

# Double Lie groupoids, Lie 2-groupoids, and integration of Courant algebroids

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# Lie bialgebroids

A *Lie bialgebroid* is a compatible dual pair  $(A, A^*)$  of Lie algebroids [Mackenzie-Xu].

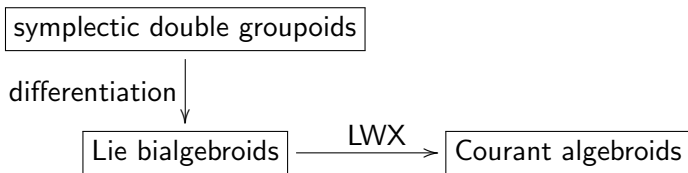
Examples: Lie bialgebras,  $(TM, T^*M)$  when  $M$  Poisson.

Lie bialgebroids are the infinitesimal objects associated to Poisson groupoids.

Poisson groupoids are the infinitesimal objects associated to *symplectic double groupoids* [Lu-Weinstein, Mackenzie, Stefanini].

## The “double” of a Lie bialgebroid

Liu-Weinstein-Xu: If  $(A, A^*)$  is a Lie bialgebroid, then there is an induced Courant algebroid structure on  $A \oplus A^*$ .



### Question (Liu-Weinstein-Xu)

“What is the global, groupoid-like object corresponding to a Courant algebroid? In particular, what is the double of a Poisson groupoid?”

Ševera, Roytenberg: symplectic 2-groupoid?

# Double Lie groupoids

A *double Lie groupoid* is a square

$$\begin{array}{ccc}
 V & \rightleftarrows & D \\
 \Downarrow & & \Downarrow \\
 M & \rightleftarrows & H
 \end{array}$$

where all four sides are Lie groupoids, satisfying a compatibility condition. An element  $\alpha \in D$  can be depicted as a square

$$\begin{array}{ccc}
 & t_V \alpha & \\
 \uparrow & \leftarrow & \uparrow \\
 t_H \alpha & \square & s_H \alpha \\
 \leftarrow & \alpha & \rightarrow \\
 & s_V \alpha & 
 \end{array}$$

and squares can be composed in two different ways:  $\square \square$  and  $\square \square$

The compatibility condition says that 4-fold products are well-defined:  $\square \square$   
 (+ filling condition)

## 2-categories

A (weak) 2-category consists of:

- Objects
- Morphisms
- 2-simplices

Each 2-simplex gives a composition of morphisms:



$$k = g \cdot_{\eta} h$$

## 2-groupoids

2-groupoid axioms:

- For any appropriately compatible pair of morphisms, composition/division exists (**but not uniquely!**).
- In equations of the form

$$(g \cdot_{\eta_1} h) \cdot_{\eta_2} k = g \cdot_{\eta_3} (h \cdot_{\eta_4} k),$$

any three of the  $\eta_i$  uniquely determines the fourth.

- There are distinguished identity morphisms and 2-simplices corresponding to left/right composition by identity.

### Definition

A *Lie 2-groupoid* is a 2-groupoid where the objects, morphisms, and 2-simplices form manifolds, and all the “face maps” are submersions.

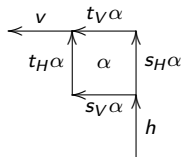
# The bar functor [Artin-Mazur]

Given a double Lie groupoid

$$\begin{array}{ccc}
 V & \rightrightarrows & D \\
 \Downarrow & & \Downarrow \\
 M & \rightrightarrows & H
 \end{array}$$

we can form a Lie 2-groupoid as follows:

- $G_0 = \{\text{objects}\} = M,$
- $G_1 = \{\text{morphisms}\} = V \times_M H,$
- $G_2 = \{\text{2-simplices}\} = V \times_M D \times_M H.$



# Symplectic double groupoids

A *symplectic double groupoid* is a double Lie groupoid

$$\begin{array}{ccc} V & \rightrightarrows & D \\ \Downarrow & & \Downarrow \\ M & \rightrightarrows & H \end{array}$$

where  $D$  is equipped with a symplectic form  $\omega$ , making both  $D \rightrightarrows V$  and  $D \rightrightarrows H$  into symplectic groupoids.

When  $M$  has the trivial Poisson structure,  $(TM, T^*M)$  integrates to

$$\begin{array}{ccc} T^*M & \rightrightarrows & T^*M \times T^*M \\ \Downarrow & & \Downarrow \\ M & \rightrightarrows & M \times M \end{array}$$



## Applying the bar functor

If we apply the bar functor to a symplectic double groupoid, then the 2-form on  $D$  can be pulled back to a 2-form  $\Omega$  on the space of 2-simplices  $G_2$ . Properties:

- Multiplicativity: Let  $G_3$  be the space of “tetrahedral quadruplets” of 2-simplices, with face maps  $f_i : G_3 \rightarrow G_2$ ,  $i = 0, \dots, 3$ . Then 
$$\sum (-1)^i f_i^* \Omega = 0.$$
- $d\Omega = 0$ .
- $\Omega$  is degenerate, but we can write down conditions (in terms of the face/degeneracy maps) that control the degeneracy (c.f. Xu’s *quasi-symplectic groupoids*).

### Definition

A *symplectic 2-groupoid* is a Lie 2-groupoid equipped with a multiplicative, closed 2-form  $\Omega$  on  $G_2$  satisfying the “controlled degeneracy” conditions.

## Integrating the standard Courant algebroid

The symplectic 2-groupoid integrating the standard Courant algebroid  $TM \oplus T^*M$  has the following form:

- $G_0 = M$ ,
- $G_1 = M \times T^*M$ ,
- $G_2 = M \times T^*M \times (T^*M \oplus T^*M)$ .

The “symplectic” form on  $G_2$  is supported along the middle two copies of  $T^*M$ .

## Related work

- Li-Bland, Severa: Differentiation process,  $TM \oplus T^*M$  twisted by closed 3-forms.
- Sheng, Zhu: Interpretation in terms of integrating representations up to homotopy (no symplectic form)

Thanks.