Representing Representations up to Homotopy

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History

- Problem: No natural adjoint representation for Lie algebroids
- Evens, Lu, Weinstein: Can define an adjoint "representation up to homotopy" on the 2-term complex $A \xrightarrow{\rho} TM$. Used it to define the *modular class* of a Lie algebroid.
- Crainic, Fernandes: Used adjoint representation up to homotopy to construct higher characteristic classes.
- Arias Abad, Crainic: Stronger definition of representation up to homotopy (strong homotopy)
- Gracia-Saz, M.: Same definition but different name "Superrepresentations", showed relationship to $\mathcal{VB}$-algebroids in the 2-term case. Gave a general construction for characteristic classes.

Definition

$A \to M$ a Lie algebroid, $(\mathcal{E} = \bigoplus E_i, \partial)$ a complex of vector bundles.

Definition

A *representation up to homotopy* of A on $\mathcal{E}$ consists of:

- An A-connection $\nabla : \Gamma(A) \otimes \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$,
- Endormorphism-valued forms $\omega_i \in \Gamma(\wedge^i A^*) \otimes \mathrm{End}_{1-i}\,\mathcal{E}$ for $i \geq 2$,

such that

1. ∇ is compatible with ∂,
2. The curvature of ∇ is $\omega_2 \partial + \partial \omega_2$,
3. Higher conditions.

Motivation

A representation of a Lie algebra $\mathfrak{g}$ on a vector space V is given by a Lie algebra morphism

$$\mathfrak{g} \to \mathfrak{gl}(V).$$

This perspective immediately implies various "naturality" results:

- Representations can be pulled back under $\mathfrak{h} \to \mathfrak{g}$
- Classes in $H^\bullet(\mathfrak{gl}(V))$ are "universal" characteristic classes
- Universal characteristic classes are natural under pullback

Question

Can a representation up to homotopy be similarly described by a morphism

$$A \to ???$$

DG Lie algebroids

M a manifold.

Definition

A *DG Lie algebroid* (DGLAoid) over M is a graded vector bundle $\mathcal{A} = \bigoplus A_i$ equipped with:

- An anchor map $\rho : A_0 \to TM$,
- A differential $\partial : A_\bullet \to A_{\bullet+1}$,
- A graded Lie bracket $[\cdot, \cdot]$ (degree 0),

such that

1. The differential is a derivation of the bracket (so $\Gamma(\mathcal{A})$ is a DGLA)
2. Brackets involving a degree 0 section satisfy a Leibniz rule, and are otherwise $C^\infty(M)$-linear

Special case of a *Q-algebroid*.

The operator DG Lie algebroid

$(\mathcal{E} = \bigoplus E_i, \partial)$ a complex of vector bundles over M. Construct the *operator DG Lie algebroid* $\mathcal{O}(\mathcal{E})$:

- Sections of $\mathcal{O}_0(\mathcal{E})$ are derivation operators on $\mathcal{E}$
- For $i \neq 0$, sections of $\mathcal{O}_i(\mathcal{E})$ are degree i endomorphisms of $\mathcal{E}$
- Anchor map $\sigma : \mathcal{O}_0(\mathcal{E}) \to TM$ is the symbol map
- Bracket is graded commutator bracket
- Differential is $\tilde{\partial} = [\partial, \cdot]$

Representing representations up to homotopy

$A \to M$ a Lie algebroid, $(\mathcal{E} = \bigoplus E_i, \partial)$ a complex of vector bundles.

Definition

An L_∞ map from A to $\mathcal{O}(\mathcal{E})$ consists of bundle maps $\beta_k : \wedge^k A \to \mathcal{O}_{1-k}(\mathcal{E})$ for $k > 0$, such that $\sigma \circ \beta_1 = \rho$ (where $\rho : A \to TM$ is the anchor map of A), and such that the induced maps of sections form an L_∞-algebra morphism.

Theorem

There is a one-to-one correspondence between representations up to homotopy of A on $(\mathcal{E}, \partial)$ and L_∞ maps from A to $\mathcal{O}(\mathcal{E})$.

Immediate consequences

- Representations up to homotopy can be pulled back under Lie algebroid morphisms
- Classes in (appropriately-defined) $H^\bullet(\mathcal{O}(\mathcal{E}))$ are "universal" characteristic classes
- Universal characteristic classes are natural under pullback

Also: equivalences of representations up to homotopy

More potential consequences

Can define maps over *different base manifolds*. So a Lie algebroid $A \to M$ can have a representation up to homotopy on $\mathcal{E} \to N$.

Could be useful for applying tools of ordinary representation theory (e.g. weights).