Differential graded contact geometry

Rajan Mehta

Smith College

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Jacobi manifolds (Kirillov, Lichnerowicz)

Definition

A Jacobi structure on a manifold M is a Lie bracket on $C^\infty(M)$ which is local:

$$\operatorname{supp}(\{f,g\}) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(g).$$

Theorem (Kirillov)

The local condition holds if and only if $\{f,\cdot\}$ is a first-order differential operator for all f.

Any skew-symmetric, first-order bracket is given by a vector field R and a bivector field Λ:

$$\{f,g\} = \Lambda(df,dg) + fR(g) - R(f)g.$$

Jacobi $\Leftrightarrow$ $[\Lambda,\Lambda] = 2R\wedge\Lambda, \quad [\Lambda,R] = 0.$

Jacobi vs. Poisson

- Poisson manifolds form the special case where $R = 0$.
- There is also a Poissonization process taking a Jacobi structure on M to a Poisson structure on $M \times \mathbb{R}$.
- Nondegenerate Jacobi structures correspond to contact forms ($R =$ Reeb vector field); the Jacobi bracket in this case is called the Legendre bracket.
- Analogy: "Jacobi" is to "contact" as "Poisson" is to "symplectic".

Other examples

- Manifolds with vector fields ($\Lambda = 0$)
- Cosymplectic manifolds
- Locally conformal symplectic manifolds

Differential view (Poisson case)

Consider the algebra of multivector fields $\mathfrak{X}^\bullet(M) = \Gamma(\wedge TM)$.

$[\Lambda, \Lambda] = 0 \rightsquigarrow$ differential $d_\Lambda := [\Lambda, \cdot]$.

Can be interpreted in terms of symplectic supergeometry: Identify $\mathfrak{X}^\bullet(M)$ with the "functions" on the graded manifold $T^*[1]M$.

Schouten bracket $\leftrightsquigarrow$ canonical symplectic form $\omega = dx^i \wedge dp_i$.

Then d_Λ is the Hamiltonian vector field associated to $\Lambda = \frac{1}{2}\Lambda^{ij} p_i p_j$.

Theorem (Ševera,Roytenberg)

There is a one-to-one correspondence between Poisson manifolds and degree 1 symplectic NQ-manifolds.

Applications of the supergeometric perspective

- Reduction (Cattaneo-Zambon, M., Bursztyn-Cattaneo-M.-Zambon)
- Deformation theory
- AKSZ field theory (Alexandrov-Kontsevich-Schwarz-Zaboronsky) $\rightsquigarrow$ Poisson sigma model $\rightsquigarrow$ Deformation quantization, integration to symplectic groupoids (Cattaneo-Felder)

Differential view (Jacobi case)

In the Jacobi case, you instead consider $\mathfrak{X}^\bullet(M)[\theta]/(\theta^2 = 0)$, where θ is a degree 1 variable.

(A differential here is equivalent to a Lie algebroid structure on $T^*M \times \mathbb{R}$)

A Jacobi structure (Λ, R) induces a differential

$$d_{\Lambda,R} = [\Lambda, \cdot] + \theta[R, \cdot] - R\varepsilon - (\Lambda + \theta R)\frac{\partial}{\partial\theta},$$

where ε is the Euler vector field: $\varepsilon(f) = |f|f$.

Which differentials come from Jacobi structures? And why is the formula so weird?

Supergeometry to the rescue

Identify $\mathfrak{X}^\bullet(M)[\theta]$ with the "functions" on the graded manifold $T^*[1]M \times \mathbb{R}[1]$.

This graded manifold has a canonical contact form: $\alpha = p_i dx^i + d\theta$.

$\rightsquigarrow$ one-to-one correspondence: functions $\leftrightsquigarrow$ contact vector fields

$\rightsquigarrow$ Legendre bracket on $\mathfrak{X}^\bullet(M)[\theta]$.

Theorem (M.)

Let Λ be a bivector field, and let R be a vector field. The above correspondence takes $\Lambda + \theta R$ to $d_{\Lambda,R}$, and
$d_{\Lambda,R}^2 = 0 \Leftrightarrow \{\Lambda + \theta R, \Lambda + \theta R\} = 0 \Leftrightarrow [\Lambda, \Lambda] = 2R \wedge \Lambda, \;\; [\Lambda, R] = 0.$
Therefore, there is a one-to-one correspondence between Jacobi manifolds and degree 1 contact NQ-manifolds.

Symplectization

Recall that there is a symplectization functor from contact manifolds to symplectic manifolds. This works for contact NQ-manifolds as well.

Theorem (M.)

Poissonization is the same thing as "super-symplectization".

Possible applications

- Jacobi Reduction (c.f. Ibort-de Leon-Marmo, Petalidou-Nunes da Costa)
- Deformation theory
- Jacobi sigma model?

Higher structures

n	deg. n symplectic NQ-manifolds	deg. n contact NQ-manifolds
1	Poisson manifolds	Jacobi manifolds
2	Courant algebroids	Jacobi-Courant algebroids?
3	H-twisted Lie algebroids	???
$\vdots$	$\vdots$	$\vdots$

Thanks.