Frobenius objects in the category of spans

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Monoids

A monoid is a set X equipped with maps

- $\eta : \{pt\} \to X$ (unit)
- $\mu : X \times X \to X$ (multiplication)

satisfying:

1. Unitality: $\mu \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = \mu \circ (\mathrm{id} \otimes \eta)$,
2. Associativity: $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$.

Monoid objects

Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. A monoid in $\mathcal{C}$ is an object X equipped with morphisms

- $\eta : I \to X$ (unit)
- $\mu : X \otimes X \to X$ (multiplication)

satisfying:

1. Unitality: $\mu \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = \mu \circ (\mathrm{id} \otimes \eta)$,
2. Associativity: $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$.

- A monoid in $(\mathbf{Set}, \times)$ is a monoid.
- A monoid in $(\mathbf{Vect}, \otimes)$ is an algebra.
- If $\mathcal{C}$ is symmetric monoidal then we can define commutativity.

A Frobenius algebra is a finite-dimensional algebra A equipped with a map $\varepsilon : A \to \mathbb{k}$ (counit) such that the bilinear form $\varepsilon \circ \mu : A \otimes A \to \mathbb{k}$ is nondegenerate.

Examples:

- $A = \{n \times n$ matrices$\}$, $\varepsilon = \mathrm{tr}$.
- G finite group $\rightsquigarrow$ group algebra $A = \mathbb{k}[G]$.
- M compact oriented manifold $\rightsquigarrow$ cohomology $H^\bullet(M)$, $\varepsilon = \int_M$.

A Frobenius object in $\mathcal{C}$ is a monoid X in $\mathcal{C}$ equipped with a morphism $\varepsilon : X \to I$ (counit) satisfying the following nondegeneracy condition:

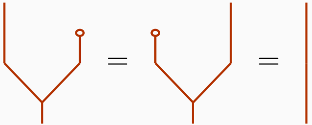
$\exists \beta : I \to X \otimes X$ such that
$((\varepsilon \circ \mu) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \beta) = \mathrm{id} = (\mathrm{id} \otimes (\varepsilon \circ \mu)) \circ (\beta \otimes \mathrm{id})$.

Frobenius objects via string diagrams

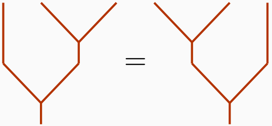
We'll use the following string diagrams to represent the unit, multiplication, and counit:

Then the axioms can be written as follows:

- Unit:

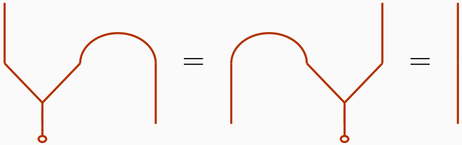


- Associativity:

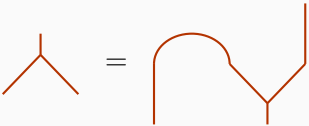


Represent $\beta : I \to X \otimes X$ by

Nondegeneracy:



Can show β is unique, so can also define a comultiplication by



and can prove it's counital & coassociative.

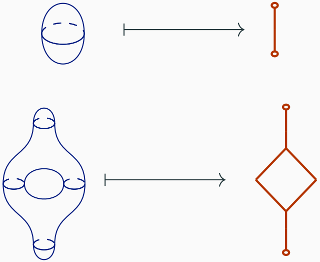
Why Frobenius objects?

- A TQFT is a symmetric monoidal functor $\mathbf{Cob} \to \mathbf{Vect}$.
- More generally, a $\mathcal{C}$-valued TFT is a symmetric monoidal functor $\mathbf{Cob} \to \mathcal{C}$.
- (Dijkgraaf, Abrams) 2D oriented TQFTs correspond to commutative Frobenius algebras.
- 2D oriented $\mathcal{C}$-valued TFTs correspond to commutative Frobenius objects in $\mathcal{C}$. (See Kock, “Frobenius algebras and 2D TQFTs”)

Surface invariants

A commutative Frobenius object gives invariants of closed surfaces:

$$\{\text{Closed surfaces}\} = \mathrm{End}_{\mathbf{Cob}}(\varnothing) \to \mathrm{End}_{\mathcal{C}}(I)$$



Note: $\mathrm{End}_{\mathbf{Vect}}(\mathbb{k}) = \mathbb{k}$ so you get numerical invariants from Frobenius algebras.

The category $\mathbf{Span}$

- Objects are sets, morphisms are isomorphism classes of spans

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

- Composition of morphisms is obtained by pullback/fiber product:

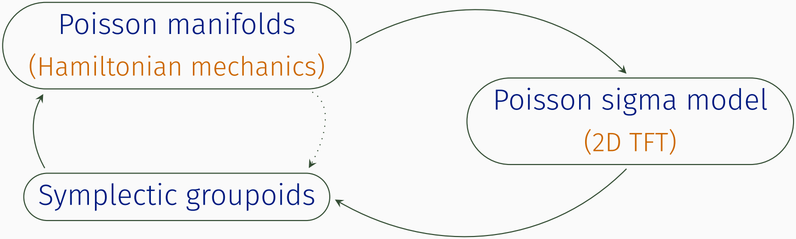
$$\begin{tikzcd} & & A \times_Y B \arrow[dl] \arrow[dr] & & \\ & A \arrow[dl, "f"'] \arrow[dr, "g"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & \\ X & & Y & & Z \end{tikzcd}$$

- Symmetric monoidal structure given by $\times$.
- $\mathrm{Hom}_{\mathbf{Span}}(\{pt\}, \{pt\}) = \{$Iso classes of sets$\}$, contains $\mathbb{N}$

Why Span?

Ultimately would like to consider the symplectic category.

- Objects: symplectic manifolds
- Morphisms (naïve): Lagrangian relations $L \subseteq \bar{M} \times N$
- Symmetric monoidal structure given by $\times$.



Observation: A symplectic groupoid is a Frobenius object in the symplectic category!

Why Span, continued

- However! Compositions of Lagrangian relations might not be smooth (require “strong transversality”).
- (Wehrheim, Woodward) Morphisms are formal sequences of Lagrangian relations, modulo strongly transversal compositions.
- This works, but it’s complicated! Even $\mathrm{Hom}_{\mathbf{Symp}}(\{pt\}, \{pt\})$ is not well-understood.
- (Li-Bland, Weinstein) There is a symmetric monoidal functor $\mathbf{Symp} \to \mathbf{Span}$, so structures in $\mathbf{Symp}$ induce structures in $\mathbf{Span}$.
- If finite, can obtain (Frobenius) algebras via “decategorification”.

Let $X_\bullet$ be a simplicial set, define unit and multiplication spans by

$$\{pt\} \leftarrow X_0 \xrightarrow{s_0} X_1 \qquad X_1 \times X_1 \xleftarrow{(d_2,d_0)} X_2 \xrightarrow{d_1} X_1$$

Lemma: The unit axiom holds if and only if the diagrams

$$\begin{tikzcd} X_1 \arrow[r, "d_0"] \arrow[d, "s_1"] & X_0 \arrow[d, "s_0"] \\ X_2 \arrow[r, "d_0"] & X_1 \end{tikzcd} \qquad \begin{tikzcd} X_1 \arrow[r, "d_1"] \arrow[d, "s_0"] & X_0 \arrow[d, "s_0"] \\ X_2 \arrow[r, "d_2"] & X_1 \end{tikzcd}$$

are pullbacks.

Associativity

Consider the “taco spaces”

$$T_{02} = X_2 \ {}_{d_1}\!\times_{d_0} X_2,$$
$$T_{13} = X_2 \ {}_{d_2}\!\times_{d_1} X_2.$$

They correspond to the two triangulations of the square:

There are four edge maps $e_1, e_2, e_3, e_{\text{out}} : T_{02} \to X_1$, and similarly for T_{13}.

Lemma: Associativity holds if and only if there exists an isomorphism $T_{02} \to T_{13}$ that commutes with the edge maps.

Monoids in $\mathbf{Span}$ come from simplicial sets

So: given a simplicial set satisfying the unit and associativity conditions, we get a monoid in $\mathbf{Span}$.

Theorem (Contreras, Keller, M.)

Every monoid in $\mathbf{Span}$ arises from a simplicial set in this way.

Let $X_\bullet$ be a simplicial set satisfying the associativity and unitality conditions.

Let τ be an automorphism of X_1. Define a counit span by

$$X_1 \xleftarrow{\tau \circ s_0} X_0 \to \{pt\}.$$

Lemma: This gives a Frobenius structure if and only if there exists $\gamma : X_1 \to X_2$ such that

1. $d_0 \circ \gamma = \tau$,
2. $d_1 \circ \gamma = \tau \circ s_0 \circ d_1$,
3. $d_2 \circ \gamma = \mathrm{id}$,
4. and the diagram $\begin{tikzcd} X_1 \arrow[r, "d_1"] \arrow[d, "\gamma"] & X_0 \arrow[d, "\tau \circ s_0"] \\ X_2 \arrow[r, "d_1"] & X_1 \end{tikzcd}$ is a pullback.

Frobenius objects in $\mathbf{Span}$ come from simplicial sets

So: given a simplicial set $X_\bullet$ equipped with an automorphism of X_1, satisfying the unit, associativity, and Frobenius conditions, we get a Frobenius object in $\mathbf{Span}$.

Theorem (Contreras, Keller, M.)

Every Frobenius object in $\mathbf{Span}$ arises from a simplicial set in this way.

Example: Groups

- If G is a group, can take the nerve

$$\cdots G \times G \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow}} G \rightrightarrows \{pt\}$$

  This satisfies the unit and associativity conditions.
- For any fixed $\omega \in G$, can take $\tau(g) = g^{-1}\omega$. This satisfies the Frobenius condition with $\gamma(g) = (g, g^{-1}\omega)$.
- If G finite, abelian, can compute surface invariants in $\mathbb{N}$:

$$Z(\Sigma_g) = \begin{cases} |G|^g & \text{if } \omega^g = \omega, \\ 0 & \text{otherwise.} \end{cases}$$

- This construction generalizes to groupoids.

- Consider a simplicial set

$$\cdots \{(1,1,1),(1,x,x),(x,x,1),\underbrace{(x,1,x)}_{n \text{ copies}}\} \substack{\rightarrow\\\rightarrow\\\rightarrow} \{1,x\} \rightrightarrows \{1\}.$$

- Set $\tau : 1 \leftrightarrow x$. Satisfies the Frobenius condition with $\gamma(1) = (x,x,1)$, $\gamma(x) = (1,x,x)$.
- Then

$$Z(\Sigma_g) = \begin{cases} 2^g n^{(g-1)/2} & \text{if } g \text{ odd,} \\ 0 & \text{if } g \text{ even.} \end{cases}$$

- When $n = 1$, this is the nerve of $\mathbb{Z}_2$. When $n \neq 1$, it is not the nerve of a groupoid.

It's more natural to define a bicategory of spans:

- objects are sets
- morphisms are spans
- 2-morphisms are maps of spans

The coherent structure in this setting is called a (Frobenius) pseudomonoid.

(Stern) Pseudomonoids in $\mathbf{Span}$ correspond to 2-Segal sets.

Theorem

(Contreras, M., Stern) Frobenius pseudomonoids in $\mathbf{Span}$ *correspond to paracyclic* 2*-Segal sets.*

In the bicategorical setting, commutativity is a structure, not a property.

- Let Φ_* be the category with objects $\langle n \rangle = \{*, 1, \ldots, n\}$ and morphisms maps $f : \langle n \rangle \to \langle m \rangle$ such that $f(*) = *$.
- There is a functor "cut" $\Delta^{\mathrm{op}} \to \Phi_*$.

Theorem

(Contreras, M., Stern) Commutative pseudomonoids in $\mathbf{Span}$ *correspond to functors* $\Phi_* \to \mathbf{Set}$ *such that the induced simplicial set is* 2*-Segal.*

Work in progress with Sophia Marx: Commutative + Frobenius $\leftrightarrow$ 2-Segal functors $\Phi \to \mathbf{Set}$.

Example: graph partitions

Let G be a graph.

- Let $X_n = \{(H; V_1, \ldots, V_n)\}$, where H is a subgraph of G, and $V_1, \ldots, V_n$ partition the vertices of H.
- Given $f : \langle n \rangle \to \langle m \rangle$, define $f_* : X_n \to X_m$ by

  $$f_*(H; V_1, \ldots, V_n) = (H'; V'_1, \ldots, V'_m),$$

  where

  $$V'_j = \bigcup_{i \in f^{-1}(j)} V_i$$

  and $H' \subseteq H$ is the full subgraph on $\bigcup V'_j$.
- This gives a functor $\Phi_* \to \mathbf{Set}$ for which the induced simplicial set is 2-Segal (Bergner, Osorno, Ozornova, Rovelli, Scheimbauer)

¡Gracias! Thanks!

This talk is based on:

- I. Contreras, M. Keller*, R. Mehta, “Frobenius objects in the category of spans,” Rev. Math. Phys. (2022), arXiv:2106.14743.
- I. Contreras, R. Mehta, W. Stern, “Frobenius and commutative pseudomonoids in the bicategory of spans,” arXiv:2311.15342.
- S. Marx, R. Mehta, “Coherent 2D TFTs in the bicategory of spans,” coming soon.