Every Combinatorial Polyhedron Can Unfold with Overlap

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Abstract

Ghomi proved that every convex polyhedron could be stretched via an affine transformation so that it has an edge-unfolding to a net [Gho14]. A net is a simple planar polygon; in particular, it does not self-overlap. One can view his result as establishing that every combinatorial polyhedron \( P \) has a metric realization \( \tilde{P} \) that allows unfolding to a net.

Joseph Malkevitch asked if the reverse holds (in some sense of “reverse”): Is there a combinatorial polyhedron \( P \) such that, for every metric realization \( \tilde{P} \) in \( \mathbb{R}^3 \), and for every spanning cut-tree \( T \) of the 1-skeleton, \( \tilde{P} \) cut by \( T \) unfolds to a net? In this note we prove the answer is no: every combinatorial polyhedron has a realization and a cut-tree that edge-unfolds the polyhedron with overlap.

1 Introduction

Joseph Malkevitch asked\(^1\) whether there is a combinatorial type \( P \) of a convex polyhedron \( P \) in \( \mathbb{R}^3 \) whose every edge-unfolding results in a net. One could imagine, to use his example, that every realization of a combinatorial cube unfolds without overlap for each of its 384 spanning cut-trees [Tuf11].\(^2\) The purpose of this note is to prove this is, alas, not true: every combinatorial type can be realized and edge-unfolded to overlap: Theorem 2 (Section 5). For an overlapping unfolding of a combinatorial cube, see ahead to Figure 12.

An implication of Theorem 2, together with [Gho14], is that a resolution of Dürer’s Problem [O’R13] must focus on the geometry rather than the combinatorial structure of convex polyhedra.

2 Proof Outline

We describe the overall proof plan in the form of a multi-step algorithm. We will illustrate the steps with an icosahedron before providing details.

Algorithm. Realizing \( G \) to unfold with overlap.

Input: A 3-connected planar graph \( G \).

Output: Polyhedron \( P \) realizing \( G \) and a cut-tree \( T \) that unfolds \( P \) with overlap.

1. Select outer face \( B \) as base.
2. Embed \( B \) as a convex polygon in the plane.
3. Apply Tutte’s theorem to calculate an equilibrium stress for \( G \).
4. Apply Maxwell-Cremona lifting stressed \( G \) to \( P \).
5. Identify special triangle \( \triangle \).
6. Compress \( P \) vertically to reduce curvatures (if necessary).
7. Stretch \( P \) horizontally to sharpen the apex of \( \triangle \) (if necessary).
8. Form cut-tree \( T \), including ‘Z’ around \( \triangle \).
9. Unfold \( P \setminus T \rightarrow \text{Overlap} \).

We are given a 3-connected planar graph \( G \), which constitutes the combinatorial type of a convex polyhedron. By Steinitz’s theorem, we know \( G \) is the 1-skeleton of a convex polyhedron. Initially assume \( G \) is triangulated; this assumption will be removed in Section 3.1.

1. Select outer face \( B \) as base. Initially, any face suffices. Later we will coordinate the choice of \( B \) with the choice of the special triangle \( \triangle \).
2. Embed \( B \) as a convex polygon in the plane. Select coordinates for the vertices of \( B \), which then pin \( B \) to the plane. \( B \) must be convex, but otherwise its shape is arbitrary.
3. Apply Tutte’s theorem [Tut63] to calculate an equilibrium stress—positive weights on each edge of \( G \)—that, when interpreted as forces, induce an equilibrium (sum to zero) at every vertex. This provides explicit coordinates for all vertices interior to \( B \). The result is a Schlegel diagram, with all interior faces convex regions. Figure 1 illustrates this for the icosahedron.\(^3\)

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\(^2\)Burnside’s Lemma can show that these 384 trees lead to 11 incongruent non-overlapping unfoldings of the cube [GSV19].
\(^3\)Here the drawing is approximate, because I did not explicitly calculate the equilibrium stresses.
(4) Apply Maxwell-Cremona lifting to $P$. The Maxwell-Cremona theorem says that any straight-line planar drawing with an equilibrium stress has a polyhedral lifting via a “reciprocal diagram.” The details are not needed here; we only need the resulting lifted polyhedron. An example from [Sch08] shows the vertical lifting of a Schlegel diagram of the dodecahedron: Figure 2. A lifting of the vertices of the icosahedron in Figure 1 is shown in Figure 3.

(5) Identify special triangle $\triangle$. This special triangle must satisfy several conditions, which we detail later (Section 3). For now, we select $\triangle = a_1 a_2 a_3 = 6, 8, 5$ in Figure 4.

(6) Compress $P$ vertically (if necessary) to reduce curvatures. Not needed in icosahedron example.

(7) Stretch $P$ horizontally (if necessary) to sharpen apex of $\triangle$. Not needed in icosahedron example.

(8) Form cut-tree $T$, including a ‘Z’-path around $\triangle$. We think of $a_1$ as the root of the spanning tree, which includes the Z-shaped (red) path $a_1 a_2 a_3 a_4$ around $\triangle$ and the adjacent triangle $\triangle'$ sharing edge $a_2 a_3$. In Figure 4, the Z vertex indices are 6, 8, 5, 11. The remainder of $T$ is completed arbitrarily.

(9) Unfold $P \setminus T$. Finally, the conditions on $\triangle$ ensure that cutting $T$ unfolds $P$ with overlap along the $a_2 a_3$ edge. See Figure 5.

3 Conditions on $\triangle$

We continue to focus on triangulated polyhedra. In order to guarantee overlap, the special triangle $\triangle = a_1 a_2 a_3$ should satisfy several conditions:

1. The angle at $a_2$ in $\triangle$ must be $\leq \pi/3 = 60^\circ$, and the edge $a_2 a_3$ at least as long as $a_1 a_2$.

2. The spanning cut-tree $T$ must contain the Z as previously explained. In addition, no other edge of $T$
is incident to either $a_1$ or $a_2$. In particular, edge $a_1a_3$ is not cut, so the triangle $\triangle$ rotates as a unit about $a_1$.

3. The curvatures at $a_1$ and $a_2$ must be small. (The curvature or “angle gap” at a vertex is $2\pi$ minus the sum of the incident face angles.) We show below that $< 20^\circ$ suffices.

4. $\Delta$ should be disjoint from the base $B$: $\Delta$ and $B$ share no vertices.

This 4th condition might be impossible to satisfy, in which case an additional argument is needed (Section 4). For now we concentrate on the first three conditions.

$\Delta$ is chosen to be the triangle disjoint from $B$ with the smallest angle $\alpha$. Clearly $\alpha \leq \pi/3 = 60^\circ$. Let $\Delta = a_1a_2a_3$ with $a_2$ the smallest angle. Chose the labels so that $|a_1a_2| \leq |a_2a_3|$. It will be easy to see that $\Delta$ an equilateral triangle is the “worst case” in that smaller $\alpha$ lead to deeper overlap, and $|a_1a_2| = |a_2a_3|$ suffices for overlap. So we will assume $\Delta$ is an equilateral triangle.

Next, we address the requirement for small curvatures, when the second condition is satisfied: no other edge of $T$ is incident to either $a_1$ or $a_2$. Let $\omega$ be the curvature at both $a_1$ and $a_2$. Then an elementary calculation shows that $\omega = \frac{1}{3} \pi = 20^\circ$ would just barely avoid overlap; see Figure 6.

One can view the flattening of $a_1$ and $a_2$ when cut as first turning the edge $a_2a_3$ by $\omega$ about $a_2$, and then rotating the rigid path $a_1a_2a'_3$ about $a_1$ by $\omega$. For any $\omega$ strictly less than $20^\circ$, overlap occurs along the $a_2a_3$ edge: Figure 6(b). The basic reason this “works” to create overlap is that the cut-path around $\Delta$ is not radially monotone, a concept introduced in [O’R16] and used in [O’R18] and [Rad21] to avoid overlap.

In the unfolded icosahedron in Figure 4, the angle at $a_2$ is $59^\circ$, and the curvatures $\omega_1, \omega_2$ at $a_1, a_2$ are $2.4^\circ$ and $8.1^\circ$ respectively.

If the two curvatures are not less than $20^\circ$, then we scale $P$ vertically, orthogonal to base $B$, step (6) of Algorithm 2. As illustrated in Figure 7, this flattens dihedral angles and reduces vertex curvatures (which reflect the spread of the normals [Hor84]) at all but the vertices of base $B$, which increase to compensate the Guss-Bonnet sum of $4\pi$. Clearly we can reduce curvatures as much as desired.

![Figure 5: Close-up views of overlap.](image1)

![Figure 6: (a) $\omega = 20^\circ$ avoids overlap. (b) $\omega = 10^\circ$ overlaps.](image2)

![Figure 7: Dihedral angle $\delta$ flattens as $z$-heights scaled: $(1, \frac{1}{2}, \frac{1}{5}) \rightarrow (90^\circ, 125^\circ, 160^\circ)$.](image3)

### 3.1 Non-Triangulated Polyhedra

If $G$ and therefore $P$ contains non-triangular faces, then we employ step (7) of Algorithm 1: Scale $P$ horizontally, parallel to the $xy$ plane containing $B$. For example, in the dodecahedron example (Figure 2), no face has an angle $\alpha \leq \pi/3$. The following lemma shows we can sharpen any selected face angle.

**Lemma 1** Any face angle $\angle a_1a_2a_3$ can be reduced via an affine stretching transformation to be arbitrarily small.

**Proof.** Adjust the coordinate system so that $a_1a_3$ lies in the $yz$-plane containing the origin, with $a_2$ in the
x-positive halfspace, wlog at \( o_2 = (1, a_{2y}, a_{2z}) \). See Figure 8. The Tutte-embedding guarantees that \( \Delta a_1a_2a_3 \) is not degenerate—the three vertices are not collinear, and Maxwell-Corona lifting guarantees the triangle is not vertical because each vertex of the Schlegel diagram lies in the relative interior of its neighbors [RG06, p.126,136]. Now stretch all vertices by \( s > 1 \) in their \( x \)-coordinate. This leaves \( a_1 \) and \( a_3 \) fixed, while \( a_2 \) stretches horizontally to \( a'_2 = (s, a_{2y}, a_{2z}) \). Eventually with large \( s \) the angle \( \angle a_1a'_2a_3 \) decreases monotonically to zero, while maintaining \( |a_1a'_2| \leq |a'_2a_3| \).

So we can identify a \( \Delta \) within any face, stretch its angle below 60°, and proceed just as in a triangulated polyhedron: Because \( a_1a_3 \) is not cut, having \( \Delta \) joined to a triangle below (4 in Figure 4) is no different than having \( \Delta \) part of a face.

![Figure 8: Stretching \( \angle a_1a_2a_3 = 108^\circ \) to \( \angle a_1a'_2a_3 = 53^\circ \).](image)

**4 No Pair of Disjoint Faces**

Finally we focus on the 4th condition that \( \Delta \) should be disjoint from the base \( B \). If \( G \) contains any two disjoint faces, triangles or \( k \)-gon faces with \( k > 3 \), we select one as \( B \) and the other to yield \( \Delta \). So what remains is those \( G \) with no pair of disjoint faces.

For example, a pyramid—a base convex polygon plus one vertex \( a \) (the apex) above the base—has no pair of disjoint faces. However, note that a pyramid has pairs of faces that share one vertex but not two vertices. It turns out that this suffices to achieve the same structure of overlap. Figure 9 illustrates why. Here \( B \) is a triangle \( b_1b_2a_3 \) and we select \( \Delta = a_1a_2a_3 \). The small-curvature requirement holds just for \( a_1, a_2 \)—the start of the \( Z \)—the curvature at \( a_3 \) could be large (117° in this example) but does not play a role, as the unfolding illustrates. Therefore, if \( G \) has no pair of disjoint faces, but does have a pair of faces that share a single vertex, we proceed just in Algorithm 1, suitably modified.

![Figure 9: (a) \( B \) and \( \Delta \) share \( a_3 \), \( Z = a_1a_2a_3b_2 \). (b) Unfolding with overlap.](image)

For the pyramid example, two triangles sharing just the apex would serve as \( \Delta \) and base \( B \). Consider the square pyramid in Figure 10(a), with \( B \) and \( \Delta \) marked. Mapping \( \Delta = 145 \) to \( \Delta = a_1a_2a_3 \) at the shared pyramid apex, (b) of the figure shows that this is equivalent to Figure 9(a). A hexagonal pyramid is illustrated in the Appendix.

![Figure 10: (a) Square pyramid Schlegel diagram, apex 5, square base 1234. (b) Relabeled to match Figure 9(a).](image)

This leaves the case where there are no two disjoint faces, nor two faces that share just a single vertex: every pair of faces shares two or more vertices. If two faces share non-adjacent vertices, they cannot both be convex. So in fact the condition is that each two faces share an edge. Then, it is not difficult to see that \( G \) can only be a tetrahedron, as the following argument shows.

Start with Euler’s formula, \( V - E + F = 2 \). Each vertex \( v \) must be incident to exactly three faces, because, if \( v \) has degree \( \geq 4 \), then each non-adjacent pair of faces incident to \( v \) cannot share an edge. So \( 3V = 2E \). Substituting into Euler’s formula yields \( F = 2 + E/3 \).

Because each pair of faces share an edge, \( F(F - 1) \)
double counts edges: 6E = F(F − 1). Substituting,

\[ F = 2 + E/3 \]
\[ E = F(F − 1)/2 \]
\[ F = 2 + F(F − 1)/6 \]

\[ F^2 − 7F + 12 = 0 \]

The two solutions of this quadratic equation are \( F = 3 \), which cannot form a closed polyhedron, and \( F = 4 \). The tetrahedron is the only polyhedron with four faces, and indeed \( F = 4 \) implies \( V = 4 \) and \( E = 6 \).

So the only case remaining is a tetrahedron. But it is well known that the thin, nearly flat tetrahedron unfolds with overlap: Figure 11. And since there is only one tetrahedron combinatorial type, this completes the inventory.

Figure 11: Figure 28.2 [detail], p.314 in [DO07]: tetrahedron overlap. Blue: exterior. Red: interior. Cut tree \( T = abcd \). \( T \) is a combinatorial ‘Z’.

5 Theorem

We have proved this theorem:

**Theorem 2** Any 3-connected planar graph \( G \) can be realized as a convex polyhedron \( P \) in \( \mathbb{R}^3 \) that has a spanning cut-tree \( T \) such that the edge-unfolding of \( P \setminus T \) overlaps in the plane.

So together with Ghomi’s result,\(^7\) any combinatorial polyhedron type can be realized to unfold and avoid overlap, or realized to unfold with overlap.

Returning to Malkevitch’s example of a combinatorial cube, consider Figure 12. Starting from the standard Schlegel diagram for a cube (one square inside another (B)), trapezoid faces between the squares), horizontal stretching (step (7) of Algorithm 1) is applied to squeeze the top and bottom squares to \( 1 \times 2 \) and \( 2 \times 4 \) diamonds, so that the angle at \( a_2 \) becomes small, in this case \( 2 \arctan(1/2) \approx 53^\circ \). The lifting leaves the curvatures at \( a_1, a_2 \) to be small enough, \( 6.0^\circ, 6.5^\circ \), so the vertical scaling step (6) of Algorithm 1 is not needed.

![Figure 12](image)

Figure 12: Unfolding of a combinatorial cube. Diagonals in the left figure are an artifact of the software; all lateral faces are planar congruent trapezoids. Base \( B \) attached left of \( b_1b_4 \) not shown. Vertex coordinates:

\[ (−1, 0, 0.5), (1, 0, 0.5), (0, −2, 0.5), (0, 2, 0.5), \]
\[ (−2, 0, 0), (2, 0, 0), (0, −4, 0), (0, 4, 0) \]

6 Open Problem

Is there a combinatorial type \( P \) of a Hamiltonian polyhedron (i.e., one with a Hamiltonian path), such that, for every metric realization \( P \subset \mathbb{R}^3 \), and every Hamiltonian path \( T, P \setminus T \) unfolds to a net?

This restricts Malkevitch’s question to combinatorial Hamiltonian polyhedra \( P \), and restricts \( T \) to a Hamiltonian path, producing a zipper unfolding [DDL+10]. Note: Some convex polyhedra are not Hamiltonian, e.g., the rhombic dodecahedron.

To rephrase the question: Is there a combinatorial Hamiltonian polyhedron whose every metric realization and zipper unfolding avoids overlap? Or is there instead an analog of Theorem 2 showing that even under these restrictions, there is always a realization and a zipper unfolding that overlaps?

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\(^6\)Similar logic is used to form Szilassi’s polyhedral torus.

\(^7\)See [SZ18] for a different proof of [Gho14].
References


A Hexagonal Prism

Figure 13 shows a hexagonal prism, following the model of the square prism in Figure 10: no pair of faces are disjoint, but $\Delta$ and $B$ marked share just one vertex. Figure 14 shows its overlapping unfolding.

Figure 13: (a) Schlegel diagram of a hexagonal prism. (b) Overhead view of combinatorial rearrangement. The cut tree $T$ is marked with red and blue paths.

Figure 14: Unfolding of Figure 13(b). Curvature at $v_6$ and $v_5$ is $5.7^\circ$. Vertices 1 and 4 are collinear with 26 and 35 respectively. Hexagon: 123456.