12

Bases for Splines on a Subdivided Domain

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Abstract

Let $S^r(\Delta)$ be the module of all splines of smoothness r on the rectilinear partition Δ which subdivides some domain D. Further, let $S^r(\Gamma)$ be the module of all splines of smoothness r on Γ which also subdivides D, where Γ is a finer subdivision of Δ. We study the relationship between a generating set of $S^r(\Delta)$ and a generating set for $S^r(\Gamma)$. This paper gives an algorithm for extending a generating set for $S^r(\Delta)$ to one for for $S^r(\Gamma)$. This method is built on algebraic properties of splines and the Gröbner Basis Algorithm.

1 Introduction

Splines are often used to approximate a function on a given domain. Given a domain $D \in \mathbb{R}^d$ choose a rectilinear partition Δ of that domain. For example, Δ may be a simplicial complex or a grid partition. One may require a certain smoothness of the spline for the given application. Once a spline approximation of these characteristics is found it is sometimes necessary to refine this with a better approximation. To do this one can change either the subdivision or the required smoothness. This paper will be concerned

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with the former of these. In particular we will consider the effect of further subdividing the existing partition.

This paper will give a technique for adapting the generating set for the splines on the old partition to a generating set for the splines on the new partition. This new generating set will essentially contain all old generating elements as well as some additional ones. This method is built on the algebraic properties of splines and Gröbner Basis Algorithm.

2 Background

Numerical analysts and others using splines have long tried to find some of the theoretical characteristics of spaces of splines. Great head way has been made in the past ten years when theses questions were attacked from a more algebraic point of view.

The set of all piecewise polynomial functions of degree m and smoothness r on a subdivided domain $\Delta \in \mathbb{R}^d$ is denoted by $S^r_m(\Delta)$ and is a vector space over $\mathbb{R}$. Two major questions for both those interested in the theoretical aspects of splines and those who use them are what are the dimension of these spaces, and what are (nice) bases for them. In 1973 Strang [St] conjectured a formula for the dimension of $S^1_m(\Delta)$, for $\Delta \in \mathbb{R}^2$ a simplicial complex, which was almostly purely combinatorial, that is, it did not depend on the specifc embedding of Δ. Numerical approaches, including the use of Beziet nets, led to some partial related results. The real breakthrough was made 10 years later when Billera approached the problem from an algebraic viewpoint. Using homological algebra, combinatorics and a key result using matroids due to Whiteley [W1] Billera proved Strang's conjecture [B].

Another important contribution of Billera to was consider $S^r(\Delta) = \cup_m S^r_m(\Delta)$ which is a module over the polynomial ring. As such, the generators for it can be found using the Gröbner Basis Algorithm. This development was presented in [B&R], [H1], and [H2]. A fascinating conection was made between $S^0(\Delta)$, for Δ a simplicial complex and the Stanley-Reisner ring [B2]. The Stanley- Reisner ring of a simplicial complex is a polynomial ring where the vertices of the complex correspond to variables, quotiented by monomials representing the non-faces of the complex. Thus this connection has led to broader interest in the structure of spline spaces from the algebraic community.

Building on this work a variety of mathematicians have used techniques from numerical anaylysis, commutative algebra, homology theory and matroid theory individually and in combination to acheive results about spline spaces. For example, in [B&H] homology was used to give a dimension formula for spaces of divergence-free splines. Alfeld , Schumaker and Whitely using Beziet nets and matroids attained a dimension formula for generically embedded simplicial complexes in $\mathbb{R}^3$ [A&W]. More recently, Schneck and Stillman [S&S1], [S&S2] have used a a different homology sequence in combination with algebraic geometry to characterize when $S^r(\Delta)$ is a free module and to obtain additional results about the dimension series.

3 Prelimnaries

Let D be a domain in $\mathbb{R}^d$. Let Δ be a rectilinear partition of D. A *face* of Δ is a part of the partition including its boundary. Two faces of the same dimension will either be disjoint or meet along their boundaries. Two d-dimensional faces (d-faces) σ and $\sigma' \in \Delta$ are *adjacent* if $\sigma \cap \sigma'$ contains a $(d-1)$-face. Δ is said to be *strongly connected* if for any pair of d-faces σ, $\sigma' \in \Delta$ there is a sequence of d-faces $\sigma = \sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_k = \sigma'$ such that for each $i < k$ σ_i and σ_{i+1} are adjacent. For any $\tau \in \Delta$ define the *star* of τ to be $\{\sigma \in \Delta | \exists \sigma' \in \Delta \text{ s.t. } \sigma, \tau \subset \sigma'\}$. In this paper it is assumed that Δ is strongly connected and has strongly connected stars. Note that this implicitly makes some assumptions about the topology of the underlying domain D.

The development here will be based on the ideas of module and Gröbner bases for spline spaces as in [B&R], [H1], and [H2]. Given a partition Δ, a degree of smoothness r, define $S^r(\Delta)$ to be the set of all splines on Δ with smoothness r. That is, $S^r(\Delta)$ consists of all functions $F : \Delta \to \mathbb{R}$ such that

$$F|_\sigma \text{ is a polynomial; for each face } \sigma \in \Delta$$

and

$$F \text{ is continuously differentiable of order } r.$$

The set $S^r(\Delta)$ is a module over the polynomial ring $\mathbb{R}[x_1, \ldots x_d]$. As in [B], [H1] and [H2] this paper will take advantage of the algebraic structure of the splines. The smoothness conditions will be treated as linear relations of the polynomials as given in the following theorem.

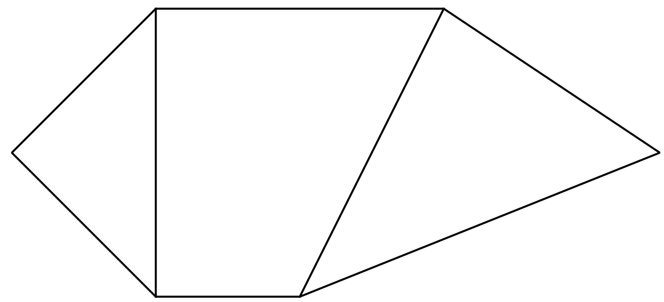

Figure 1: Δ for examples

Theorem 3.1. *[B] Given Δ as above. Let F be a piecewise polynomial function such that $F|_\sigma$ is a polynomial for each $\sigma \in \Delta$ and $r \geq 0$. Then $F \in S^r(\Delta)$ if and only if for each pair σ_i, σ_j of adjacent d-faces in Δ the following is true. If $\tau = \sigma_i \cap \sigma_j$ is of dimension $d-1$ and if l is a nontrivial affine form which vanishes on τ, then $l^{r+1}|(p_i - p_j)$ where $p_i = F|_{\sigma_i}$ and $p_j = F|_{\sigma_j}$.*

Thus we come to think of the splines as the polynomial solutions to a given set of equations over the polynomial ring, $\mathbb{R}[x_1, ...x_d]$.

Example 3.2. For example, consider the complex in figure 1. Let us assume that the two interior edges are embedded on the affine forms l_1 and l_2. A spline on this partition of smoothness r can be represented as an ordered 5-tuple $(f_1, f_2, f_3, f_4, f_5)$ of polynomials, where f_1, f_2, f_3 represent the polynomials on the three 2-faces of Δ and f_4, f_5 represent the "smoothing factors" across the two interior edges. That is the 5-tuples must satisfy the following two equations.

$$\begin{array}{ccccccc} f_1 & -f_2 & & +f_4 l_1^{r+1} & & = 0 \\ & f_2 & -f_3 & & +f_5 l_2^{r+1} & = 0 \end{array}$$

The Gröbner basis algorithm finds generating sets for syzygies of tuples of polynomials. Discussion of the Gröbner basis algorithm for finding module bases of splines appears in [B&R] and [H1]. Given n k-tuples of polynomials $G_1, G_2, ..., G_n$ (each G_i is a k-tuple of polynomials) a *syzygy* of this tuple is an n-tuple of polynomials $f_1, f_2, ..., f_n$ such that $\Sigma f_i G_i = \underline{0}$. Where $\underline{0}$ is the k-tuple of all zeros. For instance in the above example $k = 2$, $n = 5$ and $G_1 = (1, 0)$, $G_2 = (-1, 1), G_3 = (0, 1), G_4 = (l_1^{r+1}, 0)$, and $G_5 = (0, l_2^{r+1})$. Thinking of G_i as the ith column of a k by n matrix G (with entries in the polynomial ring) then a syzygy for this set of columns is an n-tuple of

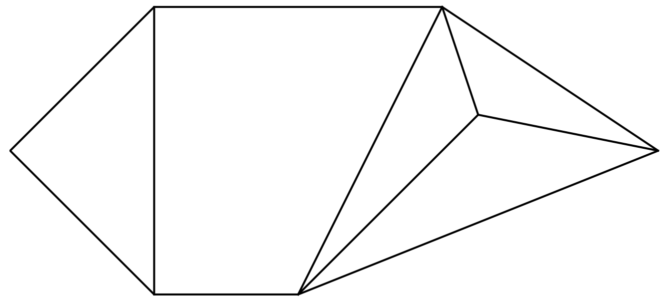

Figure 2: Further Subdivision of One face of figure 1

polynomials $F = (f_1, f_2, ..., f_n)^T$ such that the matrix equation $GF = 0$ holds.

4 Subdivision of a Partition

This paper considers the following problem. Given a partition Δ and a generating set for $S^r(\Delta)$, suppose a further subdivision of Δ is made, say to create Γ. How can that new subdivision be compared to the original one and what is the relationship between the generating elements of $S^r(\Delta)$ and the generating elements of $S^r(\Gamma)$?

Given two different partitions on the same domain D, we say that a face of one is *contained* in a face of the other, if the underlying region in D of the former is contained in the underlying region of the later. The two faces will be said to be *equal* if each is contained in the other. A partition Γ is a *finer* subdivision of Δ if each face of Γ is contained in a face of Δ.

To have any effect on the allowable spline functions a finer subdivision must subdivide some d-faces. Any such finer subdivision of Δ can be obtained by repeatedly subdividing just one d-face at a time. Consider some Γ which further subdivides exactly one d face of Δ. There are then two possibilities. First, it is possible that the only face of Δ that was divided was the d-face. The subdivision in figure 2 is such a subdivision of figure 1.

The other possibility is that lower dimensional faces were also subdivided to obtain Γ. An example of this is shown in figure 3. In this case it will be useful to define an intermediate subdivision, $\hat{\Delta}$ which subdivides the same $d-1$-faces of Δ as Γ but does not subdivide any d-face. Thus Γ is finer than $\hat{\Delta}$ which is finer than Δ. It is clear that $S^r(\hat{\Delta}) \cong S^r(\Delta)$.

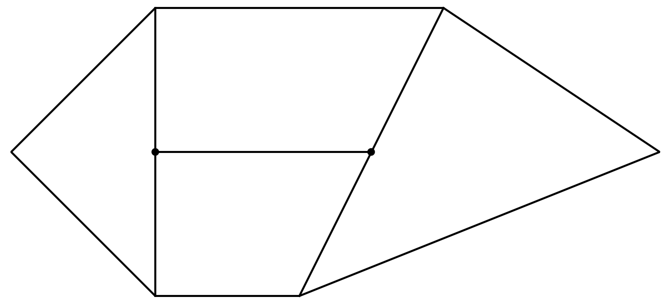

Figure 3: Further Subdivision of more than one face of figure 1

5 Computing the New Generating Set

Suppose the original partition Δ has n d-faces and k interior $(d-1)$-faces. Then as above, there is a k by $(n+k)$ matrix, G, with polynomial entries such that all elements in $S^r(\Delta)$ can be associated with an $(n+k)$-tuple, F, of polynomials such that $GF = 0$. Let Γ be a finer subdivision of Δ. Again there will be a matrix, H, such that each element of $S^r(\Gamma)$ can be associated with a tuple of polynomials J such that $HJ = 0$. We try to exploit the relationship between G and H. Ideally, there might be the following relationship: $H = \begin{pmatrix} G & A \\ 0 & B \end{pmatrix}$, for some A and B. This of course does not occur. However, in the case where only one d-face has been subdivided the following is true.

Theorem 5.1. *Let Γ be a finer subdivision of Δ that subdivides exactly one d-face of Δ. Let G, H be the matrices for $S^r(\Delta)$ and $S^r(\Gamma)$, respectively. There exists a real nonsingular matrix L such that*

$$HL = \begin{pmatrix} \hat{G} & A \\ 0 & B \end{pmatrix} := H^*.$$

Where:

(i) If no $(d-1)$-faces are subdivided then $\hat{G} = G$.

(ii) If some $(d-1)$-faces are subdivided then $\hat{G}$ is the matrix corresponding the complex $\hat{\Delta}$, as defined above.

Notice that $H^*F^* = 0$ if and only if $H(LF^*) = 0$. Thus a a generating set for the syzygies of H^* can be transformed back to a generating set for the syzygies of the columns of H.

Corollary 5.2. *Given two partitions* Δ *and* Γ *and their associated matrices* G *and* H*, as above, there exists matrices* A*,* B *and* L *such that* $S^r(\Gamma) = \{LF^* : \begin{pmatrix} \hat{G} & A \\ 0 & B \end{pmatrix} F^* = 0\}$*.*

Proof (of theorem): Let the face to be subdivided be represented by the last column of the matrix G. Write the matrix H corresponding to this subdivision as follows. Drop the last column of G (which was the face to be subdivided). Add the new columns corresponding to the new d-faces at the right of the matrix followed by the new columns corresponding to the new $(d-1)$-faces. Add new rows corresponding to the new $(d-1)$-faces as well.

An example of G and H is given for the pair of partitions in figures 1 and 2. The notation l_i is used to represent various affine forms.

$$G = \begin{pmatrix} 1 & -1 & l_1^{r+1} & 0 & 0 \\ 0 & 1 & 0 & l_2^{r+1} & -1 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & -1 & l_1^{r+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_2^{r+1} & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & l_3^{r+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & l_4^{r+1} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & l_5^{r+1} \end{pmatrix}$$

Note that summing the 3 columns that represent the 3 new 2-faces gives the column that represented the old 2-face. Define L to be the matrix which transforms the columns of H by replacing the fifth column with itself plus the sixth and seventh columns (i.e., add all columns corresponding to new faces), and leaving all other columns alone. The matrix L will be the 10×10 identity matrix except that the fifth column is now $(0,0,0,0,1,1,1,0,0,0)^T$

Return to the general case of subdividing one d-face. As in the example, adding together the new columns corresponding to all the new d-faces gives the original d-face column with zeros in all the new rows. This is because in each new row there will be exactly one 1 and one -1 in the new d-face columns. In the old rows there will be a 1 or -1 in exactly one new column only if there was one in the original column, otherwise there will be only zeros. Thus, in every case L is simply an identity (or perhaps permutation) matrix, except for one column which will sum the appropriate columns of H.

Next consider the case where Γ subdivides exactly one d-face of Δ and $(d-1)$-faces have been subdivided as well. The matrix for $S^r(\Gamma)$ will be processed in 2 steps. First derive $S^r(\hat{\Delta})$ and its associated matrix $\hat{G}$. Then $S^r(\Gamma)$ is derived from $S^r(\hat{\Delta})$ as before. $\square$

Obtaining $\hat{G}$ from G is straightforward. For each $(d-1)$-face, σ of Δ that is subdivided into p $(d-1)$-faces, $(\sigma_1, \sigma_2, \ldots, \sigma_p)$ of Γ proceed as follows. Suppose χ was the entry in G in the σ row, σ column. Replace the σ column of G with p new columns corresponding to $(\sigma_1, \sigma_2, \ldots, \sigma_p)$. Replace the σ row of G with p new rows corresponding to $(\sigma_1, \sigma_2, \ldots, \sigma_p)$. The σ_i row will be identical to the deleted row except in the new columns, where it will have χ in the σ_i column and 0 in all other new columns. The new columns will have zeros in all old rows.

6 The Syzygy Algorithm

In this section we present an algorithm for finding the new syzygies. The algorithm consists of pieces of the Gröbner Basis Algorithm. A discussion of the use of the Gröbner basis algorithm for splines can be found in [H1] and [B&R].

Recall that we are looking for the syzygies of the columns of a matrix of the form $H^* = \begin{pmatrix} G & A \\ 0 & B \end{pmatrix}$, for some A, B, and G as before. It is assumed that the syzygies for the columns $[G,0]^t$ are already known. Thus we focus our attention on syzygies which include some of the columns of $[A,B]^t$.

There are two parts to the algorithm. First the columns of $[A,B]^t$ must be *reduced* with respect to each other and the columns of $[G,0]^t$. Second, find a generating set for syzygies between the reduced columns of $[A,B]^t$ and the columns of $[G,0]^t$.

The Gröbner Basis Algorithm and this variation of it require the introduction of a total ordering of the tuples of polynomials. We avoid a general definition and just use one such term ordering here. For a k-tuple of polynomials $p = (p_1, p_2, ..., p_k)$, define t_p to be the least i such that $p_i \neq 0$. For a polynomial p_i define its leading monomial, $\mathrm{lm}(p_i)$ to be the term of p_i with the largest degree and lexicographically largest. Order k-tuples by the first nonzero coordinate. Among k-tuples with the same first nonzero coordinate, order the k-tuples by leading monomial of the first nonzero term. Define in(p), the initial term of p, to be the k-tuple with the leading monomial of p_{t_p} as the t_p-th component and 0 elsewhere.

Definition 6.1. For two $k-$tuples p, q, we say $\text{in}(p) \mid \text{in}(q)$ if $t_p = t_q$ and $\text{lm}(p_{t_p})$ divides $\text{lm}(q_{t_q})$ in the usual sense.

Given some set G of k-tuples, with coordinates in the polynomial ring. A k-tuple $a \in G$ is *reduced* with respect to G if there is no $g \in G$ for which $\text{in}(g) \mid \text{in}(a)$. A set H is *reduced* with respect to some subset $G \subseteq H$ if every k-tuple of H has been reduced. The following algorithm reduces an element a with respect to the set G.

The Reduction Algorithm:

(i) If $a = 0$ then a is reduced and the algorithm stops.

(ii) If there is no $g \in G$ such that $\text{in}(g) \mid \text{in}(a)$ then a is reduced and the algorithm stops.

(iii) If $\text{in}(g) \mid \text{in}(a)$ and the initial term of a is $m(\text{in}(g))$, for some $m \in \mathbb{R}[x_1, \ldots, x_d]$ then define $a' = a - mg$.

(iv) Set $a = a'$ and return to step (i).

The next algorithm finds a generating set for all syzygies of the columns of $H = \begin{pmatrix} G & A \\ 0 & B \end{pmatrix}$ which involve the any columns of $\begin{pmatrix} A \\ B \end{pmatrix}$.

Finding Syzygies:
Let $\hat{C} = \hat{c}_1, \hat{c}_2, ..., \hat{c}_s$, be the set of columns of H.

(0) Reduce each element $\hat{c}_i$ over $\hat{c}_1, ..., \hat{c}_s - \hat{c}_i$. Let $C = c_1, c_2, ..., c_s$ be the associated reduced set. Let $\{c_1, c_2, ..., c_k\}$, be the subset of C consisting of columns that were reduced from the columns of $\begin{pmatrix} A \\ B \end{pmatrix}$.

(i) Let $t_i = t_{p_i}$ be the position of the first nonzero coordinate of the k-tuple c_i. For all $i < k$ and each pair c_i, c_j in C such that $t_i = t_j$, do the following. Say the t_i-*th* coordinates of $\text{in}(c_i)$ and $\text{in}(c_j)$ are m_i and m_j, respectively, with m_i, $m_j \in \mathbb{R}[x_1, ...x_d]$.

   (a) Find m_{ij}, the least common multiple of m_i and m_j, and choose q_i, q_j such that $m_{ij} = q_i m_i = q_j m_j$.

(b) Let S_{ij} be the reduction of $q_i c_i - q_j c_j$ over C.

(c) If $S_{ij} = 0$ add S_{ij} to S, the list of syzygies. If $S_{ij} \neq 0$ then add S_{ij} to the set C.

(ii) For each S_{ij} in S, the list of syzygies, rewrite S_{ij} in terms of the original generators, $\hat{c}_1, \hat{c}_2, ..., \hat{c}_s$, by tracing its construction.

As this procedure is just a minor variation of the Gröbner basis algorithm we have the following result.

Theorem 6.2. *The above algorithm is finite. When it stops S is a generating set for the syzygies of $\hat{c}_1, \hat{c}_2, ..., \hat{c}_s$, which include a column of* $\begin{pmatrix} A \\ B \end{pmatrix}$.

Let us continue the earlier example. The partition of figure 2 gives the following H^*.

$$H^* = \begin{pmatrix} 1 & -1 & l_1^{r+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_2^{r+1} & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & l_3^{r+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & l_4^{r+1} & 0 \\ 0 & 0 & 0 & 0 & -0 & 0 & 1 & 0 & 0 & l_5^{r+1} \end{pmatrix}$$

The $[AB]^T$ part of this is the last 5 columns. Reducing these columns among themselves, the last three rows become:

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & l_3^{r+1} & l_4^{r+1} & l_5^{r+1} \end{pmatrix}$$

The first two of these new reduced columns can not occur in any syzygies. Thus new syzygies must use at least two of the last three columns and then perhaps some of the columns of $[G, 0]^t$.

7 Local Generators

In practice, when a partition Δ is subdivided to make Γ, it would be useful if all splines of $S^r(\Gamma)$ could be composed simply from generators of $S^r(\Delta)$,

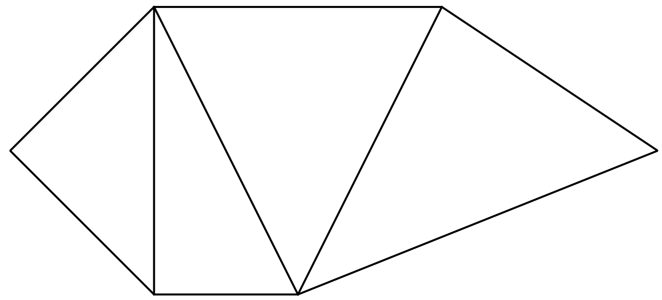

Figure 4: Example for Proposition 7.1

and from some new generators which were "local". Where local means that the new generators are to be nonzero only on the newly subdivided region of Γ. This section briefly considers local generators.

Let σ be the d-face of Δ which is subdivided to create Γ. Assume some spline function of $S^r(\Delta)$ has been constructed. We wish to change only the value on the now subdivided region σ. The specified smoothness must be retained across the boundary $(d-1)$-faces of σ in Δ. Thus we will look at splines on the subdivided σ with some boundary conditions. That is, we look for a generating set of the subspace of $S^r(\Gamma)$ which is zero on all faces outside of σ. Denote this subspace $B^r(\sigma)$.

Proposition 7.1. *$S^r(\Gamma)$ is not necessarily isomorphic to $S^r(\Delta) + B^r(\sigma)$.*

Proof. We show this result by example. This time the partition Δ given in figure 1, subdivided to Γ in figure 4. A minimal generating set for each space, interpreted as being on Γ, is given below. The coordinates of each vector correspond to the faces of Γ. We have dropped the coordinates corresponding to the edges. Notice that $B^r(\sigma)$ does not have a free basis.

$$S^r(\Gamma) =< (1,1,1,1); (0, l_1^{r+1}, l_1^{r+1}, l_1^{r+1}); (0,0,l_3^{r+1}, l_3^{r+1}); (0,0,0,l_2^{r+1}) >$$

$$S^r(\Delta) =< (1,1,1,1); (0, l_1^{r+1}, l_1^{r+1}, l_1^{r+1}); (0,0,0,l_2^{r+1}) >$$

$$B^r(\sigma) =< (0, l_1^{r+1}l_2^{r+1}, l_1^{r+1}l_2^{r+1}, 0); (0,0,l_2^{r+1}l_3^{r+1}, 0); (0, l_1^{r+1}l_3^{r+1}, 0, 0) >$$

$\square$

References

[A&W] P. Alfeld, L. L. Schumaker, and W. Whiteley, "The generic dimension of the space of C^1 splines of degree $d \geq 8$ on tetrahedral decompositions," SIAM J. Numer. Anal. 30 (1993), no. 3, 889–920.

[B] L. J. Billera, "Homology of smooth splines: generic triangulations and a conjecture of Strang," Trans. A. M. S. vol 310, 1988 pp. 325-340.

[B2] L. J. Billera, "The Algebra of continuous piecewise polynomials," Advances in Mathematics vol 76, 1989 pp. 170-183.

[B&H] L. J. Billera,and R. Haas, "The dimension and bases of divergence-free splines; a homological approach," Approx. Theory Appl. 7 (1991), no. 1, 91–99.

[B&R] L. J. Billera and L. L. Rose," Gröbner Basis Methods for Multivariate Splines," *Mathematical Methods in Computer Aided Design*, T. Lyche and L. L. Schumaker, eds., Academic Press, New York, 1989, pp.93-104.

[H1] R. Haas, *Dimension and Bases for Certain Classes of Splines: A Combinatorial and Homological Approach*, Ph.D. Thesis, Cornell University, August 1987, University Microfilms Intl., Ann Arbor, MI.

[H2] R. Haas, "Module and Vector Space Bases for Spline Spaces" *Journal of Approximation Theory*, Vol. 65, No. 1, April 1991, pp. 73-89.

[S&S1] H. Schneck and M. Stillman, "Local cohomology of bivariate splines," *Journal of Pure and Applied Algebra*, to appear.

[S&S2] H. Schneck and M. Stillman, "A family of ideals of minimal regularity and the Hilbert series of $C^r(\hat{\Delta})$," *Advances in Mathematics*, to appear.

[St] G. Strang "The dimension of piecewise polynomial spaces, and one-sided approximation," Conference on the Numerical Solution of Differential Equations (Univ. Dundee, Dundee, 1973), pp. 144–152. Lecture Notes in Math., Vol. 363, Springer, Berlin, 1974.

[W1] "A matrix for splines," *Progress in Approximation theory*, Academic Press, Boston, MA. (1991)