Unfolding Well-Separated Orthotrees

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1 Introduction

Because of the difficulty of the long-standing open problem of deciding whether every convex polyhedron can be edge-unfolded [DO05], attention has turned to various specializations or alterations of the original problem. To *edge-unfold* the surface of a polyhedron is to cut a collection of edges so that the surface may be unfolded to a planar, single-piece *net*. One line of investigation, started in [BDD$^+$98], focuses on *orthogonal polyhedra*—those whose faces meet at angles that are multiples of 90°. Although not every orthogonal polyhedron has an edge unfolding, in [BDD$^+$98] it is shown that *orthostacks* have an unfolding with some cuts interior to faces (a *general unfolding*), and *orthotubes* have an edge unfolding. Subsequent work [DM04] established that a subclass of orthostacks can be edge-unfolded. The work we report here is closest to that on orthotubes, which are polyhedra made by gluing boxes face-to-face such that the dual graph (each node a box, arcs corresponding to glued faces) is a path or cycle. An *orthotree* is a similar polyhedron, except with the condition that the dual graph be a tree. We use the convention that each box edge is an edge of the polyhedron available for cutting (i.e., edges between coplanar faces are not "erased"). Thus, our orthotrees already include the *vertex grid* of edges formed by intersecting the polyhedron with coordinate planes through every box vertex. The edges of the vertex grid offer more options for edge-unfolding; see [DIL04] and [DFO05].

Our main result is that a subclass of orthotrees, "well-separated" orthotrees, have an edge unfolding. The algorithm is naturally recursive on the tree structure, and we believe it shows promise for extension.

2 Definitions

An *orthotree* O is a polyhedron made out of boxes that are glued face-to-face such that the dual graph $G = (V, E)$ of O is a tree. We say that box $b_i \in O$ has degree d if its dual vertex has degree d in G. A box b_i is a *leaf* if it has degree one; b_i is a *connector* if it has degree two, and its two neighbors are glued to opposite faces of b_i; otherwise, b_i is a *junction.* An orthotree is *well-separated* if no neighbor of a junction is another junction, i.e., all neighbors of junctions are either leaves or connectors. See Fig. 1.

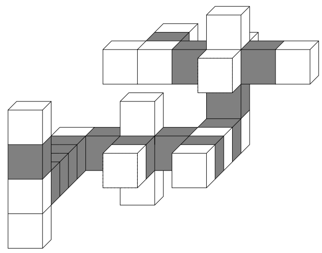

Figure 1: A well-separated orthotree; connectors are grey.

In the remainder of this paper, an edge will refer to one of the 12 edges of a box in an orthotree. Any box of degree d has $6-d$ exposed faces. In this paper we show that well-separated orthotrees can be edge-unfolded without overlap. We do allow, however, non-neighboring faces to be placed side by side.

The two faces of a box b_i in O perpendicular to the x-axis are denoted by x_i^+ and x_i^-. The face x_i^+ is the face whose outward normal is the positive x-axis. Similarly we define the faces y_i^+, y_i^-, z_i^+ and z_i^-. The four edges of x_i^+ and x_i^- are labeled f, b, u and d, for front, back, up and down. The edges of y_i^+ and y_i^- are labeled f, b, w and e, with w and e denoting west and east. Faces z_i^+ and z_i^- contain labels u, d, w and e. So an edge has multiple labels. Fig. 2 illustrates this notation and an unfolding of a single box orthotree.

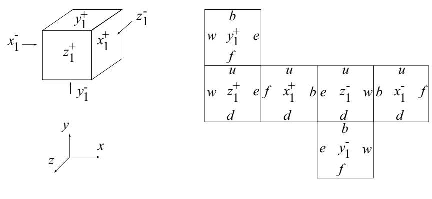

Figure 2: Notation and unfolding of a single box.

3 Unfolding Techniques

Select any leaf box as the root of O, a well-separated orthotree. Our main result is stated in Theorem 1.

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Theorem 1 *For any connector or leaf b_0 in O, the subtree rooted at b_0 can be unfolded without overlap, and in two different ways, as illustrated in Fig. 3.*

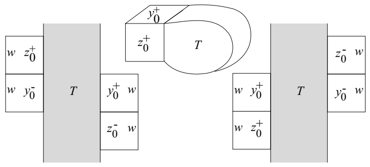

Figure 3: The unfolding of T fits within the shaded area and attaches to a z_0 or y_0 face on either side. Face x_0^- is not shown.

Proof sketch: The proof is by induction on the number of boxes in the subtree T_0 rooted at b_0. The base case corresponds to a single box subtree for which the two unfoldings can be easily derived.

The induction assumption is that Theorem 1 holds for subtrees with fewer than d nodes, giving us two unfoldings. Observe that by reversing these two unfoldings, we get unfoldings starting from the remaining two adjacent face pairs (e.g. pairs y_0^+, z_0^- and z_0^-, y_0^- in Fig 3). To prove the inductive step, consider a subtree T_0 with d nodes rooted at connector b_0. W.l.o.g, assume that $T_0 \setminus \{b_0\}$ attaches to x_0^+. Let b_1 be the box in T_0 glued to x_0^+. We distinguish six cases, depending on the degree of b_1. Here we only have space to discuss the cases when b_1 is a junction of degree 5 and 6. For any i, let T_i be the subtree rooted at b_i.

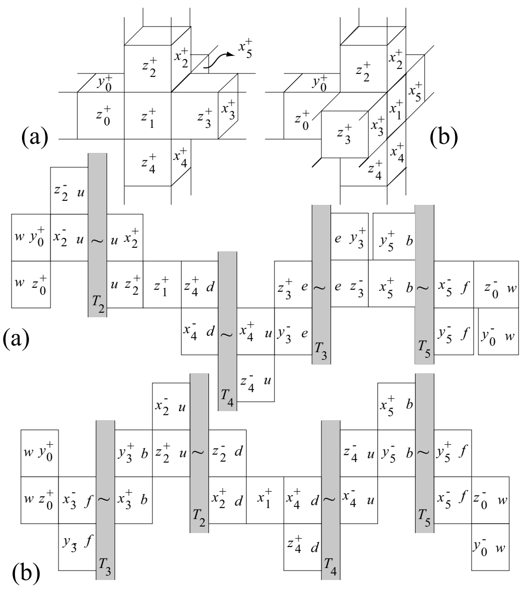

Figure 4: Box b_1 is a junction of degree 5.

Case 1: b_1 is a junction of degree 5. There are only two distinct cases for a degree 5 junction; see Figs. 4a and 4b. We can assume w.l.o.g. that the junction is oriented as shown, since we will provide unfoldings starting from each pair of adjacent faces of b_0, i.e. the two inductive step unfoldings and their reverses. The first case is shown in Fig. 4a: starting at y_0^+, unfold T_2 first, then move across z_1^+ to unfold T_4, T_3, and T_5, and finally back to b_0. Because the orthotree is well-separated, $T_2, ..., T_5$ are rooted at connectors or leaves and can be recursively handled. The second unfolding corresponding to Fig. 3a is equally easy to find. In Fig. 4b, the unfolding order is T_3, T_2, x_1^+ to get us to T_4, T_5, and back to b_0.

Case 2: b_1 is a junction of degree 6. One unfolding is shown in Fig. 5. Due to symmetry, the second unfolding is a horizontal mirror image of this one.

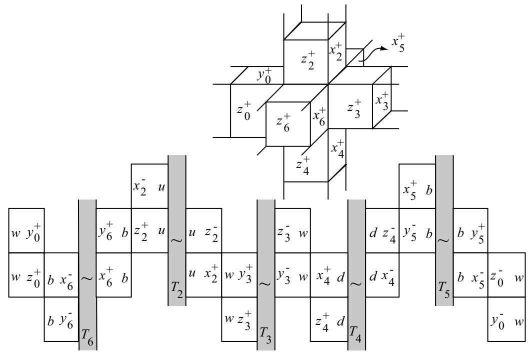

Figure 5: Interior box b_1 is a junction of degree 6.

We note that nothing in our algorithm depends on the boxes being cubes. The obvious open problem is to remove the well-separated assumption.

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