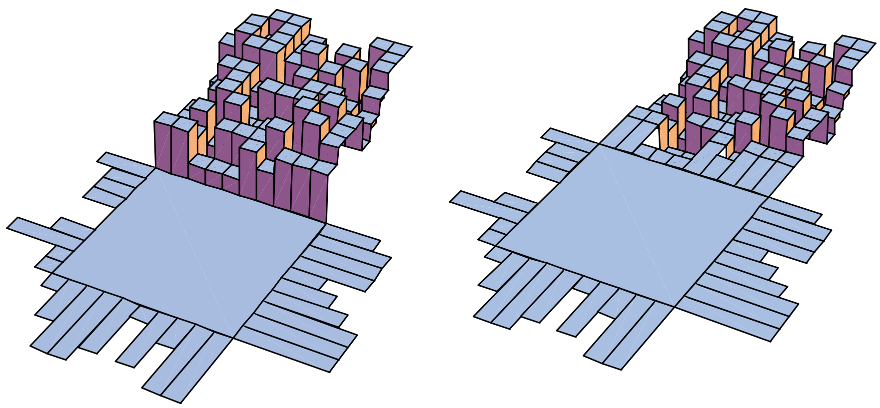

Figure 4: Flipping the base B around the line y=z=0, and then unfolding the front faces of P.

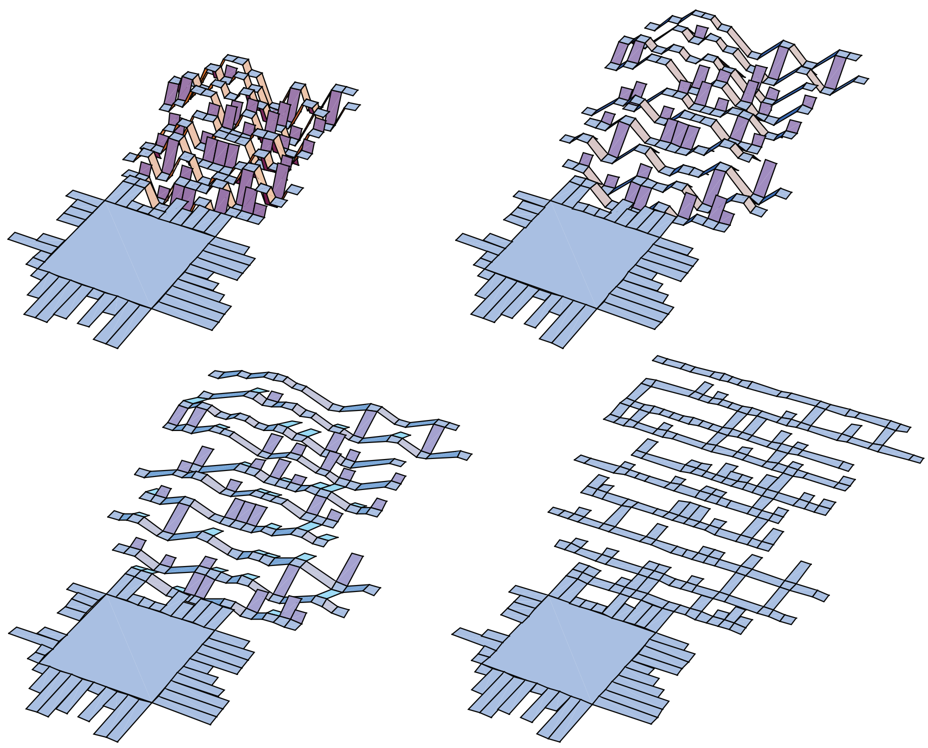

Figure 5: Unfolding the top faces of P into x-strips connected by y-bridges.

a higher y-adjacent "tower," and some descending to connect to a lower y-adjacent neighbor. Define the *bridge* b_i to be the xz-rectangle of greatest z-height between the strips, breaking ties arbitrarily. Then we lay out the x_i-strip separated from the x_{i-1} strip by the height of b_i in the planar vertical (y-) direction, and aligned horizontally so that b_i connects the two strips. Note that all the connecting xz-rectangles are attached above the x_i strip. The continuous unfolding process is depicted in Figure 5, and the final unfolding is shown in Figure 6. Note that, because of ties, the unfolding is not a simple polygon;

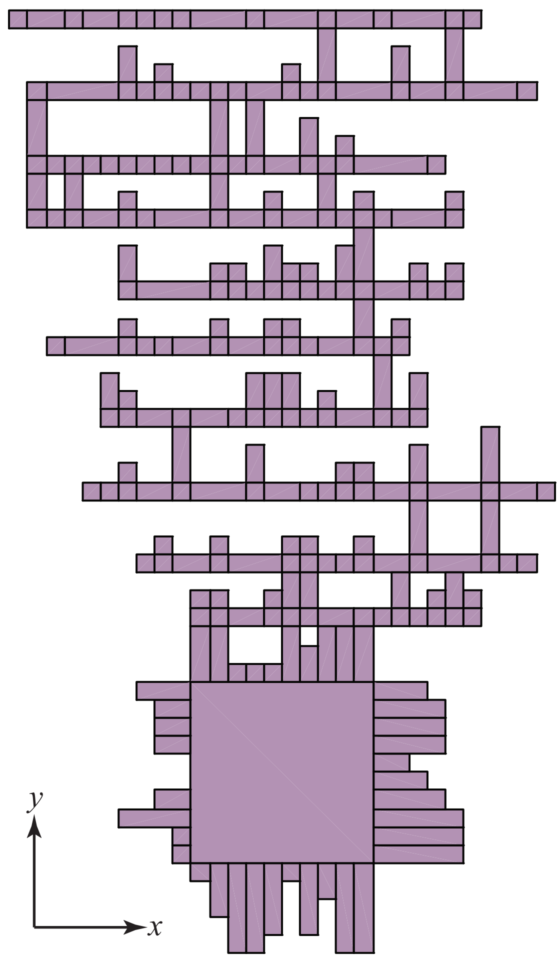

Figure 6: The final unfolding of P from Fig. 2 in the xy-plane.

rather, the boundary overlaps at several places. However, the unfolding is what is known as *weakly simple*, in that no interior points overlap, as mentioned previously.

3 Conclusion

Although our example gridded the polyhedron at every integer lattice point, it is clear that a coordinate grid plane through every vertex suffices for the algorithm.

Orthogonal terrains add to the narrow classes of orthogonal polyhedra that are known to be grid-unfoldable (orthotubes, well-separated orthotrees, orthogonally convex orthostacks; see [O'R07]), although it may be that all orthogonal polyhedra may be grid-unfoldable. Even extending this new algorithm to terrains defined by slanted axes (e.g., Figure 7) remains problematical.

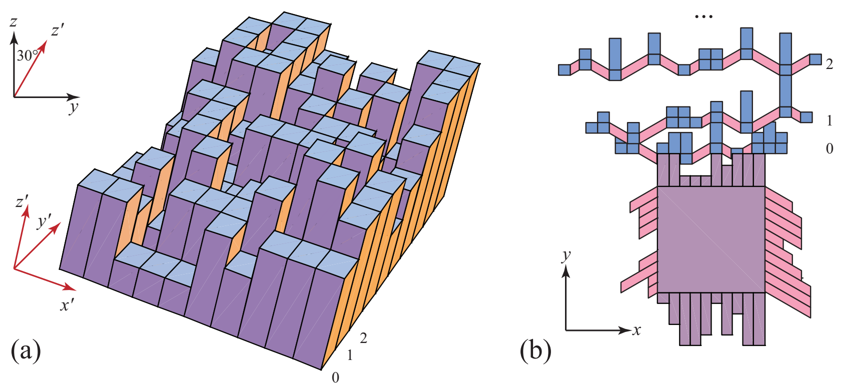

Figure 7: (a) The polyhedron from Fig. 2 with the z axis slanted 30° toward the y-axis. (b) Partial unfolding of first three strips $\{x_0, x_1, x_2\}$, showing that the algorithm that produced Fig. 6 now leads to overlap.

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References

[BDD+98] Therese Biedl, Erik D. Demaine, Martin L. Demaine, Anna Lubiw, Joseph O'Rourke, Mark Overmars, Steve Robbins, and Sue Whitesides. Unfolding some classes of orthogonal polyhedra. In *Proc. 10th Canad. Conf. Comput. Geom.*, pages 70–71, 1998. Full version in *Elec. Proc.*: `http://cgm.cs.mcgill.ca/cccg98/proceedings/cccg98-biedl-unfolding.ps.gz`.

[DFO05] Mirela Damian, Robin Flatland, and Joseph O'Rourke. Unfolding Manhattan towers. In *Proc. 17th Canad. Conf. Comput. Geom.*, pages 204–207, 2005. Full version: arXiv:0705.1541v1 [cs.CG].

[DO05] Erik D. Demaine and Joseph O'Rourke. Open problems from CCCG 2004. In *Proc. 17th Canad. Conf. Comput. Geom.*, pages 303–306, 2005.

[DO07] Erik D. Demaine and Joseph O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*. Cambridge University Press, June 2007. `http://www.gfalop.org`.

[O'R07] Joseph O'Rourke. Unfolding orthogonal polyhedra. In J.E. Goodman, J. Pach, and R. Pollack, editors, *Proc. Snowbird Conference Discrete and Computational Geometry: Twenty Years Later*. American Mathematical Society, 2007. To appear.