INSTRUCTIONS: This exam must be completed by 3:30 PM, Oct 31. This is a take home exam. You may use any text, the web, your notes, or any other reference material; you may use a computer. However, you may NOT talk to other people about the exam.

1. (35 points) The following questions refer to the rigid rotor problem.
   a. (5 pts) Show that the wave function \(|\ell m\rangle|2 0\rangle\) is orthogonal to \(|3 0\rangle\).
   b. (5 pts) Show that the wave function \(|2 0\rangle\) is an eigenfunction of the Hamiltonian for a rigid rotor.
   c. (7 pts) Show that the wave function \(|4 2\rangle\) is an eigenfunction of the Hamiltonian for a rigid rotor.
   d. (9 pts) Find the expectation value for \(\ell_z\) for the wave function \(|4 2\rangle\).
   e. (9 pts) For a system in the state \(|4 2\rangle\), what is the probability that the rotor will be found with a value of \(\theta\) between 0 and \(\pi/2\) AND a value of \(\phi\) of between 0 and \(\pi/2\)?

2. (30 points)
   a. (7 pts) Show that the function
      \[
      \Phi = \exp\left[\frac{-q^2}{2}(-15 + 90q^2 - 60q^4 + 8q^6)\right]
      \]
   is an eigenfunction of the Hamiltonian for a harmonic oscillator (in q language). Give the eigenvalue.
   b. (8 pts) For a harmonic oscillator in q language, find the expectation value for the energy if the system is described by the wave function given below.
      \[
      \psi = \exp\left[\frac{-q^2}{2}(-0.325791 - 0.390296q + 0.559589q^2 + 0.260198q^3 + 0.0613291q^4)\right]
      \]
   c. (8 pts) What is the probability that a measurement of the energy of the system described by the wave function in part b (using the Hamiltonian for the harmonic oscillator in q language) will yield a value of 5/2 (in q language, that is, with energy in units of \(\hbar\omega\))?
   d. (7 pts) Prove using Mathematica that \(a_+^3|3\rangle\) is a eigenfunction of the Hamiltonian for a harmonic oscillator with an eigenvalue of 13/2 (in q language). HINT: You know the answer; it is your process that will be graded.

3. (20 points) Show that the commutator \([\ell_z, \ell_x]\) is proportional to \(\ell_y\). HINT: You know the answer; it is your process that will be graded.
4. (40 points) A parve is trapped in a well of depth 100 units (with $\hbar = 1$ and $m = 1$) as shown. Note that the left wall of the well is at $x = -1$ and the right wall is at $x = 1$.

![Diagram of a parve in a well](image)

a. (15 pts) Find the lowest energy wave function and its energy for this situation.
b. (15 pts) What is the probability that the parve will be in a non-classical region of space?
c. (10 pts) Would the barrier penetration be larger or smaller (for the lowest energy level) if the well were less deep? Prove your answer is true.