2-Segal sets as combinatorial models for algebras

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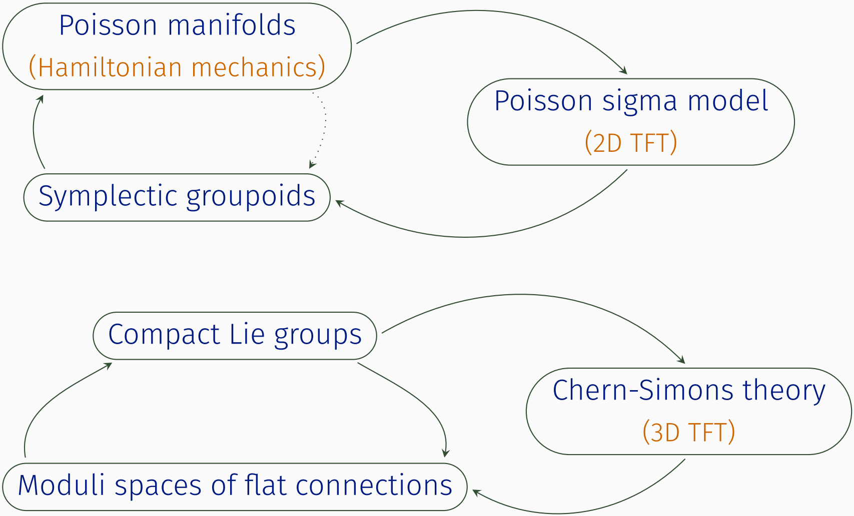
Motivation



Poisson manifolds
(Hamiltonian mechanics)
Poisson sigma model
(2D TFT)
Symplectic groupoids
Compact Lie groups
Chern-Simons theory
(3D TFT)
Moduli spaces of flat connections

The bicategory $\mathbf{Span}$ has:

- objects are sets
- morphisms are spans $X \xleftarrow{f_1} A \xrightarrow{f_2} Y$
- composition of morphisms is obtained by pullback/fiber product:

$$\begin{tikzcd}
 & & A \times_Y B \arrow[dl] \arrow[dr] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, "g"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & \\
X & & Y & & Z
\end{tikzcd}$$

- 2-morphisms are maps of spans:

$$\begin{tikzcd}
 & A \arrow[ddl, bend right, "f_1"'] \arrow[d, "h"] \arrow[ddr, bend left, "f_2"] & \\
 & B \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\
X & & Y
\end{tikzcd}$$

The Cartesian product gives $\mathbf{Span}$ the structure of a symmetric monoidal bicategory.

- In a (symmetric, monoidal) bicategory, identities (associativity, commutativity of product with composition, etc) don't hold on the nose, but there are invertible 2-morphisms (associator, tensorator, etc).
- These 2-morphisms satisfy coherence conditions.
- In $\mathbf{Span}$, this higher data is so "natural" that we often ignore it! E.g. $(A \times_X B) \times_Y C \cong A \times_X (B \times_Y C)$.

Given a set X, there are two associated vector spaces: $\Bbbk[X]$ and $C(X)$.

Given a span $X \leftarrow A \rightarrow Y$, we can (under finiteness conditions) obtain linear maps $\Bbbk[X] \rightarrow \Bbbk[Y]$ and/or $C(X) \rightarrow C(Y)$.

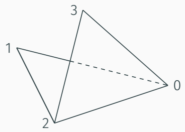
This process is functorial: composition of spans $\rightarrow$ composition of linear maps, isomorphic spans $\rightarrow$ equal linear maps, and Cartesian products $\rightarrow$ tensor products

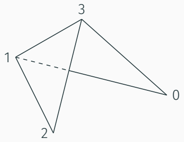
Moral: We can view $\mathbf{Span}$ as a 'categorification' of $\mathbf{Vect}$ (Baez–Hoffnung–Walker).

Let $X_\bullet$ be a simplicial set. Then we can form the following 'unit' and 'multiplication' spans:

$$\begin{tikzcd}[column sep=small] & X_0 \arrow[dl] \arrow[dr, "s_0"] & \\ \bullet & & X_1 \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & X_2 \arrow[dl, "{(d_2,d_0)}"'] \arrow[dr, "d_1"] & \\ X_1 \times X_1 & & X_1 \end{tikzcd}$$

The spans that represent the two sides of associativity are the taco spaces $T_{13} = X_2 \, {}_{d_1}\!\times_{d_2} X_2$ and $T_{02} = X_2 \, {}_{d_1}\!\times_{d_0} X_2$:





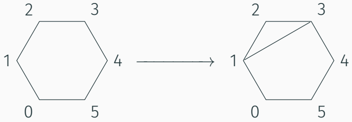
We need an isomorphism between these spaces (associator), satisfying a coherence condition (pentagon equation).

Altogether this type of structure is called a pseudomonoid.

$X_\bullet$ still a simplicial set. To any subdivision of the $(n+1)$-gon into two polygons, we can associate a map

$$X_n \to X_k \times_{X_1} X_{n-k+1}.$$

For example, the subdivision

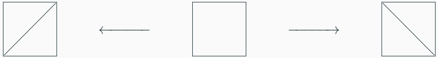


is associated to a map $X_5 \to X_2 \times_{X_1} X_4$.

Definition: (Dyckerhoff–Kapranov, Gálvez-Carrillo–Kock–Tonks) $X_\bullet$ is 2-Segal if this map is an isomorphism for all subdivisions.

2-Segal sets provide associativity data

Consider the two decompositions of the square:



The associated maps are

$$T_{13} \xleftarrow{(d_1,d_3)} X_3 \xrightarrow{(d_0,d_2)} T_{02}.$$

If $X_\bullet$ is 2-Segal then these maps are isomorphisms and provide an associator.

The pentagon equation can be proved using the maps arising from subdivisions of the pentagon.

Theorem: (Stern, also see M.–Marx) Pseudomonoids in $\mathbf{Span}$ correspond to 2-Segal sets.

- nerve of a category $\rightsquigarrow$ category algebra
- nerve of a partial monoid, e.g. $\{1, x, x^2, \ldots, x^L\}$ with partially-defined multiplication $\rightsquigarrow \mathbb{k}[x]/\langle x^{L+1}\rangle$
- $X_n = \{m; S_1, \ldots, S_n\}$ where $m \geq 0$ and $S_1, \ldots, S_n$ is a partition of $\{1, \ldots, m\}$, with nerve-like face maps

$$d_0^n(m; S_1, \ldots, S_n) = (m - |S_1|; S_2, \ldots, S_n),$$
$$d_i^n(m; S_1, \ldots, S_n) = (m; S_1, \ldots, S_i \cup S_{i+1}, \ldots, S_n), \quad 0 < i < n,$$
$$d_n^n(m; S_1, \ldots, S_n) = (m - |S_n|; S_1, \ldots, S_{n-1}),$$

and degeneracy maps insert the empty set. $\rightsquigarrow$ algebra of exponential generating functions.

Frobenius pseudomonoids

There is a notion of Frobenius pseudomonoid (Street).

What is the structure on a 2-Segal set that corresponds to a Frobenius structure on a pseudomonoid?

- In $\mathbf{Span}$, a 'pairing' on X_1 is nondegenerate iff it is isomorphic to

$$X_1 \times X_1 \xleftarrow{(\mathrm{id},\tau)} X_1 \to \bullet$$

  for some automorphism $\tau : X_1 \to X_1$.
- There are additional conditions to ensure the pairing comes from a counit, as well as coherence conditions (swallowtail equations). $\rightsquigarrow$ automorphisms $\tau^n : X_n \to X_n$ for all n, satisfying compatibility conditions.

Paracyclic sets

A paracyclic set is a simplicial set $X_\bullet$ equipped with automorphisms $\tau^n : X_n \to X_n$ such that

$$d_i^n \tau^n = \begin{cases} \tau^{n-1} d_{i+1}^n, & i < n, \\ d_0^n, & i = n, \end{cases}$$

$$s_i^n \tau^n = \begin{cases} \tau^{n+1} s_{i+1}^n, & i < n, \\ (\tau^{n+1})^2 s_0^n, & i = n. \end{cases}$$

Fancier definition: A functor $\Lambda_\infty^{\mathrm{op}} \to \mathbf{Set}$, where Λ_∞ is the paracyclic category.

The more widely known cyclic category is the quotient of the paracyclic category by the relations $(\tau^n)^{n+1} = \mathrm{id}$.

Theorem (Contreras, M., Stern)

Frobenius pseudomonoids in $\mathbf{Span}$ *correspond to* 2*-Segal paracyclic sets.*

Example: Groupoids

- Let $G_1 \rightrightarrows G_0$ be a groupoid. Let $\omega \subseteq G_1$ be a bisection.
- The nerve $G_\bullet$ is paracyclic with

$$t(g_1, \dots, g_n) = (g_2, \dots, g_n, (g_1 \cdots g_n)^{-1}\omega).$$

- It is cyclic iff ω is central, i.e. $\omega^{-1} g \omega = g$ for all $g \in G_1$.

Commutative Frobenius pseudomonoids

What is the structure on a 2-Segal set that corresponds to a commutative Frobenius structure on a pseudomonoid?

Commutative pseudomonoids were considered in [Contreras-M.-Stern] and were shown to correspond to 2-Segal Γ-sets.

So...what do you get when a 2-Segal set has both a paracyclic and a Γ-structure? And is commutative Frobenius = commutative + Frobenius?

A cosymmetric set is a simplicial set $X_\bullet$ equipped with S_{n+1}-actions on X_n satisfying the compatibility conditions

$$\theta_i s_j = \begin{cases} s_j\theta_i, & i<j, \\ s_{i-1}, & i=j, \\ s_i, & i=j+1 \\ s_j\theta_{i-1}, & i>j+1, \end{cases} \qquad \theta_i d_j = \begin{cases} d_j\theta_i, & i<j-1, \\ d_i\theta_{i+1}\theta_i, & i=j-1, \\ d_{i+1}\theta_i\theta_{i+1}, & i=j, \\ d_j\theta_{i+1}, & i>j, \end{cases}$$

$$d_i\theta_i = d_i, \qquad d_n^n = d_0^n\theta_1^n\cdots\theta_{n-1}^n$$

Fancier definition: A functor $\Phi \to \mathbf{Set}$, where Φ is the full subcategory of $\mathbf{Set}$ with the objects $[n] = \{0,\ldots,n\}$.

Theorem (Marx, M.)

Commutative Frobenius pseudomonoids in $\mathbf{Span}$ correspond to 2-Segal cosymmetric sets.

Example: Let M be a commutative partial monoid with a distinguished element $\ell \in M$. Define

$$Z_n = \{(a_0, \ldots, a_n) \in M^{n+1} \mid a_0 + \cdots + a_n = \ell\}.$$

For $f : [n] \to [m]$, define $\hat{f} : (a_0, \ldots, a_n) \mapsto (b_0, \ldots, b_m)$ by

$$b_i = \sum_{j \in f^{-1}(i)} a_j.$$

This gives a functor $\Phi \to \mathbf{Set}$, so $Z_\bullet$ is a cosymmetric set.

Proposition: $Z_\bullet$ is 2-Segal and therefore corresponds to a commutative Frobenius pseudomonoid in $\mathbf{Span}$.

If (M, ℓ) has the orthocomplement property that, for every $a \in M$, there is a unique $a^\perp \in M$ such that $a + a^\perp = \ell$, then $Z_\bullet$ is canonically isomorphic to the nerve of M. But in general, $Z_\bullet$ is different from the nerve of M.

Definition: (Foulis–Bennett) An effect algebra is a commutative monoid with a distinguished element ℓ, satisfying the orthocomplement property and the zero-one law.

Our results show that the nerves of effect algebras are 2-Segal cosymmetric sets $\rightsquigarrow$ commutative Frobenius pseudomonoids.

To do list

- Topological invariants as cardinalities
- Spans of groupoids
- Extended TQFT/semisimplicity
- 3D
- Symplectic/Lagrangian

Obrigado!

Relevant papers:

- (with Sophia Marx) “2-Segal sets and pseudomonoids in the bicategory of spans”, arXiv:2505.22832
- (with Ivan Contreras and Walker Stern) “Frobenius and commutative pseudomonoids in the bicategory of spans”, arXiv:2311.15342
- (with Sophia Marx) “Coherent 2D TQFTs in the bicategory of spans”, in preparation