Integrating exact Courant Algebroids

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The Courant bracket

The *Courant bracket* is a bracket on $\Gamma(TM \oplus T^*M)$, given by

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi.$$

Twisted version:

$$[X + \xi, Y + \eta]_H = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_X \iota_Y H,$$

where $H \in \Omega^3_{\mathrm{closed}}(M)$.

This bracket satisfies a Jacobi identity, but it is only skew-symmetric up to an exact term.

Dirac structures

A *Dirac structure* is a maximally isotropic subbundle $D \subset TM \oplus T^*M$ whose sections are closed under the Courant bracket.

Examples: Poisson structures, presymplectic structures, foliations.

If D is a Dirac structure, then the restriction of the Courant bracket is a Lie bracket, making D a Lie algebroid.

Integration of Dirac structures

Bursztyn, Crainic, Weinstein, and Zhu (2004) showed that a source-simply connected Lie groupoid G integrating a Dirac structure has a natural 2-form ω that is

1. multiplicative: $\delta\omega := p_1^*\omega - m^*\omega + p_2^*\omega = 0$,
2. H-closed: $d\omega = \delta H := s^*H - t^*H$,
3. not *too* degenerate: $\ker \omega \cap \{\text{isotropy directions}\} = \{0\}$.

Conversely, if G is a Lie groupoid of the correct dimension with a 2-form satisfying the above conditions, then its Lie algebroid can be identified with a Dirac structure.

Integrating Courant algebroids?

Liu, Weinstein, Xu (1997) gave a general definition of Courant algebroid and asked:
"What is the global, groupoid-like object corresponding to a Courant algebroid?"

Ševera (1998-2000): Morally, the answer should be a *symplectic 2-groupoid*.

Recently, integrations for the exact Courant algebroids were constructed by Li-Bland & Ševera, Sheng & Zhu, Tang & myself. But they are too "simple" to contain *all* the presymplectic groupoids.

Lie 2-groupoids

Definition

A *Lie 2-groupoid* is a Kan simplicial manifold $X_\bullet$ for which the n-dimensional horn-fillings are unique for $n > 2$.

Notation: d_i for face maps, s_i for degeneracy maps.

Duskin (1979): Any Kan simplicial manifold $X_\bullet$ can be *truncated* to a 2-groupoid $\tau_{\leq 2}X$. In particular, $(\tau_{\leq 2}X)_2 = X_2 \,/ \sim$, where $x \sim y$ if there exists $z \in X_3$ such that $d_2 z = x$, $d_3 z = y$, and $d_0 z, d_1 z \in \mathrm{im}(s_1)$.

...but you have to worry about whether $X_2 \,/ \sim$ is smooth.

Cotangent simplices

For $n = 0, 1, \ldots,$ let $\mathfrak{C}_n(M)$ be the space of ($C^{2,1}$) bundle maps from $T\Delta^n$ to T^*M.

Proposition

$\mathfrak{C}_\bullet(M)$ is a Kan simplicial Banach manifold.

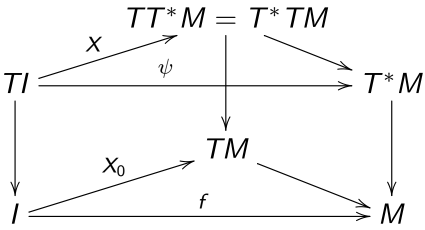
Theorem

$(\tau_{\leq 2}\mathfrak{C}(M))_2$ is a Banach manifold, and therefore $\tau_{\leq 2}\mathfrak{C}(M)$ is a Lie 2-groupoid. We'll call it the Liu-Weinstein-Xu 2-groupoid $\mathrm{LWX}(M)$.

- $\mathfrak{C}_0(M) = \mathrm{LWX}_0(M) = M$.
- $\mathfrak{C}_1(M) = \mathrm{LWX}_1(M)$ can be identified with $\mathrm{Paths}(T^*M)$ (but maybe you shouldn't).
- An element of $\mathrm{LWX}_2(M)$ is given by a homotopy class of maps $\Delta^2 \to M$ together with lifts of the edges to $\mathfrak{C}_1(M)$.

Lifting forms

For $\psi \in \mathfrak{C}_1(M)$, a tangent vector at ψ is a linear lift $X : TI \to T^*M$:



For each X, define a 1-form θ_X on I by

$$\theta_X(v) = \lambda(X(v)),$$

and let $\lambda_1 \in \Omega^1(\mathfrak{C}_1(M))$ be given by

$$\lambda_1(X) = \int_I \theta_X.$$

$\mathsf{LWX}(M)$ is a symplectic 2-groupoid

Definition

A *symplectic 2-groupoid* is a Lie 2-groupoid equipped with a closed, "nondegenerate" 2-form $\omega \in \Omega^2(X_2)$ satisfying the multiplicativity condition $\delta\omega := \sum_{i=0}^{3}(-1)^i d_i^*\omega = 0$.

Lemma

1. $\omega_1 := d\lambda_1$ *is (weakly) nondegenerate.*
2. $\omega_2 := \delta\omega_1$ *is (weakly) nondegenerate on* $\mathsf{LWX}_2(M)$.

Theorem

$\mathsf{LWX}(M)$ *is a symplectic 2-groupoid.*

Lifting forms 2: twisting forms

For $H \in \Omega^3_{\mathrm{closed}}(M)$ and $X, Y \in T_\psi \mathfrak{C}_1(M)$, define a 1-form $H_{X,Y}$ on I by

$$H_{X,Y} = f^* H(X_0, Y_0, \cdot),$$

and let $\phi_1^H \in \Omega^2(\mathfrak{C}_1(M))$ be given by

$$\phi_1^H(X, Y) = \int_I H_{X,Y}.$$

Lemma

ϕ_1^H is H-closed, i.e. $d\phi_1^H = \delta H$.

Let $\phi_2^H := \delta \phi_1^H$.

Theorem

$\mathrm{LWX}(M)$, equipped with the 2-form $\omega_2 + \phi_2^H$ is a symplectic 2-groupoid.

Simplicial integration of Dirac structures

Let D be a Dirac structure that integrates to a source-simply connected Lie groupoid G.
For $n = 0, 1, \ldots,$ let $\mathfrak{G}(D)_n$ be the space of (C^2) groupoid morphisms from $\Delta^n \times \Delta^n$ to G (which can be identified with the space of ($C^{2,1}$) Lie algebroid morphism from $T\Delta^n$ to D).

Proposition

$\mathfrak{G}(D)_n$ is a Kan simplicial Banach manifold.

G can be recovered as the 1-truncation of $\mathfrak{G}(D)$.

Dirac structures in LWX(M)

There is a natural simplicial embedding $F_\bullet : \mathfrak{G}(D)_\bullet \hookrightarrow \mathfrak{C}(M)_\bullet$.

Proposition

$F_2^*\omega_2 = 0$, and $F_2^*\phi_2^H = 0$.

Corollary

$F_1^*\omega_1$ is a closed, multiplicative 2-form on $\mathfrak{G}(D)_1$, and $F_1^*\phi_1^H$ is an H-closed, multiplicative 2-form on $\mathfrak{G}(D)_1$.

Proposition

The image of $\mathfrak{G}(D)_2$ in LWX(M) is Lagrangian at the constant maps.

Conjecture

The image of $\mathfrak{G}(D)_2$ in LWX(M) is Lagrangian.

Further questions

- Where does the "not too degenerate" condition appear in this picture? Probably related to the Lagrangian property.
- What is the relationship between $\mathsf{LWX}(M)$ and the finite-dimensional integrations? What is the correct notion of equivalence for symplectic 2-groupoids?
- What is the general construction for arbitrary Courant algebroids? Are there obstructions to integrability, in general?
- If $\{X_\bullet\}$ is a symplectic 2-groupoid, is there an induced geometric structure on X_1?