Homotopy Poisson actions

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Conventional perspectives

Definition
A *Poisson structure* on a manifold M is a Lie bracket on $C^\infty(M)$ that satisfies the Leibniz rule.

Equivalently,

Definition
A *Poisson structure* on a manifold M is a bivector field $\pi \in \mathfrak{X}^2(M) = \Gamma(\wedge^2 TM)$ such that $[\pi, \pi]_{\mathrm{Schouten}} = 0$.

Derived bracket formula:

$$\{f, g\}_\pi = [[\pi, f], g].$$

Differential perspective

$d_\pi := [\pi, \cdot]$ is a degree 1 operator on $\mathfrak{X}^\bullet(M) = \Gamma(\wedge TM)$.

- $[\pi, \pi] = 0 \iff d_\pi^2 = 0$ ($\leftrightsquigarrow$ Lie algebroid T^*M).
- d_π is a graded derivation with respect to the wedge product and the Schouten bracket.

Derived bracket formula:

$$\{f, g\}_\pi = [[\pi, f], g] = [d_\pi f, g].$$

Graded geometry perspective

$\mathfrak{X}^\bullet(M)$ = algebra of "smooth functions" on $T^*[1]M$.

d_π is a derivation of the product structure $\iff$ d_π is a vector field on $T^*[1]M$.

- d_π is deg. 1 and $d_\pi^2 = 0$ $\iff$ d_π is homological (($T^*[1]M, d_\pi$) is an *NQ*-manifold).
- d_π is a derivation of Schouten $\iff$ d_π is symplectic.

Definition

A *Poisson structure* on M is a homological symplectic vector field on $T^*[1]M$. (($T^*[1]M, \omega, d_\pi$) is a deg. 1 symplectic *NQ*-manifold.)

Definition

A *Poisson structure* on M is a degree 2 function π on $T^*[1]M$ such that $[\pi, \pi] = 0$.

Poisson reduction via supersymplectic reduction

Cattaneo-Zambon: Poisson reduction = (super)symplectic reduction of $T^*[1]M$

For moment map reduction, they considered DGLA actions. If the comoment map $\mathfrak{g} \to C^\infty(T^*[1]M)$ is a DGLA map, then π passes to the quotient.

We also want to include Poisson-Lie group/Lie bialgebra actions.

- dg-group = Q-group = (graded) Lie group with multiplicative vector field, $[Q, Q] = 2Q^2 = 0$.
- Poisson-Lie group = Lie group with multiplicative bivector field, $[\pi, \pi] = 0$.
- homotopy Poisson-Lie group = Lie group with multiplicative multivector field, $[\pi, \pi] = 0$.

Homotopy Poisson manifolds

Let $\mathcal{M}$ be a graded manifold.

Definition

A *homotopy Poisson (hPoisson)* structure on $\mathcal{M}$ is any of the following equivalent things:

- an L_∞ algebra structure on $C^\infty(\mathcal{M})$ where the brackets satisfy the Leibniz rule.
- a homological symplectic vector field on $T^*[1]\mathcal{M}$.
- a degree 2 function π on $T^*[1]\mathcal{M}$ such that $[\pi, \pi] = 0$.

Write $\pi = \sum \pi_k$, where $\pi_k \in \mathfrak{X}^k(\mathcal{M})$. Then we have the derived bracket formula

$$\{f_1, \dots, f_k\}_\pi = [\cdots[[\pi_k, f_1], f_2], \cdots f_k] = [\cdots[d_\pi f_1, f_2], \cdots f_k].$$

Note: the "homological" degree of π_k is $2 - k$.

Examples

Example

A graded (deg. 0) Poisson manifold is an hPoisson manifold. Note: For ordinary manifolds, then hPoisson = Poisson.

Example

Q-manifolds/dg-manifolds, e.g. $A[1]$ if A is a Lie algebroid.

Example

A QP-manifold is a Poisson manifold equipped with a homological Poisson vector field, e.g. $T^*(A[1])$ if A is a Lie algebroid.

Another example

Example

If $\mathcal{V} = \bigoplus V_i[i]$ is an L_∞-algebra, then $\mathcal{V}^* = \bigoplus V_i^*[-i]$ is a (linear) hPoisson manifold. $T^*[1](\mathcal{V}[1]) = T^*[1](\mathcal{V}^*)$.

Remark

If $\mathcal{M}$ is hPoisson, then $T^*[1]\mathcal{M}$ is a degree 1 symplectic Q-manifold, but generally has negative degree coordinates even if $\mathcal{M}$ is $\mathbb{N}$-graded.

c.f. Roytenberg-Severa correspondence

$$\{\textit{Poisson manifolds}\} \leftrightsquigarrow \{\textit{deg. } 1 \textit{ symplectic NQ-manifolds}\}$$

Morphisms

Definition

A (strict) *morphism* of hPoisson manifolds from $(\mathcal{M}, \pi)$ to $(\mathcal{M}', \pi')$ is a graded manifold morphism $\psi : \mathcal{M} \to \mathcal{M}'$ such that

$$\psi^*\{f_1, \ldots, f_k\}_{\pi'} = \{\psi^* f_1, \ldots, \psi^* f_k\}_\pi$$

for $f_1, \ldots f_k \in C^\infty(\mathcal{M}')$.

Equivalently, $\pi \overset{\psi}{\sim} \pi'$.

Weak morphisms??

hPoisson-Lie groups

Definition

A *hPoisson-Lie group* is a graded Lie group $\mathcal{G}$ equipped with a hPoisson structure such that the multiplication map $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a hPoisson morphism.

Examples

Poisson-Lie groups, Q-groups/dg-groups,...

Definition

A *hPoisson-Lie group* is a graded Lie group $\mathcal{G}$ where $T^*[1]\mathcal{G}$ is equipped with a multiplicative homological symplectic vector field, or equivalently, a degree 2 multiplicative function ϕ such that $[\phi, \phi] = 0$.

"Multiplicative" refers to the groupoid structure $T^*[1]\mathcal{G} \rightrightarrows \mathfrak{g}^*[1]$.

Homotopy Lie bialgebras

A multiplicative homological symplectic vector field d_ϕ on $T^*[1]\mathcal{G} \rightrightarrows \mathfrak{g}^*[1]$ lives over a homological Poisson vector field $\hat{d}_\phi$ on $\mathfrak{g}^*[1]$, which can be thought of as a differential on $C^\infty(\mathfrak{g}^*[1]) = \mathrm{S}(\mathfrak{g}[-1])$ (think $\bigwedge \mathfrak{g}$).

$\hat{d}_\phi$ Poisson $\iff$ derivation of the Schouten-Lie bracket.

Definition

A *homotopy Lie bialgebra* is a graded Lie algebra $\mathfrak{g}$ equipped with a differential δ on $\mathrm{S}(\mathfrak{g}[-1])$ that is a derivation of symmetric product and the Schouten-Lie bracket.

- If δ is linear, then $\mathfrak{g}$ is a DGLA ($=$ Lie Q-algebra).
- If δ is quadratic, then $\mathfrak{g}$ is a graded Lie bialgebra.
- In general, the derivation property expresses a compatibility between a graded Lie algebra structure on $\mathfrak{g}$ and an L_∞-algebra structure on $\mathfrak{g}^*$.

hPoisson actions

Let $\mathcal{M}$ be a hPoisson manifold, and let $\mathcal{G}$ be a hPoisson-Lie group.

Definition

An action $\sigma : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$ is *hPoisson* if σ is a hPoisson morphism.

Infinitesimal version: Let $\mathfrak{g}$ be a homotopy Lie bialgebra.

Definition

An action $\rho : \mathfrak{g} \to \mathfrak{X}(\mathcal{M})$ is a *homotopy Lie bialgebra action* if the extension $\hat{\rho} : \mathrm{S}(\mathfrak{g}[-1]) \to \mathfrak{X}^\bullet(\mathcal{M})$ respects differentials.

Lemma

Suppose that $\mathcal{G}$ has a free and proper hPoisson action on $\mathcal{M}$. Then the quotient $\mathcal{M}/\mathcal{G}$ inherits a hPoisson structure.

Hamiltonian actions

Let $\mathcal{S}$ be a degree 1 symplectic Q-manifold. Let $(\mathcal{G}, \phi)$ be a connected hPoisson-Lie group with a Hamiltonian action on $\mathcal{S}$ with moment map $\mu : \mathcal{S} \to \mathfrak{g}^*[1]$.

Recall that $\mathfrak{g}^*[1]$ has a homological vector field $\hat{d}_\phi$.

Definition

The action is called *Q-Hamiltonian* if μ is a Q-manifold morphism. Equivalently, $\mu^* : \mathrm{S}(\mathfrak{g}[-1]) \to C^\infty(\mathcal{S})$ respects differentials.

Theorem

If $\mathcal{G}$ is flat and the action is Q-Hamiltonian (+ regular value, etc.), then the homological vector field on $\mathcal{S}$ descends to the quotient $\mu^{-1}(0)/\mathcal{G}$.

Nonflat $\leftrightsquigarrow$ reduction at nonzero values?

hPoisson actions revisited

Let $\mathcal{M}$ be a hPoisson manifold, and let $\mathcal{G}$ be a flat hPoisson-Lie group with a free and proper hPoisson action on $\mathcal{M}$.

$\rightsquigarrow$ (shifted) cotangent lift action $\mathcal{G} \circlearrowright T^*[1]\mathcal{M}$.

Theorem

The cotangent lift action is Q-Hamiltonian, and the reduced symplectic Q-manifold is $T^*[1](\mathcal{M}/\mathcal{G})$.

Example

If M is a Poisson manifold and G is a Poisson-Lie group with a free and proper Poisson action on M, then the Poisson quotient M/G can be interpreted as arising from the "Q-symplectic quotient" $T^*[1]M//G$.

Higher hPoisson structures

Let $\mathcal{M}$ be a graded manifold.

Definition

A *degree n hPoisson structure* on $\mathcal{M}$ is a degree $n+1$ function π on $T^*[n]\mathcal{M}$ such that $[\pi, \pi] = 0$.

degree n hPoisson-Lie groups can do Q-symplectic reduction on degree n symplectic Q-manifolds.

Example

Bursztyn-Cavalcanti-Gualtieri notion of "extended action with moment map" for reduction of Courant algebroids. (In this case, the deg. 2 homotopy Lie bialgebra is a DGLA.)

The quadratic case

Example

Quadratic deg. 2 homotopy Lie bialgebras correspond to “matched pairs” of Lie algebras.

Interesting example of Courant reduction by “matched pair action”?

Thanks.