INTEGRATION OF DG SYMPLECTIC MANIFOLDS VIA MAPPING SPACES

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Abstract. Severa and Roytenberg observed that Courant algebroids are in one-to-one correspondence with differential graded (DG) symplectic manifolds of degree 2. I will describe this correspondence, as well as an integration procedure (due to Severa, following Sullivan) involving mapping spaces. The result of the integration procedure is a symplectic 2-groupoid, but it is infinite-dimensional. Nonetheless, in the case of exact Courant algebroids, the process can be explicitly carried out and described in ordinary terms. This construction gives a nice conceptual explanation for why (twisted) Dirac structures integrate to (twisted) presymplectic groupoids. This talk is based on joint work with Xiang Tang (arXiv:1310.6587).

Outline:

(1) Warmup: Integration of Poisson manifolds from the DG perspective

(2) Courant algebroids

(3) Integrating exact Courant algebroids

(4) Dirac structures

1. Warmup: Poisson manifolds

1.1. **Poisson structures in the DG language.** Let M be a manifold. A *Poisson structure* on M is a Lie bracket on $C^\infty(M)$ satisfying the Leibniz rule $\{f, gh\} = \{f, g\}h + g\{f, h\}$. Motivation: Hamiltonian mechanics.

Let $\mathfrak{X}^\bullet(M) = \Gamma(\wedge TM)$ denote the algebra of multivector fields. The Lie bracket of vector fields naturally extends to a bracket (called the *Schouten bracket*) on $\mathfrak{X}^\bullet(M)$, making $\mathfrak{X}^\bullet(M)$ into a Gerstenhaber algebra.

A Poisson structure on M can be equivalently described by a bivector field $\pi \in \mathfrak{X}^2(M)$ satisfying the integrability condition $[\pi, \pi] = 0$. So a Poisson structure on M induces a differential $d_\pi := [\pi, \cdot]$ on $\mathfrak{X}^\bullet(M)$:

$$C^\infty(M) \xrightarrow{d_\pi} \mathfrak{X}(M) \xrightarrow{d_\pi} \mathfrak{X}^2(M) \longrightarrow \cdots$$

Conversely, given a differential d on $\mathfrak{X}^\bullet(M)$ (compatible with the Gerstenhaber algebra structure), we can recover a Poisson bracket as a derived bracket:

$$\{f, g\} = [df, g].$$

From the perspective of graded geometry, we can view $\mathfrak{X}^\bullet(M)$ as the "smooth functions" on the shifted cotangent bundle $T^*[1]M$, the Schouten bracket as the (degree -1) Poisson bracket corresponding to the canonical (degree 1) symplectic structure, π as a degree 2 function, and d_π as the Hamiltonian vector field of π.

Conversely, it can be shown that any symplectic graded manifold with coordinates in degrees 0 and 1 is *canonically* isomorphic to $T^*[1]M$ for some manifold M, giving a correspondence between Poisson manifolds and "degree 1 symplectic dg-manifolds".

1.2. **Integration via mapping spaces.** Given a Poisson manifold M, one can build an associated simplicial space as follows. The k-simplices are dg-manifold maps $T[1]\Delta^k \to T^*[1]M$ (= dga maps $\mathfrak{X}^\bullet(M) \to \Omega(\Delta^k)$), and the face and degeneracy maps come from the natural maps between simplices.

Let $\mathcal{X}_k$ denote the space of k-simplices. In low degrees:

- $\mathcal{X}_0 = M$.
- $\mathcal{X}_1$ is a certain subspace of paths in T^*M.
- $\mathcal{X}_2$ consists of certain homotopies between paths in T^*M.

Letting $G = \mathcal{X}_1/\sim$, we obtain (up to smoothness issues) a Lie groupoid $G \rightrightarrows M$ (Cattaneo-Felder, Crainic-Fernandes).

1.3. **Symplectic structure of the integration.** The symplectic structure on T^*M induces a symplectic structure on G as follows. Fix $\gamma \in \mathcal{X}_k$, and let v_1, v_2 be tangent vectors in $T_\gamma \mathcal{X}_k$. Using the symplectic form on $T^*[1]M$, we can pair v_1 and v_2 to get a degree 1 function on $T[1]\Delta^k$, i.e. a 1-form on Δ^k.

When $k = 1$, we can integrate the 1-form to get a number. Thus, $\mathcal{X}_1$ is equipped with a 2-form, and it is both closed and multiplicative. Furthermore, the kernel of the 2-form precisely coincides with the homotopies, so that it descends to multiplicative symplectic form on the quotient G.

To summarize, we can say that Poisson manifolds integrate to symplectic groupoids.

2. Courant algebroids

2.1. **Definition.** A *Courant algebroid* is a vector bundle $E \to M$ equipped with a nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle$, a bundle map $\rho : E \to TM$, and a bracket $[\![\cdot, \cdot]\!]$ such that

(1) $[\![[\![e_1, e_2]\!], e_3]\!] = [\![e_1, [\![e_2, e_3]\!]]\!] - [\![e_2, [\![e_1, e_3]\!]]\!]$,

(2) $[\![e_1, fe_2]\!] = \rho(e_1)(f)e_2 + f[\![e_1, e_2]\!]$,

(3) $\rho(e_1)\langle e_2, e_3\rangle = \langle [\![e_1, e_2]\!], e_3\rangle + \langle e_2, [\![e_1, e_3]\!]\rangle$,

(4) $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \mathcal{D}\langle e_1, e_2\rangle$,

where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is given by $\langle \mathcal{D}f, e\rangle = \rho(e)(f)$.

Examples:

(1) On $E = TM \oplus T^*M$, use the obvious symmetric pairing and the bracket
$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_X \iota_Y H,$$
where H is any closed 3-form on M. "Exact Courant algebroids"

(2) $E = \mathfrak{g}$, any Lie algebra equipped with an invariant scalar product.

2.2. **dg perspective.** (Severa, Roytenberg) Let $E \to M$ be a Courant algebroid. There is an associated dga in this setting as well:

$$C^\infty(M) \xrightarrow{\mathcal{D}} \Gamma(E) \xrightarrow{\mathcal{L}} \mathfrak{o}(E) \longrightarrow \cdots$$

Here,

- $\mathfrak{o}(E)$ is the space of first-order skew-symmetric operators on $\Gamma(E)$, and
- $\mathcal{L}$ is given by $\mathcal{L}_{e_1} e_2 = [\![e_1, e_2]\!]$.

There is also a degree -2 Poisson bracket $\{,\}$, where

- $\{e_1, e_2\}$ coincides with the inner product for $e_1, e_2 \in \Gamma(E)$,
- $\{\phi, e\} = \phi(e)$ and $\{\phi, f\} = \sigma(\phi)(f)$ for $\phi \in \mathfrak{o}(E)$, $e \in \Gamma(E)$, and $f \in C^\infty(M)$ (here σ is the symbol map),
- $\{\phi_1, \phi_2\}$ coincides with the commutator bracket,
- $\{f, g\} = \{f, e\} = 0$.

As in the Poisson case, the anchor and Courant bracket can be recovered from the differential as derived brackets:

$$\rho(e)(f) = \{\mathcal{L}_e, f\} = \{\mathcal{D}f, e\}$$
$$[\![e_1, e_2]\!] = \{\mathcal{L}_{e_1}, e_2\}.$$

In fact, the Courant algebroid axioms are equivalent to the statement that this is a dg Poisson algebra.

From the perspective of graded geometry, we view the dg Poisson algebra as the "smooth functions" on a degree 2 symplectic dg-manifold $\mathcal{E}$. Conversely, any degree 2 symplectic dg-manifold gives a Courant algebroid via the derived bracket construction, so there is a correspondence (Severa, Roytenberg).

3. Integrating Courant algebroids

The integration procedure is similar to the Poisson case; the k-simplices are dg-manifold maps $T[1]\Delta^k \to \mathcal{E}$. Because the symplectic structure on $\mathcal{E}$ is degree 2, the resulting symplectic manifold is a quotient of $\mathcal{X}_2$, so we get a *symplectic 2-groupoid.*

3.1. **Exact Courant algebroids.**

For the standard Courant algebroid $E = TM \oplus T^*M$ (set $H = 0$ for now), we can make the identification

$$\mathcal{X}_k = \mathrm{Hom}_{dg}(T[1]\Delta^k, \mathcal{E}) = \mathrm{Hom}(T[1]\Delta^k, T^*[1]M),$$

where the last Hom is just in the category of graded manifolds. In other words, the k-simplices are bundle maps from $T\Delta^k$ to T^*M. In low degrees, we have:

- $\mathcal{X}_0 = M$,
- $\mathcal{X}_1$ can be identified with the space of paths in T^*M.
- The points of the quotient $G_2 := \mathcal{X}_2/\sim$ can be described by a homotopy class of maps $\alpha : \Delta^2 \to M$ and maps ξ_0, ξ_1, ξ_2 lifting each edge of α to T^*M. (The ξ_i are not required to agree at the vertices.)

Theorem 1. *The quotient is smooth (Banach), so $G_2 \Rrightarrow \mathcal{X}_1 \rightrightarrows M$ is a Lie 2-groupoid.*

We call it the *Liu-Weinstein-Xu 2-groupoid $LWX(M)$*, because it answers (at least for exact Courant algebroids) a question posed in their 1995 paper: "What is the global, groupoid-like object corresponding to a Courant algebroid?"

Note that, in this case, there is a natural symplectic structure ω_1 on $\mathcal{X}_1$ which is not part of the general theory. The symplectic form ω_2 on G_2 (which *is* part of the general theory) is the simplicial coboundary (alternating sum of pullbacks by the face maps) of ω_1.

If $H \neq 0$, then the integrating 2-groupoid is the same, but the symplectic form is different. Specifically, $\omega_1' = \omega_1 + \hat{H}$, where $\hat{H}$ is the "transgression" of H to the path space. But now, ω_1' isn't closed. Instead, letting δ be the simplicial coboundary, we have

$$\begin{aligned} \omega_2' &= \delta\omega_1', \\ d\omega_1' &= \delta H, \\ dH &= 0. \end{aligned}$$

4. Dirac structures

A *Dirac structure* in a Courant algebroid E is a subbundle $D \subseteq E$ that is maximally isotropic, and whose sections are closed under the Courant bracket. Examples: Poisson structures, closed 2-forms, and foliations all have corresponding Dirac structures in $TM \oplus T^*M$.

The Courant bracket restricts to a Lie bracket on $\Gamma(D)$, making D into a Lie algebroid.

Theorem 2. *To every Dirac structure in $TM \oplus T^*M$, there is an associated "Lagrangian" sub-2-groupoid of $LWX(M)$.*

The quotes are because the Lagrangian part is conjectural. We proved that it's isotropic, and that it's Lagrangian at the units.

Corollary 2.1. *The 2-form ω_1 induces a multiplicative H-closed 2-form on the Lie groupoid integrating a Dirac structure.*

Here, H-closed means that $d\omega = \delta H$. This recovers a result due to Bursztyn-Crainic-Weinstein-Xu, that (twisted) Dirac structures integrate to (twisted) presymplectic groupoids.