Picker Engineering Program

THE LAPLACE TRANSFORM II

EGR 320: Signals & Systems
Lectures 26/27: April 11 and 13, 2011

Topics for Today and Wednesday

- Properties of the ROC
- The Inverse Laplace Xform and partial fraction expansion
- Properties of the Laplace Xform
- Analysis of LTI systems w/ the Laplace Xform
- Unilateral Laplace Xform

Properties of the Laplace Transform's ROC

Property 1: The ROC of $X(s)$ consists of strips parallel to the $j\omega$ axis in the s-plane.

Property 2: For rational Laplace transforms, the ROC does not contain any poles.

Property 3: If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

Properties of the Laplace Transform's ROC

Property 4: If $x(t)$ is right sided, and if the line $\Re\{s\} = \sigma_o$ is in the ROC, then all values of s for which $\Re\{s\} > \sigma_o$ will also be in the ROC.

Property 5: If $x(t)$ is left sided, and if the line $\Re\{s\} = \sigma_o$ is in the ROC, then all values of s for which $\Re\{s\} < \sigma_o$ will also be in the ROC.

Property 6: If $x(t)$ is two sided, and if the line $\Re\{s\} = \sigma_o$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $\Re\{s\} = \sigma_o$.

Property 7: If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

Property 8: If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the ROC is the region in the s-plane to the right of the rightmost pole. If $x(t)$ is left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

The Inverse Laplace Transform

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

- Evaluate at a constant σ and from $\omega=-\infty$ to $\omega=\infty$.
- The contour of integration is a line parallel to the $j\omega$ axis.
- General evaluation requires contour integration in the complex plane and is difficult!
- We will use technique of partial-fraction expansion combined with Laplace transform tables to determine inverse transforms.

The Inverse Laplace Transform

$$\begin{aligned} X(s) &= \frac{B(s)}{A(s)} \\ &= \frac{b_M s^M + b_{M-1}s^{M-1} + \cdots + b_1 s + b_o}{s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_o} \\ &= \underbrace{\sum_{k=0}^{M-N} c_k s^k}_{\text{term exists only when } M>N} + \sum_{k=1}^{N} \frac{A_k}{s-d_k} \end{aligned}$$

Inverse Transform Techniques for Rational Functions

Given an s-domain expression

$$\mathbf{V(s)} = \frac{\mathbf{N(s)}}{\mathbf{D(s)}} = \frac{(s+a_1)(s+a_2)\dots(s+a_m)}{(s+\alpha_1)(s+\alpha_2)\dots(s+\alpha_n)} = \frac{A}{s+\alpha_1} + \frac{B}{s+\alpha_2} + \dots + \frac{N}{s+\alpha_n}$$

- *N(s)* and *D(s)* are polynomials in *s*
- Zeros: values of s for which *N(s)=0*
- Poles: values of s for which *D(s)=0*
- The degree of the numerator is less than the degree of the denominator (i.e., $n>m$)

Partial Fraction Expansion for Simple Poles

$$\mathbf{F(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$\mathbf{F(s)} = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \cdots + \frac{k_n}{s+p_n}$$

We need to find the $k_1, k_2, \cdots k_n$.

$$(s+p_1)\mathbf{F(s)} = k_1 + \frac{(s+p_1)k_2}{s+p_2} + \cdots + \frac{(s+p_1)k_n}{s+p_n}$$

Since $p_i \neq p_j$, setting $s = -p_i$ leaves only k_1 on the right-hand side.

$$(s+p_1)\mathbf{F(s)}|_{s=-p_1} = k_1$$

In general,

$$k_i = (s+p_i)\mathbf{F(s)}|_{s=-p_i}$$

Partial Fraction Expansion Example: Distinct Pole Case *(H & K Text Example 8.16)*

Determine the inverse Laplace transform $x(t)$ when

$$X(s) = \frac{s+2}{s^3+4s^2+3s} = \frac{s+2}{s(s+1)(s+3)}$$

Re{s}>0

$$X(s) = \frac{c_1}{s} + \frac{c_2}{s+1} + \frac{c_3}{s+3}$$

Partial Fraction Expansion Example: Repeated Pole Case

Determine the inverse Laplace transform $x(t)$ when

$$X(s) = \frac{3s+4}{(s+1)(s+2)^2}$$

Re{s}>-1

$$= \frac{A_1}{s+1} + \frac{A_2}{(s+2)} + \frac{A_3}{(s+2)^2}$$

Text examples 9.9, 9.10, 9.11 (you should do them)

Laplace Transform Properties

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

Special Case of Time Scaling: Time Reversal

Time Reversal: $x(-t) \leftrightarrow X(-s)$,   with ROC=-R

Example: Determine $x(t)$ that corresponds to $X(s)$.

$$X(s) = \frac{s}{s^2+9}, \quad \text{ROC}: \Re\{s\} < 0$$
$$X(s) = \frac{s+1}{(s+1)^2+9}, \quad \text{ROC}: \Re\{s\} < -1$$

See HW problems 9.22 (a,b,e)

Convolution Example *(adapted from H&K 8.14)*

Determine $x(t) * x(t)$ given $x(t) = u(t) - u(t-1)$, and sketch both $x(t)$ and $x(t) * x(t)$.

Approach 1: Use the property $x(t) * x(t) \leftrightarrow X^2(s)$.

Approach 2: Compute $x(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$.

Initial- and Final-Value Theorems

(requires that x(t) = 0 for t<0)

$$\lim_{s\to\infty} sX(s) = x(0^+)$$
$$\lim_{s\to 0} sX(s) = x(t\to\infty)$$

Determine the final and intial values for $x(t)$ where $x(t) \leftrightarrow X(s)$ and

$$X(s) = \frac{3s^2+4s+1}{s^4+3s^3+3s^2+2s}$$

Final-Value Theorem Example

Determine the final and intial values for $x(t)$ where $x(t) \leftrightarrow X(s)$ and

$$X(s) = \frac{4}{s(s-1)}$$

Analysis and characterization of LTI systems using the Laplace Transform (9.7)

Important points from Section 9.7

1. **System Function (transfer function)**

$$Y(s) \quad = \quad H(s)X(s)$$

2. **Causality:** The ROC associated with the system function for a causal system is a right-half plane to the right of the rightmost pole.

3. **Stability:** An LTI system is stable if and only if the ROC of its system function $H(s)$ inclues the entire $j\omega$ axis.

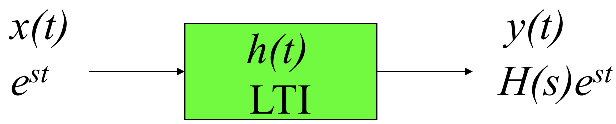
4. **Causality and Stability:** A causal system with system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half of the s-plane – i.e., all of the poles have negative real parts.

Eigenfunction Property of e^{st}

The response of an LTI system to a complex exponential is the same complex exponential multiplied by a complex amplitude

- Recall: $e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t} = e^{\sigma t}[\cos \omega t + j \sin \omega t]$
- e^{st} is an eigenfunction (eigenvector) of the system
- H(s) is a complex amplitude and an eigenvalue of the system

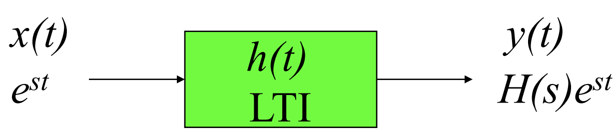
Proof: The response of an LTI system to a complex exponential is the same complex exponential multiplied by a complex amplitude.

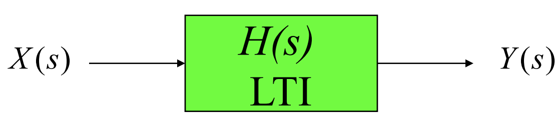


$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau \\
&= e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau}_{H(s)} \\
&= H(s)e^{st}
\end{aligned}
$$

For a specific value of s, the complex constant H(s) is the eigenvalue associated with the eigenfunction e^{st}.

Transfer Function Representation





- H(s) is the transfer function when the system is linear and time invariant
- H(s) is the Laplace Transform of the impulse response h(t)
- H(s)=Y(s)/X(s)

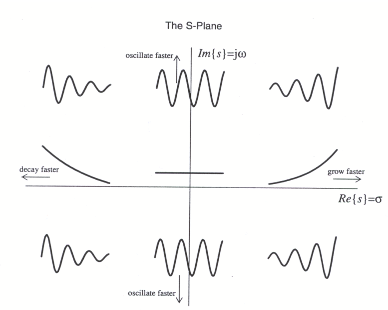
The poles of the transfer function H(s) determine the form of the system's impulse response h(t). For a causal system:

1. If $H(s)$ has a nonrepeated pole p that is real, then $h(t)$ contains a term of the form ce^{pt} for some constant c.
2. If $H(s)$ has a pole p that is real and repeated two times, then $h(t)$ contains a term of the form $c_1e^{pt} + c_2te^{pt}$ for some constant c.
3. If $H(s)$ has a nonrepeated pair of complex poles $p = \sigma \pm j\omega$, then $h(t)$ contains a term of the form $ce^{\sigma t}\cos(\omega t + \theta)$ for some constants c and θ.

What condition is required for the impulse response to converge to zero as $t \to \infty$?

$$Re(p_i) < 0 \qquad \text{for } i = 1, 2, \cdots, N$$

The condition requires that all poles of the system are located in the open left-half plane.



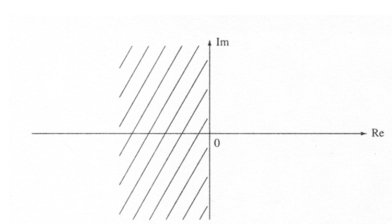


Figure 9.1 Open left-half plane.

Causality and Stability

These statements assume the system is causal.

1. **Stable:** A system with transfer function $H(s)$ is stable if its impulse response $h(t)$ converges to zero as $t \to \infty$. A system is stable if and only if all its poles are located in the open left-half plane.
2. **Marginally stable:** A system with transfer function $H(s)$ is marginally stable if its impulse response $h(t)$ is bounded; that is, there exists a finite positive constant c such that $|h(t)| \leq c$ for all t. A system is stable if and only if all its poles are located in the open left-half plane and non-repeated poles only can be located on the $j\omega$ axis.
3. **Unstable:** A system with transfer function $H(s)$ is unstable if its impulse response $h(t)$ grows without bound as $t \to \infty$. Thus, a system is unstable if and only if there is at least one pole located in the open right-half plane or if there are repeated poles on the $j\omega$-axis.

The One-sided or uni-lateral Laplace Transform

$$X(s) \quad = \quad \int_0^{\infty} x(t)e^{-st}dt$$

- Useful for solving differential equations with initial conditions (section 9.9.3).
- ROC is always a right half plane

Properties of the Laplace Transform

D.2 *Laplace Transform Properties*

Signal	*Unilateral Transform* $x(t) \xleftrightarrow{\mathcal{L}_u} X(s)$ $y(t) \xleftrightarrow{\mathcal{L}_u} Y(s)$	*Bilateral Transform* $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ $y(t) \xleftrightarrow{\mathcal{L}} Y(s)$	*ROC* $s \in R_x$ $s \in R_y$
$ax(t) + by(t)$	$aX(s) + bY(s)$	$aX(s) + bY(s)$	At least $R_x \cap R_y$
$x(t - \tau)$	$e^{-s\tau}X(s)$ if $x(t - \tau)u(t) = x(t - \tau)u(t - \tau)$	$e^{-s\tau}X(s)$	R_x
$e^{s_o t}x(t)$	$X(s - s_o)$	$X(s - s_o)$	$R_x + \mathrm{Re}\{s_o\}$
$x(at)$	$\frac{1}{a}X\left(\frac{s}{a}\right), \quad a > 0$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	$\frac{R_x}{\lvert a\rvert}$
$x(t) * y(t)$	$X(s)Y(s)$ if $x(t) = y(t) = 0$ for $t < 0$	$X(s)Y(s)$	At least $R_x \cap R_y$
$-tx(t)$	$\frac{d}{ds}X(s)$	$\frac{d}{ds}X(s)$	R_x
$\frac{d}{dt}x(t)$	$sX(s) - x(0^-)$	$sX(s)$	At least R_x
$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{1}{s}\int_{-\infty}^{0^-} x(\tau)\, d\tau + \frac{X(s)}{s}$	$\frac{X(s)}{s}$	At least $R_x \cap \{\mathrm{Re}\{s\} > 0\}$

Transform of the Input/Output Differential Equation (Haykin and Van Veen 6.10)

(Unilateral)

Use the Laplace transform to solve the differential equation with the given conditions

$$
\begin{aligned}
\frac{d}{dt}y(t) + 5y(t) &= 5x(t) \\
x(t) &= \frac{3}{5}e^{-2t}u(t) \\
y(0^-) &= -2
\end{aligned}
$$

Transform of the Input/Output Differential Equation

Problem 8.14 (g) (H&K text). Compute the solution to the differential equation using the Laplace Transform.

$$\begin{aligned} \frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 13y(t) &= x(t) \\ x(t) &= u(t) \\ y(0) &= 1 \\ dy(0)/dt &= 1 \end{aligned}$$

Take the Laplace Transform of both sides.

$$\begin{aligned} \left[s^2Y(s) - sy(0-) - dy(0-)/dt\right] + 6\left[sY(s) - y(0-)\right] + 13Y(s)] &= X(s) \\ Y(s)\left[s^2 + 6s + 13\right] - \left[sy(0-) + dy(0-)/dt + 6y(0-)\right] &= X(s) \end{aligned}$$

$$Y(s) = \underbrace{\frac{X(s)}{s^2 + 6s + 13}}_{\text{forced response}} + \underbrace{\frac{sy(0-) + dy(0-)/dt + 6y(0-)}{s^2 + 6s + 13}}_{\text{natural response}}$$

Transform of the Input/Output Differential Equation

$$Y(s) = \underbrace{\frac{X(s)}{s^2 + 6s + 13}}_{\text{forced response}} + \underbrace{\frac{sy(0-) + dy(0-)/dt + 6y(0-)}{s^2 + 6s + 13}}_{\text{natural response}}$$

Substitute for $X(s) = 1/s$, $y(0-) = 1$, and $dy(0-)/dt = 1$.

$$\begin{aligned} Y(s) &= \frac{1}{s(s^2 + 6s + 13)} + \frac{s + 7}{s^2 + 6s + 13} \\ Y(s) &= \frac{s^2 + 7s + 1}{s(s^2 + 6s + 13)} \\ Y(s) &= \frac{s^2 + 7s + 1}{s\left[(s + 3)^2 + 4\right]} \\ Y(s) &= \frac{A}{s} + \frac{Bs + C}{(s + 3)^2 + 4} \end{aligned}$$

Transform of the Input/Output Differential Equation

$$
\begin{aligned}
Y(s) &= \frac{1}{s(s^2+6s+13)} + \frac{s+7}{s^2+6s+13} \\
Y(s) &= \frac{A}{s} + \frac{Bs+C}{(s+3)^2+4}
\end{aligned}
$$

$$
\begin{aligned}
Y(s) &= \frac{\frac{1}{13}}{s} + \frac{\frac{12}{13}s + \frac{85}{13}}{(s+3)^2+4} \\
Y(s) &= \frac{1}{13}\frac{1}{s} + \frac{12}{13}\frac{s+3}{(s+3)^2+4} + \frac{49}{26}\frac{2}{(s+3)^2+4}
\end{aligned}
$$

$$
y(t) = \frac{1}{13}u(t) + \frac{12}{13}e^{-3t}\cos(2t)u(t) + \frac{49}{26}e^{-3t}\sin(2t)u(t)
$$

Pole Locations and the Form of a Signal

1. If $X(s)$ has a nonrepeated pole p that is real, then $x(t)$ contains a term of the form ce^{pt} for some constant c.
2. If $X(s)$ has a pole p that is real and repeated two times, then $x(t)$ contains a term of the form $c_1e^{pt} + c_2te^{pt}$ for some constant c.
3. If $X(s)$ has a nonrepeated pair of complex poles $p = \sigma \pm j\omega$, then $x(t)$ contains a term of the form $ce^{\sigma t}\cos(\omega t + \theta)$ for some constants c and θ.