Double Lie groupoids, Lie 2-groupoids, and integration of Courant algebroids

Rajan Mehta (joint with X. Tang)

Pennsylvania State University

October 30, 2011

Lie bialgebroids

A *Lie bialgebroid* is a compatible dual pair (A, A^*) of Lie algebroids [Mackenzie-Xu].

Examples: Lie bialgebras, (TM, T^*M) when M Poisson.

Lie bialgebroids are the infinitesimal objects associated to Poisson groupoids.

Poisson groupoids are the infinitesimal objects associated to *symplectic double groupoids* [Lu-Weinstein, Mackenzie, Stefanini].

The "double" of a Lie bialgebroid

Liu-Weinstein-Xu: If (A, A^*) is a Lie bialgebroid, then there is an induced Courant algebroid structure on $A \oplus A^*$.

$$
\begin{tikzcd}
\boxed{\text{symplectic double groupoids}} \arrow[d, "\text{differentiation}"'] & \\
\boxed{\text{Lie bialgebroids}} \arrow[r, "\text{LWX}"] & \boxed{\text{Courant algebroids}}
\end{tikzcd}
$$

Question (Liu-Weinstein-Xu)

"What is the global, groupoid-like object corresponding to a Courant algebroid? In particular, what is the double of a Poisson groupoid?"

Ševera, Roytenberg: symplectic 2-groupoid?

Double Lie groupoids

A *double Lie groupoid* is a square

$$\begin{tikzcd} V & D \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift left] \arrow[d, shift right] \\ M & H \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

where all four sides are Lie groupoids, satisfying a compatibility condition.
An element $\alpha \in D$ can be depicted as a square

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "t_V\alpha"'] \\ \cdot \arrow[u, "t_H\alpha"] & \cdot \arrow[l, "s_V\alpha"] \arrow[u, "s_H\alpha"'] \arrow[ul, phantom, "\alpha"] \end{tikzcd}$$

and squares can be composed in two different ways: $\square\square$ and $\begin{array}{c}\square\\\square\end{array}$

The compatibility condition says that 4-fold products are well-defined: $\begin{array}{cc}\square&\square\\\square&\square\end{array}$
(+ filling condition)

2-categories

A *(weak) 2-category* consists of:

- Objects
- Morphisms
- 2-simplices

Each 2-simplex gives a composition of morphisms:

$$\begin{tikzcd} {} & \\ {} \arrow[u, "g"] & {} \arrow[l, "h"] \arrow[ul, "k"'] \arrow[ul, phantom, shift right=3ex, "\eta"] \end{tikzcd} \qquad\qquad k = g \cdot_\eta h$$

2-groupoids

2-groupoid axioms:

- For any appropriately compatible pair of morphisms, composition/division exists (but not uniquely!).
- In equations of the form

$$(g \cdot_{\eta_1} h) \cdot_{\eta_2} k = g \cdot_{\eta_3} (h \cdot_{\eta_4} k),$$

  any three of the η_i uniquely determines the fourth.
- There are distinguished identity morphisms and 2-simplices corresponding to left/right composition by identity.

Definition

A *Lie 2-groupoid* is a 2-groupoid where the objects, morphisms, and 2-simplices form manifolds, and all the "face maps" are submersions.

The bar functor [Artin-Mazur]

Given a double Lie groupoid

$$\begin{tikzcd} V \arrow[d, shift left] \arrow[d, shift right] & D \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift left] \arrow[d, shift right] \\ M & H \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

we can form a Lie 2-groupoid as follows:

- $G_0 = \{\text{objects}\} = M$,
- $G_1 = \{\text{morphisms}\} = V \times_M H$,

$$\begin{tikzcd} {} & {} \arrow[l, "v"'] \\ & {} \arrow[u, "h"'] \end{tikzcd}$$

- $G_2 = \{\text{2-simplices}\} = V \times_M D \times_M H$.

$$\begin{tikzcd} {} & {} \arrow[l, "v"'] & {} \arrow[l, "t_V\alpha"'] \\ & {} \arrow[u, "t_H\alpha"] \arrow[ur, phantom, "\alpha"] & {} \arrow[u, "s_H\alpha"'] \arrow[l, "s_V\alpha"] \\ & & {} \arrow[u, "h"'] \end{tikzcd}$$

Symplectic double groupoids

A *symplectic double groupoid* is a double Lie groupoid

$$\begin{tikzcd} V & D \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[from=u, shift left] \arrow[from=u, shift right] & H \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

where D is equipped with a symplectic form ω, making both $D \rightrightarrows V$ and $D \rightrightarrows H$ into symplectic groupoids.

When M has the trivial Poisson structure, (TM, T^*M) integrates to

$$\begin{tikzcd} T^*M & T^*M \times T^*M \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[from=u, shift left] \arrow[from=u, shift right] & M \times M \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

Applying the bar functor

If we apply the bar functor to a symplectic double groupoid, then the 2-form on D can be pulled back to a 2-form Ω on the space of 2-simplices G_2. Properties:

- Multiplicativity: Let G_3 be the space of "tetrahedral quadruplets" of 2-simplices, with face maps $f_i : G_3 \to G_2$, $i = 0, \dots, 3$. Then $\sum (-1)^i f_i^* \Omega = 0$.
- $d\Omega = 0$.
- Ω is degenerate, but we can write down conditions (in terms of the face/degeneracy maps) that control the degeneracy (c.f. Xu's *quasi-symplectic groupoids*).

Definition

A *symplectic 2-groupoid* is a Lie 2-groupoid equipped with a multiplicative, closed 2-form Ω on G_2 satisfying the "controlled degeneracy" conditions.

Integrating the standard Courant algebroid

The symplectic 2-groupoid integrating the standard Courant algebroid $TM \oplus T^*M$ has the following form:

- $G_0 = M$,
- $G_1 = M \times T^*M$,
- $G_2 = M \times T^*M \times (T^*M \oplus T^*M)$.

The "symplectic" form on G_2 is supported along the middle two copies of T^*M.

Related work

- Li-Bland, Severa: Differentiation process, $TM \oplus T^*M$ twisted by closed 3-forms.
- Sheng, Zhu: Interpretation in terms of integrating representations up to homotopy (no symplectic form)

Thanks.