Courant cohomology and Cartan calculus

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Warm-up: de Rham theory

Let M be a manifold.

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Cartan calculus in de Rham theory

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\begin{aligned}
[d,d] &= 2d^2 = 0, & [d,\mathcal{L}_X] &= 0, \\
[\iota_X, d] &= \mathcal{L}_X, & [\mathcal{L}_X,\mathcal{L}_Y] &= \mathcal{L}_{[X,Y]}, \\
[\iota_X, \iota_Y] &= 0, & [\mathcal{L}_X, \iota_Y] &= \iota_{[X,Y]}.
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- Using these relations, can derive the Cartan formula

$$\begin{aligned} d\omega(X_0,\dots,X_k) = &\sum_i (-1)^i X_i(\omega(X_1, \overset{\hat{}}{\dots}, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i,X_j], X_1, \overset{\hat{}}{\dots}\overset{\hat{}}{\dots}, X_k). \end{aligned}$$

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- The Cartan formula explicitly connects the differential structure of $\Omega(M)$ to the Lie structure of $\mathfrak{X}(M)$.
- This relationship holds more generally for Lie algebroids.

Courant algebroids and cohomology

Definition

A *Courant algebroid* is a vector bundle $E \to M$ equipped with a nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle$, a bundle map $\rho : E \to TM$, and a bracket $[\![\cdot, \cdot]\!]$ such that

1. $[\![[\![e_1, e_2]\!], e_3]\!] = [\![e_1, [\![e_2, e_3]\!]]\!] - [\![e_2, [\![e_1, e_3]\!]]\!]$,
2. $[\![e_1, fe_2]\!] = \rho(e_1)(f)e_2 + f[\![e_1, e_2]\!]$,
3. $\rho(e_1)\langle e_2, e_3 \rangle = \langle [\![e_1, e_2]\!], e_3 \rangle + \langle e_2, [\![e_1, e_3]\!] \rangle$,
4. $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \mathcal{D}\langle e_1, e_2 \rangle$,

where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is given by $\langle \mathcal{D}f, e \rangle = \rho(e)(f)$.

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- Examples: $TM \oplus T^*M$, quadratic Lie algebras
- Motivations: Dirac constraints, generalized geometry, 3d AKSZ theory,...

Theorem (Severa, Roytenberg)

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For general Courant algebroids, there is an explicit description in low degrees:

$$C^\infty(M) \xrightarrow{\mathcal{D}} \Gamma(E) \xrightarrow{\mathcal{L}} \mathcal{O}^{\langle,\rangle}(E) \to \cdots$$

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but in general the known descriptions were suboptimal (local coords, connection, etc.).

The Keller-Waldmann algebra

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- For $k \geq 2$, there exists a map

$$\sigma_\omega : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{k-2} \to \mathfrak{X}(M)$$

  such that

$$\omega(e_1, \dots, e_i, e_{i+1}, \dots, e_k) + \omega(e_1, \dots, e_{i+1}, e_i, \dots, e_k)$$
$$= \sigma_\omega(e_1, \overset{\wedge}{\dots}\overset{\wedge}{\dots}, e_k)(\langle e_i, e_{i+1} \rangle).$$

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  - Well-defined since ω is $C^\infty(M)$-linear in last entry,
  - $\langle \hat{\omega}(e), e' \rangle = \sigma_\omega(\langle e, e' \rangle) - \langle \hat{\omega}(e'), e \rangle$,
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- Degree 3: For $\omega \in C^3(E)$, define $\hat{\omega} : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ by

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Exercise 2: If E has a Courant structure, then can define a 3-cochain T by $T(e, e', e'') = \langle [\![e, e']\!], e'' \rangle$.

Keller-Waldmann=Severa-Roytenberg

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In the smooth setting, the Keller-Waldmann algebra is isomorphic to the algebra of functions on the corresponding symplectic graded manifold.

Idea: If ψ is a degree k function, then the corresponding k-cochain ω is given by

$$\omega(e_1, \dots, e_k) = \{e_k, \dots, \{e_2, \{e_1, \psi\}\} \dots\}.$$

Cartan calculus

Cartan calculus, part 1

- (Severa, Roytenberg) Courant structures on $E \to M$ are in correspondence with degree 3 functions θ such that $\{\theta, \theta\} = 0$.

$$\rho(e)(f) = \{\{e, \theta\}, f\}, \qquad [\![e_1, e_2]\!] = \{\{e_1, \theta\}, e_2\}.$$

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- These operators satisfy the graded commutation relations

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but not $[\iota_e, \iota_{e'}] = 0$.

Cartan calculus, part 2

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Theorem (Cueca-M)

The differential satisfies the following Cartan formula:

$$\begin{aligned} d_E\omega(e_0, \dots, e_k) = &\sum_i (-1)^i \rho(e_i)(\omega(e_1, \dots, \widehat{e}_i, \dots, e_k)) \\ &+ \sum_{i<j} (-1)^{i+1} \omega(e_1, \dots, \widehat{e}_i, \dots, [\![e_i, e_j]\!], \dots, e_k) \end{aligned}$$

Connections and curvature

Let $E \to M$ be a Courant algebroid.

Definition (Alekseev-Xu)

An E-connection on a vector bundle $B \to M$ is a map $\nabla : \Gamma(E) \times \Gamma(B) \to \Gamma(B)$ such that

- $\nabla_e(fb) = f\nabla_e b + \rho(e)(f)b$
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The curvature of an E-connection is defined as usual:

$$F_\nabla(e_1, e_2) = \nabla_{e_1}\nabla_{e_2} - \nabla_{e_2}\nabla_{e_1} - \nabla_{[\![e_1,e_2]\!]}.$$

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Exercise 3: F_∇ is an $\mathrm{End}(B)$-valued 2-cochain.

- Given an E-connection ∇ on B, we can define an operator D_∇ on $C^\bullet(E) \otimes \Gamma(B)$:

$$D_\nabla \omega(e_0, \ldots, e_k) = \sum_i (-1)^i \nabla_{e_i} \omega(e_1, \ldots, \widehat{e_i}, \ldots, e_k) + \sum_{i<j} (-1)^{i+1} \omega(e_1, \ldots, \widehat{e_i}, \ldots, [\![e_i, e_j]\!], \ldots, e_k)$$

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Exercise 4: The Bianchi identity $D_\nabla F_\nabla = 0$ holds.

Adjoint connection

- Let $\hat{\nabla}$ be a linear connection on E. Then we can define an *adjoint* E-connection ∇^E on E by:

$$\nabla^E_{e_1} e_2 = [\![e_1, e_2]\!] + \hat{\nabla}_{\rho(e_2)} e_1 - \rho^* \langle D_{\hat{\nabla}} e_1, e_2 \rangle$$

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- The adjoint E-connection is compatible with the pairing:

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- Also: $\operatorname{tr}(F^k_{\nabla^E}) = 0$ when k odd.

Characteristic classes

Let $E \to M$ be a Courant algebroid, ∇ a flat E-connection on a line bundle $L \to M$.

Modular class

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- There is a canonical flat E-connection ∇^{top} on $\bigwedge^{\mathrm{top}} E$. The (intrinsic) modular class of E is the modular class of $(\bigwedge^{\mathrm{top}} E, \nabla^{\mathrm{top}})$.

Modular class

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Proposition (Cueca-M)

The modular class vanishes for all E.

- Given an E-connection ∇ on a vector bundle B, let $\mathrm{ch}_k(\nabla) = \mathrm{tr}(F_\nabla^k) \in C^{2k}(E)$.

Chern-Weil, Chern-Simons

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- As in the classical theory:
  - $d_E \mathrm{ch}_k(\nabla) = 0$,
  - The cohomology class is independent of the connection.

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  - $d_E \mathrm{ch}_k(\nabla) = 0$,
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  so if $\mathrm{ch}_k(\nabla_0) = \mathrm{ch}_k(\nabla_1) = 0$, the transgression form is closed.

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Theorem (Cueca-M)

The classes $[\mathrm{cs}_k(\nabla^E, \nabla^{E,g})] \in H^{2k-1}(E)$ are independent of the choices.

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Remarks

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- For Lie algebroids, characteristic class constructions require reps up to homotopy, but for Courant algebroids we only need a connection!
- Explanation: the T^*M and TM components “cancel”.

Miquel Cueca and Rajan Amit Mehta, “Courant cohomology, Cartan calculus, connections, curvature, characteristic classes,” arXiv:1911.05898