Partial List Colorings

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Abstract

Suppose G is an s-choosable graph with n vertices, and every vertex of G is assigned a list of t colors. We conjecture that at least $\frac{t}{s} \cdot n$ of the vertices of G can be colored from these lists. We provide lower bounds and consider related questions. For instance we show that if G is χ-colorable (rather than being s-choosable), then more than $\left(1 - \left(\frac{\chi-1}{\chi}\right)^t\right) \cdot n$ of the vertices of G can be colored from the lists and that this is asymptotically best possible. We include a number of open questions.

Key words: list coloring; graph coloring; choice number; chromatic number

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1 Introduction

Suppose G is a graph with n vertices, chromatic number χ, and independence number α. Whenever the vertices of G are properly colored with r colors, at least one color class must contain at least $\frac{n}{r}$ vertices. This immediately implies that $\alpha \geq \frac{n}{\chi}$. This inequality shows up in the theory of perfect graphs (7) as well as in Erdős' groundbreaking contribution that there are graphs of arbitrarily large girth and chromatic number (5).

A natural extension of the independence number is to define $\alpha_t = \alpha_t(G)$ to be the maximum number of vertices in G that can be t-colored (1). Considering the largest t color classes in an r-coloring of G immediately implies that $\alpha_t \geq \frac{t \cdot n}{\chi}$.

The goal of this paper is frankly mischievous - to introduce a list coloring analogue of the parameter α_t and to incite further work.

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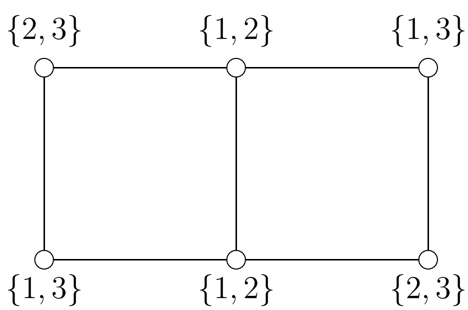
Let $R = \{1, 2, \ldots, r\}$ be a set of colors. The function $\ell : V(G) \to 2^R$ assigns to each vertex x a list of possible colors $\ell(x)$. A proper coloring $c : V(G) \to R$ is a *list coloring* if $c(x) \in \ell(x)$ for all $x \in V(G)$. G is said to be *s-choosable* if there is a list coloring for every assignment function ℓ with $|\ell(x)| = s$ for all $x \in V(G)$. Sometimes we write ℓ_s to emphasize the constant list sizes. The *list chromatic number* of the graph G, denoted by $\chi_\ell(G)$ is defined to be the minimum s such that G is s-choosable. Note that $\chi(G) \leq \chi_\ell(G)$. For $t \leq \chi_\ell(G)$ let ℓ_t be an assignment of t colors to every vertex of G. We use λ_t to denote the list coloring analogue of α_t. Formally $\lambda_t = \min_{\ell_t}\{$maximum number of vertices of G that can be colored from the lists $\ell_t\}$.

Conjecture 1 $\lambda_t \geq \frac{t \cdot n}{\chi_\ell}$

For example the graph G shown below has $2 = \chi(G) < \chi_\ell(G) = 3$. It is impossible to color G from the lists shown; however, five of the vertices of G can be colored from these lists (or any others of size two). Thus $\lambda_2(G) = 5$.



If $t = 1$ or $t = \chi_\ell$ the above conjecture is satisfied. Consequently the simplest open case is when lists of size 2 are assigned to the vertices of a 3-choosable graph. Here it remains unknown whether in all cases $\lambda_2 \geq \frac{2n}{3}$.

2 What we do know

We have two techniques for establishing lower bounds for $\lambda_t(G)$. The first relies on $\chi(G)$ while the second relies on $\chi_\ell(G)$.

Theorem 2 If G is a graph with n vertices and chromatic number χ, then

$$\frac{\lambda_t}{n} > 1 - \left(\frac{\chi - 1}{\chi}\right)^t .$$

PROOF. Suppose G is χ-colored and $C_1, C_2, \ldots, C_\chi$ are the color classes. Furthermore suppose each vertex x has a list $\ell(x)$ of t colors with r colors used on the union of the lists. Assume for the moment that r is an integral multiple of χ. We imagine using $\frac{r}{\chi}$ of these colors to list color some of the vertices in each color class. We need to show that some partition of the colors leaves only a few of the vertices uncolored.

Our accounting will be accomplished by a bipartite graph. The graph $N(G)$ will contain a red vertex, x, for each vertex in G. $N(G)$ will also have a blue vertex, y, for every partition of R, the set of colors, into χ parts, each containing exactly $\frac{r}{\chi}$ colors. Suppose the red vertex x is in color class C_j. Then x will be joined to a blue vertex y precisely when the j-th part, of the color partition represented by y has no color in common with $\ell(x)$. Thus if the colors available are partitioned according to y, then x will remain uncolored.

Since $|\ell(x)| = t$ there are $\binom{r-t}{\frac{r}{\chi}}$ sets of colors available for the j-th part of the partition associated with y. Therefore in $N(G)$

$$deg(x) = \binom{r-t}{\frac{r}{\chi}} \cdot \binom{r-\frac{r}{\chi}}{\frac{r}{\chi}} \cdot \binom{r-\frac{2r}{\chi}}{\frac{r}{\chi}} \cdots \binom{r-\frac{(\chi-1)r}{\chi}}{\frac{r}{\chi}} \quad .$$

Counting the number of edges in $N(G)$ from the point of view of the red vertices we have $|E(N(G))| = deg(x) \cdot n$. Looking at the other side of the bipartition we see that in $N(G)$ some blue vertex, say y, has

$$deg(y) \leq \frac{|E(N(G))|}{\binom{r}{\frac{r}{\chi}} \cdot \binom{r-\frac{r}{\chi}}{\frac{r}{\chi}} \cdot \binom{r-\frac{2r}{\chi}}{\frac{r}{\chi}} \cdots \binom{r-\frac{(\chi-1)r}{\chi}}{\frac{r}{\chi}}} \quad .$$

Canceling common factors from the numerator and denominator we see that

$$deg(y) \leq n \cdot \frac{\binom{r-t}{\frac{r}{\chi}}}{\binom{r}{\frac{r}{\chi}}} \quad . \tag{1}$$

This simplifies to

$$deg(y) \leq n \cdot \frac{(r-\frac{r}{\chi}) \cdot (r-\frac{r}{\chi}-1) \cdots (r-\frac{r}{\chi}-t+1)}{r \cdot (r-1) \cdots (r-t+1)} < n \cdot \left(\frac{\chi-1}{\chi}\right)^t \quad .$$

Since $\deg_{N(G)}(y)$ counts the number of vertices that cannot be colored from the lists given the partition associated with y, the theorem follows. If r is not an integral multiple of χ the asymptotics are the same.

Corollary 3 $\frac{\lambda_\chi}{n} > 1 - \frac{1}{e}$

Corollary 4 There exists a graph G with chromatic number χ such that $\lambda_t(G)$ is asymptotically close to $n \cdot (1 - (\frac{\chi-1}{\chi})^t)$.

PROOF. Let G be a complete χ-partite graph with $\binom{r}{t}$ vertices in each part. Each part contains exactly one vertex with each possible list of t colors.

Consider the proportion of uncolored vertices in the construction of the preceding corollary. This is the case of equality in Inequality 1. If $r = p \cdot q$ and $t = \chi = p$, then this proportion is $\frac{\binom{p \cdot q - p}{q}}{\binom{p \cdot q}{q}}$. It is straightforward to check that this equals $\frac{\binom{p \cdot q - q}{p}}{\binom{p \cdot q}{p}}$. This is the proportion of uncolored vertices when $r = p \cdot q$ and $t = \chi = q$. The bijection between the uncolored vertices in the graphs corresponding to these two different cases remains elusive.

Of course the list chromatic number of the graph constructed in the above corollary would be enormous. Erdős, Rubin and Taylor showed that bipartite graphs can have arbitrarily large list chromatic numbers (6). However, for bipartite graphs $\frac{\lambda_t}{n}$ is close to 1.

Corollary 5 If G is bipartite, then $\lambda_t > \frac{2^t - 1}{2^t} \cdot n$.

Corollary 6 If $\chi(G) < \chi_\ell(G)$, then $\lambda_2(G) \geq \frac{2n}{\chi_\ell}$.

PROOF. We have that $\lambda_2(G) \geq n \cdot (1 - (\frac{\chi-1}{\chi})^2) = n \cdot (\frac{2\chi-1}{\chi^2})$. Since $(\frac{2\chi-1}{\chi^2}) > \frac{2}{\chi+1} \geq \frac{2}{\chi_\ell}$, the result follows.

Corollary 7 If $\chi_\ell = 3$, then $\lambda_2 > \frac{5n}{9}$.

Our second proof technique will enable us to improve the lower bound in the preceding corollary.

Theorem 8 If $\chi_\ell(G) = 3$, then $\lambda_2(G) > \frac{\sqrt{5}-1}{2}\, n$.

PROOF. We are given a graph G with lists of two colors assigned to every vertex. We augment every list by adding a new color, say π. There is a 3-list coloring from the augmented lists. In this list coloring some of the vertices may be assigned the color π. These must form an independent set in the graph which we will call I_π. Let $\beta n = |I_\pi|$ and $H = G - I_\pi$. In the 3-list coloring of G the vertices of H have been colored with the original r colors.

The idea of the proof is to use k of the original r colors on vertices in H and $r-k$ of the original colors on vertices in I_π. By an argument similar to that presented in the proof of Theorem 1, it can be checked for any $1 \le k \le r$ that some choice of k colors from R assigned to H results in

$$\lambda_2 \;\ge\; \frac{k}{r}\cdot(n-\beta n) \;+\; \beta n\cdot\left(1-\frac{\binom{k}{2}}{\binom{r}{2}}\right) \quad .$$

From here, we resort to some algebra and calculus to achieve our result. Note that if $\beta \le \frac{1}{3}$ then by setting $k = r$ we get $\lambda_2 \ge (1-\beta)n \ge \frac{2n}{3}$. On the other hand, if $\beta \ge \frac{2}{3}$ setting $k = 0$ gives $\lambda_2 \ge \beta n \ge \frac{2n}{3}$. Thus we examine only the case where $\frac{1}{3} \le \beta \le \frac{2}{3}$. The calculations are straightforward and easily obtained with the help of a symbolic computer language. Simplifying and dividing by n the right hand side becomes

$$\Lambda(k,\beta,r) := \;\frac{k}{r}\cdot(1-\beta) \;+\; \beta\cdot\left(1-\frac{k(k-1)}{r(r-1)}\right) \quad .$$

We proceed by fixing β and maximizing $\Lambda(k,\beta,r)$ with respect to k. Recall k is the number of colors we choose to retain to color the vertices of H. As $\Lambda(k,\beta,r)$ is quadratic in k the maximum occurs where the $\frac{d\Lambda}{dk} = 0$. While the exact maximum occurs at $k = \frac{(r-\beta r+2\beta-1)}{2\beta}$, we will use the close, but simpler value $\hat{k} = \frac{(1-\beta)}{2\beta}\cdot r$. Thus

$$\Lambda(\hat{k},\beta,r) = \;\frac{(1-\beta)^2}{2\beta} \;+\; \beta\cdot\left(1-\frac{(1-\beta)}{4\beta^2}\cdot\left(\frac{r(1-\beta)}{(r-1)}-\frac{2\beta}{(r-1)}\right)\right) \quad .$$

Noting that $\left(\frac{r(1-\beta)}{(r-1)}-\frac{2\beta}{(r-1)}\right) \le (1-\beta)$ when $\beta \ge \frac{1}{3}$ we have

$$\Lambda(\hat{k},\beta,r) \ge \;\frac{(1-\beta)^2}{2\beta} \;+\; \beta\cdot\left(1-\frac{(1-\beta)}{4\beta^2}\cdot(1-\beta)\right) \;=\frac{(1-\beta)^2}{4\beta}+\beta.$$

Find the minimum value of this with respect to β. This will occur at $\beta = 1/\sqrt{5}$, and so

$$\frac{\lambda_2}{n} \ge \Lambda(\hat{k},\beta,r) \ge \frac{\sqrt{5}-1}{2} \;.$$

3 Open Questions

One might instead consider the size of the largest induced subgraph that is t-list colorable. The cube Q_3 shows that this parameter is not identical with λ_t.

While $K_{n,n}$ shows that this parameter does not yield an analogue to Theorem 1. It remains open whether there is an analogue to Theorem 2.

The rest of our open problems follow directly from Conjecture 1. Here we restrict ourselves to planar graphs where the results ought to be stronger. If G is planar we know that $\chi(G) \leq 4$ and $\chi_\ell(G) \leq 5$ (9). That planar graphs are acyclically 5-colorable (3) implies that $\lambda_2 \geq \frac{2n}{5}$. Our Theorem 1 improves the lower bound to $\frac{7n}{16}$. We believe the truth to be:

Conjecture 9 *If G is planar, then $\lambda_2 \geq \frac{n}{2}$.*

This would be implied by the induced forest conjecture (2). If G is bipartite we believe even more.

Conjecture 10 *If G is planar and bipartite, then $\lambda_2 \geq \frac{5n}{6}$.*

For planar graphs perhaps λ_4 is most provocative. Our Theorem 1 yields $\lambda_4(G) \geq \frac{175n}{256}$. The techniques used in the proof of Theorem 2 might improve this to about $\frac{7n}{10}$. Mirzakhani has the smallest example of a planar graph that is not 4-choosable (8). For this graph $n = 63$ and $\lambda_4 = 62$. This is quite a gap!

4 Postscript

Responding to an early draft of this paper, Chappell has established a Theorem 2 type lower bound for λ_t for all t and all s-choosable graphs (4).

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