The fundamental group(oid) of a Lie groupoid

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October 15, 2011

Lie groupoids

Definition

A *groupoid* is a category where every morphism is invertible.

In other words, there is a set of objects and a set of morphisms. Each morphism has a source and a target object, and two morphisms can be composed if they have compatible source and target. (+ associativity and inverses)

Definition

A *Lie groupoid* is a groupoid where the objects and the morphisms both form manifolds, and the source, target, and composition maps are smooth submersions.

Notation: $G \rightrightarrows M$, where $M = \{\text{objects}\}$, $G = \{\text{morphisms}\}$.
$s =$ source map, $t =$ target map.

Fundamental groupoids

Example

The *fundamental groupoid* $\Pi_1(M)$ of a manifold M. The "objects" are points in M. The "morphisms" are paths in M modulo endpoint-fixing homotopies.

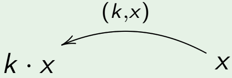
The fundamental groupoid contains all the fundamental groups (at every basepoint) and the natural isomorphisms between them.

“Manifolds with symmetry”

Group actions, principal bundles, orbifolds, and foliations can all be described in terms of Lie groupoids.

Example

Let K be a Lie group acting on a manifold M. The *transformation groupoid* is $K \times M \rightrightarrows M$.



Classifying spaces

Recall: If G is a Lie group, then the *classifying space* BG is the quotient of a contractible space by a free G action.
Examples: $B\mathbb{Z} \cong S^1$, $B\mathbb{Z}_2 \cong \mathbb{R}P^\infty$, $BS^1 \cong \mathbb{C}P^\infty$.

Lie groupoids also have classifying spaces.

- For a transformation groupoid $K \times M \rightrightarrows M$, the classifying space is the "homotopy quotient."
- When M is connected, $B\Pi_1(M) \cong B\pi_1(M)$.

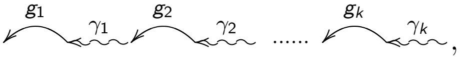
For Lie groups, $\pi_i(BG) = \pi_{i-1}(G)$. But this does not hold for Lie groupoids.

Question

How can the fundamental group of BG be described directly in terms of G?

Haefliger's G-paths

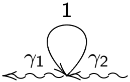
A *G-path* consists of a formal product of morphisms in G and connecting paths in M:



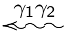
$g_i \in G, \gamma_i : [0,1] \to M.$

Homotopies are generated by those of three different types.

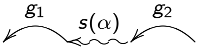
1. Identity insertion:



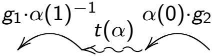
is equivalent to



2. Given a path of morphisms $\alpha : [0,1] \to G$,



is equivalent to



3. If $(\alpha_1, \sigma_1, \dots, \alpha_k, \sigma_k)$ is a path of G-paths, then $(\alpha_1(0), \sigma_1(0), \dots, \alpha_k(0), \sigma_k(0))$ is equivalent to $(\alpha_1(1), \sigma_1(1), \dots, \alpha_k(1), \sigma_k(1))$.

Fundamental group(oid)

The Haefliger fundamental groupoid of G consists of G-paths modulo endpoint-fixing homotopies.

At any basepoint $x \in M$, we have the Haefliger fundamental group of G.

Theorem (Haefliger, Moerdijk-Mrčun)

The Haefliger fundamental group of G is isomorphic to the fundamental group of BG.

Fundamental double groupoid

Take the fundamental groupoid of G and M as manifolds:

$$\begin{array}{ccc} G & \leftleftarrows & \Pi_1(G) \\ \downdownarrows & & \downdownarrows \\ M & \leftleftarrows & \Pi_1(M) \end{array}$$

If the source map is a fibration, then this is a *double Lie groupoid*: each edge of the square is a Lie groupoid, and all the groupoid structures are compatible.

Question

How is the fundamental double groupoid related to the Haefliger fundamental groupoid?

There are functors:

$$\boxed{\text{Double Lie groupoids}} \longrightarrow \boxed{\text{Lie 2-groupoids}} \longrightarrow \boxed{\text{Groupoids}}$$

2-categories

A *(weak) 2-category* consists of:

- Objects
- Morphisms
- 2-morphisms ("homotopies")

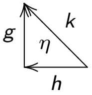
Two ways to impose "weakness":

1. Composition of morphisms is well-defined but only associative up to homotopy.
2. Composition of morphisms is not uniquely defined but satisfies a generalized associativity condition.

We use the latter perspective. Here, the information about composition and homotopies is implicitly encoded in a set of 2-*simplices*.

2-groupoids

Each 2-simplex gives a composition of morphisms:



$$k = g \cdot_\eta h$$

2-groupoid axioms:

1. For any compatible pair of morphisms, there exists some "filling" η giving a composition or division (but not uniquely!).
2. In equations of the form

$$(g \cdot_{\eta_1} h) \cdot_{\eta_2} k = g \cdot_{\eta_3} (h \cdot_{\eta_4} k),$$

any three of the η_i uniquely determines the fourth.

3. There are distinguished identity morphisms and 2-simplices corresponding to left/right composition by identity.

Lie 2-groupoids

Example

"Fundamental 2-groupoid" of a manifold M. Morphisms = paths, 2-simplices = (maps $\Delta \to M$) modulo boundary-fixing homotopies.

Definition

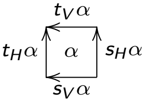
A *Lie 2-groupoid* is a 2-groupoid where the objects, morphisms, and 2-simplices form manifolds, and the "face maps" are submersions.

Double Lie groupoids

Consider a double Lie groupoid

$$\begin{array}{ccc} V & \leftleftarrows & D \\ \downdownarrows & & \downdownarrows \\ M & \leftleftarrows & H \end{array}$$

An element $\alpha \in D$ can be depicted as a square

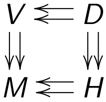


and squares can be composed in two different ways: $\square\square$ and $\begin{array}{c}\square\\\square\end{array}$

The compatibility condition says that 4-fold products are well-defined: $\begin{array}{c}\square\square\\\square\square\end{array}$
(+ filling condition)

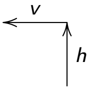
From double Lie groupoids to Lie 2-groupoids

Given a double Lie groupoid

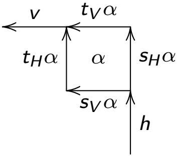


we can form a Lie 2-groupoid as follows:

- $\{\text{objects}\} = M$,
- $\{\text{morphisms}\} = V \times_M H$,



- $\{\text{2-simplices}\} = V \times_M D \times_M H$.



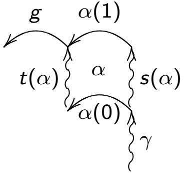
Back to the fundamental double groupoid

Let's apply the construction to the fundamental double groupoid:

$$\begin{array}{ccc} G & \leftleftarrows & \Pi_1(G) \\ \downdownarrows & & \downdownarrows \\ M & \leftleftarrows & \Pi_1(M) \end{array}$$

- $\{\text{objects}\} = M$,
- $\{\text{morphisms}\} = G \times_M \Pi_1(M)$,

- $\{\text{2-simplices}\} = G \times_M \Pi_1(G) \times_M \Pi_1(M)$.



From Lie 2-groupoids to groupoids

Given a Lie 2-groupoid, we can produce a groupoid by "truncation": Morphisms are quotiented by relations coming from the 2-simplices. In this case, the morphisms have the equivalence relation

$$(t_V\alpha, s_H\alpha) \sim (t_V\alpha', s_H\alpha')$$

if $t_H\alpha = t_H\alpha'$ and $s_V\alpha = s_V\alpha'$.

Theorem

Applying this process to the fundamental double groupoid produces the Haefliger fundamental groupoid.

Transformation groupoids

Example

Consider a transformation groupoid $K \times M \rightrightarrows M$.
If K is connected, then the Haefliger fundamental groupoid is $\Pi_1(M)/\pi_1(K, e)$.
If K is discrete, it is $K \ltimes \Pi_1(M)$.

Thanks.