Puzzling and Apuzzling Graphs [1]

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Abstract

Let G be a graph with chromatic number $\chi(G)$ and consider a partition P of G into connected subgraphs. P is a **puzzle** on G if there is a unique vertex coloring of G using 1, 2, ..., $\chi(G)$ such that the sums of the numbers assigned to the partition pieces are all the same. P is an **apuzzle** if there is a unique vertex coloring such that the sums are all different.

We investigate the concept of puzzling and apuzzling graphs, detailing classes of graphs that are puzzling, apuzzling and neither.

1. Introduction

This research grew out of earlier work on variants of sudoku [2],[3]. Instead of the usual partition of a square, the sudoku board was divided into different regions. No numbers were given, but the challenge was to place numbers so that no number appeared twice in a row or column, and the sums of the numbers in each region were the same. From puzzles such as these, it is only a few steps to puzzling graphs.

By "vertex coloring," we mean a minimal coloring of the vertices using the numbers 1,2, . . . , $\chi(G)$. For this paper, "partition" will mean a partition where the pieces are all connected.

Definition 1. *Let G be a connected graph with chromatic number $\chi(G)$. A* **puzzle** *on G is a partition*

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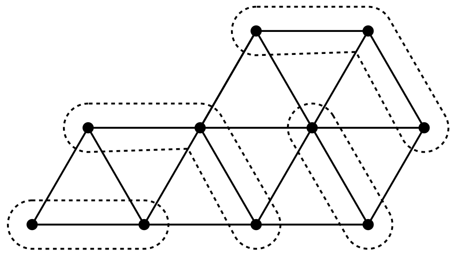
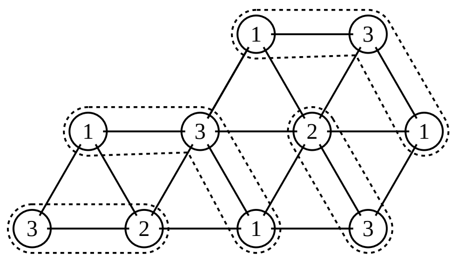


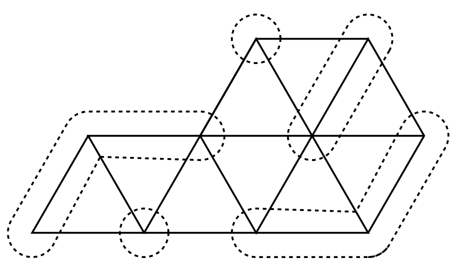
such that there is exactly one vertex coloring with the property that the sums of the vertex labels of the partition pieces are all the same.



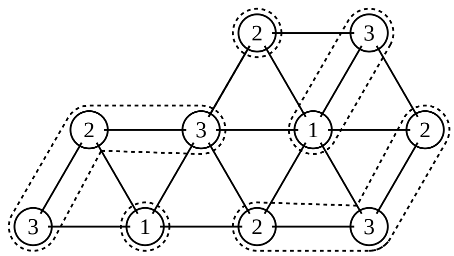
A graph is **puzzling** *if there is a puzzle on the graph.*

In [2] and [3], a second sort of puzzle made an appearance. Here the challenge was to place numbers so that the sums of the numbers in the regions were all different.

Definition 2. *An* **apuzzle** *on a graph G is a partition of G*



such that there is exactly one vertex coloring such the sums of the vertex labels of the partition pieces are all different.



A graph is **apuzzling** *if there is an apuzzle on the graph.*

Many graphs are neither puzzling nor apuzzling. It is easy to see, for example, that for $n > 1$, K_n, the complete graph on n vertices, is not puzzling. A puzzle must have a piece with at least two vertices. Any solution to the puzzle generates another solution by switching the colors of the two vertices in the piece. A similar argument shows that K_n is not apuzzling.

The work here is in a tradition of research into graphs with special colorings, a tradition that includes "uniquely colorable" graphs ([1]), "magic" graphs ([4],[5],[6]), and more recently, "anti-magic" graphs.

In this paper we determine the puzzlicity and apuzzlicity of a number of common classes of graphs. In section 1 we show that no path and no cycle is either puzzling or apuzzling. In section 2 we determine for all m and n which complete bipartite graphs $K_{m,n}$ are puzzling and which are apuzzling. In section 3 we find infinite puzzling and apuzzling graphs and puzzles and apuzzles of arbitrarily large chromatic number. Section 4 is devoted to a miscellany of open questions and partial results.

2. Paths and Cycles

2.1. Paths

By "path," we simply mean a connected graph where the degree of no vertex is greater than 2. Paths have only two vertex colorings, making their analysis fairly simple.

Proposition 1. *For all n, P_n, the path on n vertices, is not puzzling.*

Proof. Suppose a path has been partitioned into connected pieces. In any equal coloring, a piece of length $2k$ must have a vertex sum of $3k$. A piece of length $2k+1$ must have a vertex sum of either $3k+1$ or $3k+2$. Thus, if an equal coloring exists, all pieces must have the same length. If that length is even, clearly both vertex colorings are equal colorings, hence there is no unique solution.

If the length of all pieces of the partition is odd, then adjacent pieces must have different sums (one begins and ends with '1'; the other begins and ends with '2'). If the partition has more than one piece then, there is no equal coloring. If the partition has only one piece, there is no unique equal coloring.

□

Proposition 2. *Paths are not apuzzling*

Proof. A partition of a path has an unequal coloring iff the pieces are all of different lengths. In that case, both vertex colorings are unequal so there is no unique unequal coloring. □

2.2. Cycles

Proposition 3. *For all n, C_n, the cycle on n vertices, is not puzzling.*

Proof. If n is even, the chromatic number of C_n is 2. In this case the argument in the proof of Proposition 1 can be applied to show there is no puzzle. We may assume then, that the number of vertices is odd and the chromatic number is 3.

Suppose that an odd cycle is partitioned and there is a equal coloring and it is unique.

Claim 1. *If one piece of the partition is colored with all three colors, then the coloring is not unique.*

Proof of Claim 1: If a piece has all three colors, then somewhere in the piece all three colors will be next to each other. There are four possibilities for the neighbors of colors abc. In each case the colors can be rearranged to form a second, different equal coloring.

$$\begin{matrix} babca \\ babcb \\ cabca \\ cabcb \end{matrix} \quad \text{which can be rearranged:} \quad \begin{matrix} bacba \\ bcbab \\ cacba \\ cbacb \end{matrix}$$

This contradicts the assumption that the equal coloring is unique. □ (Claim 1)

This result tells us that a puzzling partition must contain only two distinct colors in each piece. The possible pairs of colors are 1 & 2, 2 & 3, and 1 & 3. For each of these the possible sums of a piece with $2k$ vertices are as follows:

colors	possible sums
$1,2$	$3k,\ 3k+1,\ 3k+2$
$1,3$	$4k,\ 4k+1,\ 4k+3$
$2,3$	$5k,\ 5k+2,\ 5k+3$

For a piece with an odd number of vertices, the highest sum results when the larger of the two numbers bookends the sequence, that is, borders the sequence at both ends. It is clear from the table that within a color pair if two pieces have the same sum they must contain the same number of vertices. Thus, in a puzzling graph we can have pieces of only one size from each color pair. In particular, they will be either all odd or all even.

In addition, in order for the whole cycle to have an odd number of vertices there must be an odd number of pieces containing an odd number of vertices.

Note also that two pieces from the same color pair with an odd number of vertices and the same sum cannot be adjacent to each other.

Claim 2. *A puzzling graph must contain pieces of all three color pairs.*

Proof of Claim 2: Suppose there are only two different color pairs. Since the cycle is odd, at least one of the pairs must consist of odd-length pieces (call it A). Two identical odd pieces can't be adjacent to each other, so between two A pieces there must be non-A pieces with a total length that is odd. The only way this can occur is if the second pair (call it B) is also of odd length and between any two A pieces there is exactly one B piece and vice-versa. But then there will be exactly as many A pieces as B pieces and the total length of the cycle will be even, a contradiction. $\square$ (Claim 2)

Now note that for the pair $\langle 1,3\rangle$ if there are more than four vertices in a piece then we can make the following substitutions and maintain the same sum:

$$\ldots 3\underbrace{1\ 3\ 1}_{2\ 1\ 2}3\ldots \text{ or } \ldots 1\underbrace{3\ 1\ 3}_{2\ 3\ 2}1\ldots$$

Thus pieces of the $\langle 1,3\rangle$ color pair must have at most four vertices. This in turn limits the possible sums of the pieces to 4, 5, 7 and 8, as these are the only totals obtainable by a $\langle 1,3\rangle$ piece containing four or fewer vertices.

<u>Case 1:</u> *Each piece sums to 4*

This is not possible because a puzzling partition must contain pieces of all three color pairs and this sum cannot be attained by the $\langle 2,3\rangle$ pair.

<u>Case 2:</u> *Each piece sums to 5*

In this case, the partition must contain the following pieces: (1 3 1), (2 1 2) and (2 3). In a puzzling partition, every (2 3) piece or sequence of adjacent (2 3) pieces must be sandwiched between a (1 3 1) and a (2 1 2). This is the case because a group of (2 3)'s cannot be sandwiched between (2 1 2)'s without violating coloring rules and if it is located between two (1 3 1)'s then the (2 3) could be changed to a (3 2), giving multiple solutions. However, the total number of (1 3 1)'s and (2 1 2)'s must be odd so there must be more of one of these than the other, i.e. there are at least two (1 3 1)'s without a (2 1 2) between them or at least two (2 1 2)'s without a (1 3 1) between them. Since odd cycles of the same sum cannot be placed next to each other, we must place a (2 3) in between them. But as demonstrated above, a puzzling partition cannot contain a (2 3) between two (2 1 2)'s or (1 3 1)'s. Thus the pieces of the partition cannot sum to 5.

<u>Case 3:</u> *Each piece sums to 7*

The possible pieces that sum to 7 are (3 1 3), (1 2 1 2 1) and (2 3 2). Any of these pieces can be placed next to one another and there must be at least least one piece from each color pair. Thus multiple solutions can be obtained by swapping all of the (3 1 3)'s for (2 3 2)'s and vice versa.

<u>*Case 4:*</u> *Each piece sums to 8*

Here, the possible pieces are (2 1 2 1 2), (3 2 3), (1 3 1 3), and (3 1 3 1). To ensure the graph is an odd cycle there must be an odd number of (2 1 2 1 2)'s and (3 2 3)'s, hence there must be more of one than the other.

First suppose that there are more (3 2 3)'s than (2 1 2 1 2)'s. There must be at least two (3 2 3) pieces that do have a (2 1 2 1 2) in between them. However, (3 1 3)'s cannot be adjacent to each other, nor can they be separated by only (1 3 1 3)'s and (3 1 3 1)'s. So there cannot be more (3 2 3)'s than (2 1 2 1 2)'s.

Now suppose that there are more (2 1 2 1 2)'s than (3 2 3)'s. Similarly, there must be at least two (2 1 2 1 2) pieces that have a (3 2 3) in between them. Because there cannot be adjacent (2 1 2 1 2) pieces, there must be one of (1 3 1 3) or (3 1 3 1)'s between them. But then the 1's and 3's in these pieces can be switched, resulting in a second solution.

□

Proposition 4. *Cycles are not apuzzling*

Proof. As with puzzles, if the number of vertices is even, the proof of Proposition 2 shows that the cycle is not apuzzling.

Suppose then that $\mathcal{C}$ is an odd cycle and that $\bigcup_i^n P_i = \mathcal{C}$ is an apuzzle. Suppose that S is the unique solution to the apuzzle. The chromatic number of C is 3 and we know from Claim 1 of Proposition 3 that in no P_i do all three colors, 1,2,3 appear. Let m be the largest sum of any of the P_i in the solution S.

Claim 1. $m < 9$

Proof of Claim 1: Suppose $m \geq 9$. Let P_k be the piece with sum m.

<u>*Case 1:*</u> *P_k is a piece using '1's and '2's.*

Then we can change one number to a '3' and get a different solution—a contradiction.

<u>*Case 2:*</u> *P_k is a piece using '1's and '3's.*

Since $m \geq 9$ there will be one inner '1' in P_k. We can replace that with a '2' to get a different solution—again, a contradiction.

<u>*Case 3:*</u> *P_k is a piece using '2's and '3's.*

<u>*Case 3a:*</u> $m \equiv 0 \mod 5$.

We can replace one '2' with a '1'. Since the solution is unique, there must be a piece, P_q with sum $m-1$. Since $m-1 \equiv 4 \mod 5$, P_q must either use '1's and '2's or '1's and '3's. If it uses '1's and '2's, we can replace a '1' with a '3' and get a new solution. If it uses '1's and '3's, we can replace a '1' with a '2'. The two changes result in a solution that is new (note that P_q will have all three colors in it, since $m-1 \geq 8$).

<u>*Case 3b:*</u> $m \equiv 2 \mod 5$.

We can replace one '2' with a '1'. Since the solution is unique, there must be a piece, P_q with sum $m-1$. Since $m-1 \equiv 1 \mod 5$, P_q must either use '1's and '2's or '1's and '3's. The rest of the proof for this case follows that for case 3a.

<u>*Case 3c:*</u> $m \equiv 3 \mod 5$.

Note first that as $m \geq 9$, m must be at least 13. We can replace one '3' with a '1'. Since the solution is unique, there must be a piece, P_q with sum $m-2$. Since $m-1 \equiv 1 \mod 5$, P_q must either use '1's and '2's or '1's and '3's. If it uses '1's and '2's, we can replace a '1' with a '3' and get a new solution (note that P_q will have all three colors in it, since $m-2 \geq 11$). If it uses '1's and '3's, there will be at least two '1's that can be changed to '2's. This a new solution because P_k will have all three colors. $\square$ (Claim 3)

Solution S can't include a piece summing to 6. The pieces that sum to 6 (and don't use all three colors) are (1 2 1 2) and (2 1 2 1). But a (1 2 1 2) piece could be replaced by a piece with a sum of 9, contradicting Claim 3. This can be done no matter what colors appear at the ends of (1 2 1 2) as follows:

$$\begin{array}{c} 2\ (1\ 2\ 1\ 2)\ 1 \\ 3\ (1\ 2\ 1\ 2)\ 1 \\ 2\ (1\ 2\ 1\ 2)\ 3 \\ 3\ (1\ 2\ 1\ 2)\ 3 \end{array} \Longrightarrow \begin{array}{c} 2\ (3\ 1\ 3\ 2)\ 1 \\ 3\ (2\ 3\ 1\ 3)\ 1 \\ 2\ (3\ 1\ 3\ 2)\ 3 \\ 3\ (2\ 3\ 3\ 1)\ 3 \end{array}$$

S can't include a piece summing to 7. The pieces that sum to 7 are (3 1 3) and (2 3 2). A (3 1 3) piece can be replaced by a piece summing to 6:

$$\begin{array}{c} 1\ (3\ 1\ 3) \\ 2\ (3\ 1\ 3) \end{array} \Longrightarrow \begin{array}{c} 1\ (2\ 1\ 3) \\ 2\ (1\ 2\ 3) \end{array}$$

The same is true for (3 1 3).

S can't include a piece summing to 8. The only pieces that sum to 8 are (3 2 3), (3 1 3 1), (1 3 1 3), and (2,1,2,1,2). The first can be replaced by (3 1 3), which we have seen is not possible. The remaining examples can be replaced by pieces with larger sums, contradicting Claim 3.

Finally, consider now that S has only pieces summing to 1, 2, 3, 4, and 5. Such pieces may only have 1 or two vertices, for pieces summing to 4 or 5, (1 2 1), (2 1 2) and (1 3 1) can all be replaced by pieces summing to 6 by the methods above. But if five or fewer pieces of length 1 and 2 are arranged in a circle, the arrangement will have bilateral symmetry. By reflecting the colors across the axis of symmetry, a new solution can be obtained, contradicting the uniqueness of S.

$\square$

3. Complete Bipartite Graphs

3.1. Puzzling Bipartite Graphs

Proposition 5. *The complete bipartite graph, $K_{n,k}$ is puzzling if and only if at least one of $2n+k$, $n+2k$ has a factor greater than 1 and less than both n and k, and if both n, k are even, one of the two is greater than 4.*

Proof. For $K_{n,k}$, let A be a set of n vertices and B be a set of k vertices such that every edge of $K_{n,k}$ connects a member of A with a member of B. $\chi(K_{n,k}) = 2$. Any coloring of $K_{n,k}$ must either color all vertices in A **1** and all vertices in B **2** or the reverse.

Any puzzle on $K_{n,k}$ is a partition of $A \cup B$ into i sets. Note that each piece of the partition must contain points of both A and B. Otherwise such a piece would have only one member. Then the sum of the labels of each piece would have to be either 1 or 2. It should be clear that neither of these is possible.

One direction of the theorem is fairly simple. Suppose that $K_{n,k}$ is puzzling. Let P be a puzzle on $K_{n,k}$ with i pieces and let us say that in the solution, the vertices of A are colored 1. Then the sum of all colors in the graph is $n+2k$ and since all pieces have the same sum, i divides $n+2k$. We have $1 < i$ and since each pieces contains points of both A and B, $i \leq n, k$. Finally, we can't have i equal to either n or k: say $n = i$. Then each piece contains exactly one element of A, hence to have equal sums, every piece must contain the same number of elements of B, so there won't be a unique solution (switching all the colors yields a second solution).

For the last clause, it is easy to check that $K_{4,4}$ is not puzzling.

For the other direction, we must break the proof up into cases.

<u>Case 1:</u> *n and k are even.*

Let $n \leq k$. Consider then, the following two-piece partition: One piece has $\frac{n}{2}+1$ vertices from A and $\frac{k}{2}-2$ vertices from B. The other piece has $\frac{n}{2}-1$ vertices from A and $\frac{k}{2}+2$ vertices from B. Since $k > 4$ these are connected subgraphs. Furthermore, coloring the vertices in A **2** and the vertices in B **1** is the unique solution, as

$$2\left(\frac{n}{2}+1\right)+1\left(\frac{k}{2}-2\right) = 2\left(\frac{n}{2}-1\right)+1\left(\frac{k}{2}+2\right).$$

<u>Case 2:</u> *One of n, k is even, the other odd.*

Suppose n is even and $k = 2j+1$ (we do assume this since $k > 2$). Consider the following two-piece partition: One piece has $\frac{n}{2}+1$ vertices from A and j vertices from B. The other piece has $\frac{n}{2}-1$ vertices from A and $j+1$ vertices from B. The pieces are connected (since $n \geq 4$) and coloring the vertices in A **1** and the vertices in B **2** is the unique solution, as

$$1\left(\frac{n}{2}+1\right)+2\left(j\right) = 1\left(\frac{n}{2}-1\right)+2\left(j+1\right).$$

<u>Case 3:</u> *Both n, k are odd.*

We can assume without loss of generality that i is a factor of $n+2k$, $1 < i < n, k$. Let $h = \frac{n+2k}{i}$. Note that h must be odd.

<u>Case 3a:</u> *i doesn't divide n*

Claim 2. *We can write $n = ir + t$ with t even and $1 < \frac{t}{2} < i$.*

Proof of the Claim: We certainly have $n = ir + t$ with $1 < t < i$. Suppose t is odd. Then since $i|(n+2k)$, i is odd, so r must be even and at least 2. Then letting $r' = r - 1$ and $t' = i + t$ we can have $n = ir' + t'$ with $1 < t' < 2i$, or $1 < \frac{t}{2} < i$.

From the Claim we may assume $2k + t$ is even. Since i divides $n + 2k$ and $n - t$, we have that i divides $2k + t$ and we can let $s = \frac{2k+t}{2i} \geq 1$. Now consider the partition of $A \cup B$ with i pieces:

$\frac{t}{2}$ of the pieces have $r + 2$ vertices from A plus $s - 1$ vertices from B and $i - \frac{t}{2}$ of the pieces each have r vertices from A plus s vertices from B.

It follows that coloring vertices in A **1** and vertices in B **2** is the unique solution since

$$1(r+2) + 2(s-1) = 1(r) + 2(s).$$

In A,

$$\frac{t}{2}(r+2) + (i - \frac{t}{2})r = ir + t = n$$

and in B,

$$\frac{t}{2}(s-1) + (i - \frac{t}{2})(s) = -\frac{t}{2} + is = -\frac{t}{2} + i\frac{2k+t}{2i} = k.$$

<u>Case 3b:</u> *i divides n*

Say $ir = n$. Since i divides n, i is odd and i divides $n + 2k$, so i divides k. Let $s = \frac{k}{i} - 1$. This is positive since $is = k - i$ and $k > i$.

Consider, then, the partition of $A \cup B$ with i pieces:

$i - 2$ of the pieces have r vertices from A plus $s + 1$ vertices from B and one piece has $r + 2$ vertices from A plus s vertices from B, and one piece has $r - 2$ vertices from A plus $s + 2$ vertices from B.

Note that as n is odd, $n = ir$, $n > i$, we must have r odd and greater than 1, hence at least 3. Thus all these pieces are connected.

It follows that coloring vertices in A **1** and vertices in B **2** is the unique solution since

$$1(r) + 2(s+1) = 1(r+2) + 2(s) = 1(r-2) + 2(s+2).$$

In A,

$$(i-2)(r) + (r+2) + (r-2) = ir = n$$

and in B,

$$(i-2)(s+1) + (s) + (s+2) = i(s+1) = i\frac{k}{i} = k.$$

□

Tripartite graphs are much more complicated. Partial results include that $K_{1,1,r}$ is neither puzzling nor apuzzling; $K_{1,2,r}$ is not apuzzling, and $K_{1,2,2r}$ is puzzling.

3.2. Apuzzling Bipartite Graphs

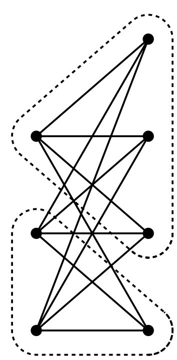
Proposition 6. *The complete bipartite graph, $K_{n,k}$ is apuzzling if and only if $n \geq 4$ and $k \geq 3$ or $n \geq 3$ and $k \geq 4$.*

Proof. One direction is easily seen by inspection. We will discuss only the largest case, $K_{3,3}$. If the three vertices from one side are separated into three pieces, each piece must have a different number of vertices from the other side, 0, 1, and 2. The resulting partition has two solutions.

If the vertices are placed in a single piece, it must include a piece from the other side. If it includes only 1, there are no solutions. If it includes two or more, there are two solutions.

Finally, if the vertices are split, 1 and 2. the piece with 2 must include at least one from the other side. If it includes 2 or 3, there will be two solutions. If it includes only one, again there will be two solutions.

For the other direction, note first that every two-piece puzzle on a graph of chromatic number 2 is also an apuzzle. The smallest example of this in complete bipartite graphs is $K_{3,4}$.



The unique solution results in sums of 4 and 7. Using this as a base we can construct an puzzle on $K_{3+a,4+b}$ by putting all additional vertices in a new piece. That is possible as long as both a and b are non-zero. The new piece will have a sum different from 4 and 7 with only four exceptions: $K_{5,5}$, where the new piece has sum 4, and $K_{4,7}$, $K_{6,6}$, $K_{8,5}$, where the new piece has sum 7.

For $K_{5,5}$, we can create an apuzzle with a new piece containing a vertex from each side and a fourth piece with a single vertex.

There are two-piece puzzles on $K_{4,7}$, $K_{6,6}$ and $K_{8,5}$ by the proof of Proposition 5. These are also apuzzles.

For $b = 0$, and for $a = 0$, b even, $K_{3+a,4+b}$ has a two-piece puzzle, hence a two-piece apuzzle.

We are left with the case $a = 0$, $b = 2p + 1$, odd. For this, take the partition of $K_{3,4+2p}$ which is both a puzzle and an apuzzle. Let the remaining vertex be the third piece. This is an apuzzle and concludes the proof. □

We note in passing the following:

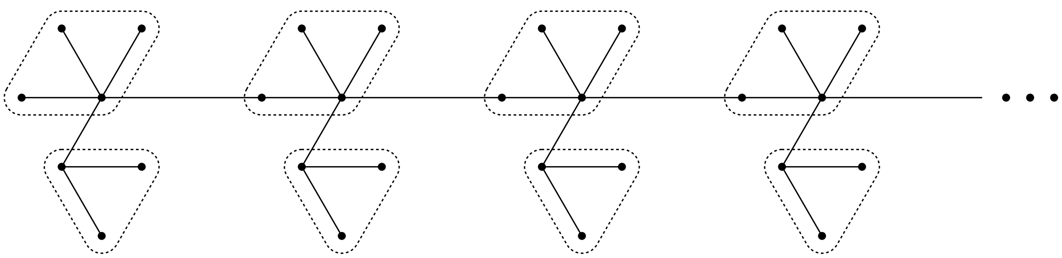
Proposition 7. *A graph G with chromatic number 2 is puzzling (apuzzling), then the complete bipartite graph on the vertex set is also puzzling (apuzzling).*

4. Infinite Puzzling and Apuzzling Graphs and Graphs of Arbitrarily Large Chromatic Number

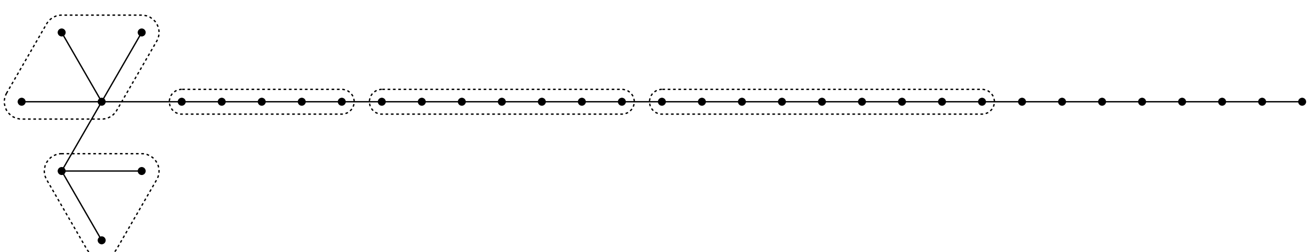
4.1. Infinite Graphs

Proposition 8. *There exist infinite puzzling graphs and infinite apuzzling graphs.*

Proof. For an infinite puzzling graph, we can simply link together infinitely many copies of a single puzzling graph.



For an infinite apuzzling graph, we can add an infinite path with larger and larger pieces.



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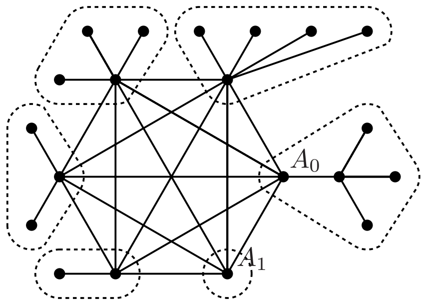
4.2. Arbitrarily Large Chromatic Numbers

Proposition 9. *For all n there is a puzzling graph with chromatic number n.*

Proof. An example will illustrate the proof. For $\chi(G) = 6$:



The sum of the labels of the vertices in piece with A_0 is at least 6. The sum for piece with A_1 is at most 6, so all pieces must have sum 6. That is sufficient to deduce the coloring of all the vertices.

For a puzzling graph with chromatic number n in general, we take K_n with vertices $\{A_i\}_{i<n}$. We attach to A_0 a single vertex with $n-3$ leaves. This is one piece of the partition. We then attach to A_i, $i > 0$, $i-1$ leaves. We set each A_i with its leaves as a piece of the partition. As in the example, the piece with A_0 must have sum at least n while the piece with A_1 has sum at most n, thus the sums of all the pieces are n. It is not hard to show then that A_0 must be labelled 1, the vertex attached to it must be labelled 2, all the leaves added must be labelled 1, and for each $i > 0$, A_i must be labelled $n+1-i$.

□

Proposition 10. *For all n there is an apuzzling graph with chromatic number n.*

Proof. The graph G with $\chi(G) = n$ will have n sets of vertices, V_i. It will be constructed in stages. Each stage will add vertices to the V_i. In the end, the graph will be the complete n-partite graph, $K_{a_1,\ldots,a_n}$, where $|V_i| = a_i$, for $1 \le i \le n$.

Each stage of the construction will add to the number of vertices in each V_i. Each stage will also construct pieces of the apuzzle. Stage k will, by the pieces it adds, ensure that the vertices in $V_{n-(k-1)}$ are colored $n-(k-1)$. In addition, there will be pieces whose sums comprise the set of numbers from 1 to $(k+1)n - \sum_{i=1}^{k} i$.

Stage **0**: In this stage we start each V_i with one vertex and construct n apuzzle pieces each consisting of a single vertex. In this way, the sums of the pieces are exactly the numbers from 1 to n. No color has been forced.

Stage **1**: We add $n-1$ vertices to V_n, one vertex each to V_i, $i < n$, and construct $n-1$ pieces each consisting of two vertices, one from V_n and one from V_i, $i < n$. This forces the vertices of V_n to be colored n, for otherwise one of the sums of the new pieces will be n, already the sum of a piece. With the vertices of V_n colored n, the sums are now all the numbers from 1 to $2n-1$.

Assuming, after Stage k, that we have forced each V_i, $i > n-(k-1)$, to be colored i and the sums of the pieces constructed so far are the number from

1 to $(k+1)n - \sum_{i=1}^{k} i$. Then for Stage $k+1$, we add $n-(k+1)$ vertices to V_i, $i > n-k$, and one vertex each to V_i, $i \leq n-k$. We also construct $n-k$ new pieces, each consisting of a vertex from each V_i, $i > n-k$, plus one vertex from a V_i, $i \leq n-k$. This forces the vertices of V_{n-k} to be colored $n-k$, for otherwise the smallest sum of a new piece will be less than $\sum_{i=n-k+1}^{n} i$ plus 1 plus $n-k-1$, that is,

$$\begin{aligned}
\textstyle\sum_{i=n-k+1}^{n} i + 1 + (n-k-1) &= kn - \textstyle\sum_{i=1}^{k-1} i \ + 1 \ + (n-k-1) \\
&= (k+1)n - \textstyle\sum_{i=1}^{k-1} i \ - k \\
&= (k+1)n - \textstyle\sum_{i=1}^{k} i
\end{aligned}$$

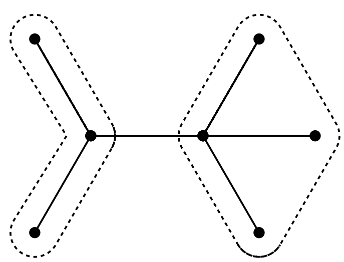
and this is already the sum of an earlier constructed piece.

The last stage is Stage $n-1$. With this the color of each V_i, $i > 1$ is forced to be i and by elimination, for $i = 1$ as well.

□

5. Open questions and partial results

1. The partitioned graph



   has the property that it is both a puzzle and an apuzzle. It is not hard to see that a two-piece partition on a bipartite graph is an apuzzle if and only if it is a puzzle. With some effort we found a partitioned graph with chromatic number 3 that is also simultaneously a puzzle and an apuzzle. Are there such examples of all chromatic numbers?
2. The graph above, without the partition, has the property that there is a unique partition that forms a puzzle. Are there graphs with this property of all chromatic numbers? The same graph is uniquely partitioned to form an apuzzle. Again we can ask if examples exist of higher chromatic number.
3. In [3] another sort of puzzle was explored in which the sums of the regions could be the same or different, but all had to be prime numbers. We can similarly define a graph to be **prime puzzling** if it can be partitioned so that there is a unique coloring in which the sums of the regions are all prime. This definition turns out to be too easy to satisfy; restricting partitions to no more than three pieces makes it interesting. With this

restriction, we conjecture that P_n is prime puzzling for all $n > 4$. The conjecture follows from the Goldbach conjecture.

Not all trees are prime puzzling, but most appear to be. We conjecture that for some n, all trees with at least n vertices of degree greater than 1 are prime puzzling.

4. Some puzzles are "parsimonious" in that the graph partition has only two pieces. We can prove that there are graphs with such puzzles of every chromatic number. A parsimonious apuzzle of chromatic number 3 is not possible. But in general, how many pieces does an apuzzle of chromatic number k have to have?

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