Characterizations of Arboricity of Graphs

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Abstract

The aim of this paper is to give several characterizations for the following two classes of graphs: (i) graphs for which adding *any* l edges produces a graph which is decomposible into k spanning trees and (ii) graphs for which adding *some* l edges produces a graph which is decomposible into k spanning trees.

Introduction and Theorems

The concept of decomposing a graph into the minimum number of trees or forests dates back to Nash-Williams and Tutte [6, 7, 11]. Since then, many authors have examined various tree decompositions of classes of graphs (for example [2, 8]). The aim of this paper is to give several characterizations for the following two classes of graphs: (i) graphs for which adding *any* l edges produces a graph which is decomposible into k spanning trees and (ii) graphs for which adding *some* l edges produces a graph which is decomposible into k spanning trees. Graphs in this paper will include those with multiple edges but no loops. Let V_G and E_G be respectively, the number of vertices and edges in the graph G.

In [1], Albertson and Haas define a graph G to be *bounded* by the function $f(n)$ if $E_G = f(V_G)$ and each subgraph $H \subset G$ satisfies $E_H \leq f(V_H)$. That paper begins the study of which functions bound graphs, and which bounding functions correspond to properties of graphs. In [3], Catlin et al. characterize uniformly dense graphs by a bounding function. This paper characterizes graphs bounded by functions of the form $k(V_G - 1) - l$ for integers $k \geq l \geq 0$. There are many cases in which the condition that G is bounded by a function of the form $k(V_G-1)-l$ is necessary, sufficient or equivalent to the statement that G represents some sort of rigid structure (see for example [4, 10]).

In [4], Crapo gives a new condition equivalent to a graph being realizable as a generically rigid bar and joint framework in the plane. He defines a qTk decomposition of a graph to be the decomposition of the edges of G into q edge-disjoint trees such that each vertex is contained in exactly k trees. He proves that G is minimally rigid if and only if it has a $3T2$ decomposition such that for every subgraph of G the trees in the subgraph have distinct spans, which he calls a *proper* $3T2$. In [9], Tay uses this result to give a proof of Laman's theorem and mentions that similar results may be obtainable for other types of rigidity.

In 1961 Tutte and Nash-Williams independently gave a condition for when a graph could be decomposed into k forests. The *arboricity* of a graph G is defined to be the least number of disjoint forests whose union covers the edge set of G. Nash-Williams [7] showed this number to be

$$k = \max \left\lceil \frac{E_H}{(V_H - 1)} \right\rceil$$

where the maximum is taken over all subgraphs H on at least two vertices. We reword this condition and give two additional equivilant conditions.

Theorem 1 *The following are equivalent for a graph G, and integers $k > 0$ and $l > 0$.*

1. $E_G = k(V_G - 1) - l$, *and for subgraphs $H \subset G$ with at least 2 vertices* $E_H \leq k(V_H - 1)$.

2. *There exist some l edges which when added to G result in a graph that can be decomposed into k spanning trees.*

3. *G admits a $(k+l)Tk$ decomposition.*

We next give a constructive method to build graphs of this type. The graph consisting of 2 vertices and $(k-l)$ parallel edges is the only graph on 2 vertices that meets conditions 1-3. Call this graph K_2^{k-l}. If $\hat{G}$ satisfies 1-3 and G is created by adding a vertex to $\hat{G}$ then G must also have k additional edges. The proper method of adding a vertex and the required edges follows.

Operation $\mathcal{O}$: Remove any $0 \leq i < k$ edges from $\hat{G}$. Add a new vertex v which will have degree $k+i$. Add $2i$ new edges, joining v to each end of each deleted edges. Note that if two or more removed edges are incident to vertex u, then the previous step will create multiple copies of the edge uv. Add edges from v to some additional $k - i$ vertices of $\hat{G}$ such that no edge has multiplicity greater than k. For convenience we label the following property for a graph G.

4. *G can be constructed by repeated application of the operation $\mathcal{O}$, starting with K_2^{k-l}.*

Theorem 2 *A graph satifying 4. will satisfy properties 1-3.*
Further, if G is a graph satisfying 1-3 and $l \leq 2$, $k > l$ then G also satisfies 4.

For $l > 2$ there are graphs which satisfy 1-3 and cannot be constructed by operation $\mathcal{O}$. An example with $k = 4$ and $l = 3$ is the graph on 4 vertices, say $\{a, b, c, d\}$, with 3 copies each of(a, b),(a, c) and (a, d). A graph constructed by repeated application of $\mathcal{O}$ will necessarily contain a vertex of degree at least k and not more than $2k$. If G satisfies 1-3 and for all vertices v, either $\deg(v) < k$ or $\deg(v) \geq 2k$ then it can be shown that $V_G \leq kl - 2k$ (see lemma 6). Thus if $l \leq 2$ there will be a vertex v with $k \leq \deg(v) \leq 2k$.

The next theorems characterizes graphs for which adding any l edges results in the decomposition into k spanning trees. These theorems are restricted to the case that $0 \leq l < k$ to allow the case of adding l edges by simply increasing the multiplicity of an existing edge by l.

Given a tree τ in G and a subgraph H of G, define a *subtree* of τ in H to be a connected component of $\tau \cap H$. Thus we speak of a set of subtrees of τ in H. Let $\mathcal{T}$ be a qTk decomposition consisting of trees $\tau_1, \ldots, \tau_q$. The *set of subtrees of $\mathcal{T}$ in subgraph H* is the union over all i of the sets of subtrees of τ_i in H.

Theorem 3 *The following are equivalent for a graph G, and integers $0 \leq l < k$.*

$1'$. *$E_G = k(V_G - 1) - l$ and for any subgraph H of G with $V_H \geq 2$, $E_H \leq k(V_H - 1) - l$.*

$2'$. *Adding any l edges to G (including multiple edges) results in a graph that can be decomposed into k spanning trees.*

$3'$. *G can be decomposed into a $(k + l)Tk$, $\mathcal{T}$, such that for every subgraph H the set of subtrees of $\mathcal{T}$ in H has cardinality at least $k + l$.*

A version of the construction rule of theorem 2 also applies to this case. Again, the smallest example of a graph satisfying $1'$-$3'$ is K_2^{k-l}. However, for $k > 2l$, there are no graphs satisfying $1'$-$3'$ on 3 vertices. Thus the basis for the construction would need to be the set of graphs which satisfy $1'$-$3'$ on the minimum number of vertices $n > 2$, for which there are members of this class of graphs. The correct operation is very similar to the previous case.
Operation $\mathcal{O}'$: Remove any $0 \leq i < k$ edges from $\hat{G}$. Add a new vertex v which will have degree $k + i$. Add $2i$ new edges, joining v to each end of each deleted edges. Note that if two or more removed edges are incident to the same vertex u, then the previous step will create multiple copies of the edge uv. Add edges from v to some additional $k - i$ vertices of $\hat{G}$ such that no edge has multiplicity greater than $(k - l)$.

Theorem 4 *If $k \leq 2l$ then the following is equivilant to statements $1'$-$3'$.*

$4'$. *G can be constructed by repeated application of the operation $\mathcal{O}'$ to K_2^{k-l}.*

Proofs

To prove these theorems the following lemma will be used repeatedly.

Lemma 5 *Let H be any subgraph of G. If $\mathcal{T}$ is a qTk decomposition of G, then the number of subtrees in the set of subtrees of $\mathcal{T}$ in H is precisely $kV_H - E_H$.*

This lemma implies that the number of subtrees of any qTk in a subgraph H will be the same. Consequently, if one qTk satisfies the subgraph condition of $3'$ then every qTk will.

Proof of Lemma 5: Let R_H be the set of subtrees of $\mathcal{T}$ in H. Note that some of the subtrees in R_H may come from the same tree of $\mathcal{T}$. We count the vertex tree incidences in H in two ways. Let t_i be the number of edges in the ith subtree of R_H. Since every vertex is in k trees of a qTk and thus k subtrees of R_H we have $kV_H = \sum(t_i + 1) = E_H + |R_H|$.

Proof of Theorem 1: That 1 and 2 are equivilant follows immediately from the arboricity results of Nash-Williams [7].

$2 \Rightarrow 3$. Suppose after adding l edges to G we get the k spanning trees $T_1, \ldots T_k$. Removal of the l edges will break up some of the trees into spanning forests. Since removing i edges from a tree leaves $i+1$ trees, after removing the l edges we will have $k+l$ trees, some of which may be single vertex trees. Since the trees come from k spanning forests, each vertex will be incident to exactly k of the $k+l$ trees.

$3 \Rightarrow 1$. By lemma 5, $E_G = kV_G - |R_G|$ and by 3 we have $|R_G| = k+l$. Thus $E_G = k(V_G - 1) - l$. If H is a subgraph on at least 2 vertices, then since every vertex is incident to exactly k trees, $|R_H| \geq k$. Using lemma 5 once more gives $E_H = kV_H - |R_H| \leq k(V_H - 1)$ as desired.

Proof of Theorem 3 We show $1'$ implies $2'$ implies $3'$ implies $1'$.

$1' \Rightarrow 2'$. Add l edges to G to create G'. $E_{G'} = k(V_{G'} - 1)$ and for any subgraph H' of G', $E_{H'} \leq E_H + l \leq k(V_H - 1) - l + l = k(V_{H'} - 1)$. Thus by [7] G' can be decomposed into k disjoint spanning trees (see also [1]).

$2' \Rightarrow 3'$. By theorem 1, it remains to show the proper conditions on the subgraph hold. Consider any subgraph H. If we add all l edges within that subgraph then the resulting $(k+l)Tk$ will have at least $(k+l)$ subtrees in that subgraph. Thus by lemma 5, every $(k+l)Tk$ for G will have at least $k+l$ trees in that (and in every) subgraph.

$3' \Rightarrow 1'$. Again, by theorem 1 it remains only to show that the correct conditions on the subgraphs hold. If H is a subgraph on at least 2 vertices, then by $3'$ $|R_H| \geq k+l$. Using lemma 5 gives $E_H = kV_H - |R_H| \leq k(V_H - 1) - l$ as desired.

Construction Theorems

$4 \Rightarrow 1$ and $4' \Rightarrow 1'$. We first show by induction that a graph on n vertices that was constructed by rule $\mathcal{O}$ (resp.$\mathcal{O}'$) will satisfy the conditions of 1 (resp. $1'$). The graph on 2 vertices with $k-l$ edges does trivially. Now suppose G has n vertices and was constructed by rule $\mathcal{O}$ (resp.$\mathcal{O}'$). Let v be the last vertex added, and assume that $\hat{G}$, the graph to which v is added, satisfies 1 ($1'$). Clearly every subgraph of G that does not contain v still satisfies the subgraph condition of 1 ($1'$).

Suppose H is a subgraph of G that contains v and at least 2 other vertices. Define $\hat{H}$ to be the induced subgraph of $\hat{G}$ on the set of vertices of $H - v$.

Let $q \leq k$ be the number of edges removed from $\hat{G}$ in the construction of G with both endpoints in $\hat{H}$, and r be the number of edges removed from $\hat{G}$ in the construction of G with exactly one endpoint in $\hat{H}$ and s be the number of edges removed from $\hat{G}$ in the construction of G with no endpoint in $\hat{H}$. Now, the number of edges $E_H \leq E_{\hat{H}} - q + 2q + r + (k - q - r - s) \leq E_{\hat{H}} + k$. Which for 1 gives $E_H \leq k(V_{\hat{H}} - 1) + k = k(V_H - 1)$ and for $1'$ gives $E_H \leq k(V_{\hat{H}} - 1) - l + k = k(V_H - 1) - l$.

Finally, if H is a subgraph of G on exactly 2 vertices, v and one other, then by constructon $\mathcal{O}$, $E_H \leq k$ and by $\mathcal{O}'$, $E_H \leq k - l$.

Lemma 6 *If G satisfies 1-3 and all its vertices are of degree either less than k or greater than $2k$ then $l > 2$.*

Proof of Lemma 6 Suppose there are V_1 vertices of degree less than or equal $k-1$ (vertices of low degree) and V_2 vertices of degree at least $2k$ (vertices of high degree). Let E_1 denote the number of edges between vertices of low degree E_2 denote the number of edges between vertices of high degree and E_3 denote the number of edges between a vertex of high degree and a vertex of low degree. The total number of edges is thus

$$E_1 + E_2 + E_3 = k(V_1 + V_2 - 1) - l. \tag{1}$$

That the low degree vertices all have degree $\leq k - 1$ gives

$$2E_1 + E_3 \leq (k-1)V_1. \tag{2}$$

That no subgraph has more than $k(V_H - 1)$ edges gives

$$E_2 \leq k(V_2 - 1). \tag{3}$$

Writing the equations (1) and (2) in terms of E_3 and then substituting in (3) we get

$$-V_1 - E_1 \geq k(V_2 - 1) - l - E_2 \geq k(V_2 - 1) - l - k(V_2 - 1).$$

Which gives us that $l \geq V_1 + E_1$. Similarly, that the high degree vertices all have degree $> 2k$ gives

$$2E_2 + E_3 \geq (2k+1)V_2.$$

This can be combined in a similar manner with equations (1) and then (3) to get

$$V_2 \leq kV_1 - 2k - l - E_1.$$

Thus the total number of vertices $V_1 + V_2 \leq kl - 2k$. Hence, such a graph exists only for $l > 2$.

Proof of Theorem 2 1, 3 $\Rightarrow$ 4. We show by induction that a graph on n vertices with properties 1 and 3 can be constructed by rule $\mathcal{O}$.

The smallest graph with a $(k+l)Tk$ has two vertices and $k-l$ parallel edges. This graph can be decomposed into $k-l$ single edge trees and $2l$ single vertex trees, l of each of the two vertices. If G is a graph on $n>2$ vertices with property 1, then the average degree is $= \frac{2k(n-1)-l}{n} < 2k$. By lemma 6, there exists a vertex v with $k \leq \deg(v) < 2k$.

We first show there is a $(k+l)Tk$ of G such that the vertex v does not occur as a single vertex tree. We have assumed only that there is some $(k+l)Tk$ say $\mathcal{T}$. Suppose that v occurs q times as single vertex tree $\mathcal{T}$. Since $\deg(v) \geq k > k-q$ this forces at least one tree say $T_i \in \mathcal{T}$ to have more than one edge incident to v. Create a new family of trees for G, say $\mathcal{T}'$ by splitting T_i into two trees at v and deleting one occurrence of v as a single vertex tree. $\mathcal{T}'$ has $(k+l)$ trees and every vertex that was contained in T_i before, will be contained in exactly one of the two trees created from T_i. This process can be repeated to obtain a $(k+l)Tk$ for G in which v does not occur as a single vertex tree.

We now use this $(k+l)TK$ to construct a graph $\hat{G}$ on $n-1$ vertices with properties 1 and 3 such that the addition of a new vertex v by the operation $\mathcal{O}'$ gives the graph G. Say $\deg(v) = k+i$, $0 \leq i < k$ and the trees occurring at v are $\{T_1, \ldots, T_k\}$. We delete v and preserve the span of the trees $\{T_1, \ldots, T_k\}$ as follows. For each tree T_j consider the set of vertices $\{v_{j1}, \ldots, v_{ji_j}\}$ adjacent to v in T_j. Add $i_j - 1$ edges that form a spanning tree of $\{v_{j1}, \ldots, v_{ji_j}\}$. The span of each of the $k+l$ trees is thus preserved. I.e., each vertex remains in exactly k of the $k+l$ trees. Note that in general there will be many choices as to how to do this. Any choice will give an appropriate $\hat{G}$.

Proof that 1′ implies 4′ If G is a graph on $n>2$ vertices with property 1′, then the average degree is $= \frac{2k(n-1)-l}{n} < 2k$. Further, if a vertex, v, of G had degree $< k$ then $G-v$, the subgraph obtained from G after deletion of v, has $n-1$ vertices and $E_{G-v} \geq k(n-2)-l$ which contradicts property 1′. Thus there exists a vertex v such that $k \leq \deg(v) < 2k$.

We now show there exists a graph $\hat{G}$ on $n-1$ vertices that satisfies 1′ and which becomes G when the vertex v is added by the operation. Assume that $\deg(v) = k+r$ and that the neighbors of v are $u_1, \ldots u_{r+k}$. We need to show that r edges of the form u_iu_j can be added to $G-v$ such that each subgraph H still satisfies $E_H \leq k(V_H - 1) - l$. Call the edges u_iu_j the *potential* edges. The potential edge, $u_i u_j$, cannot be added if and only if there exists a subgraph $H \subset G - v$ containing the vertices u_i and u_j such that $E_H = k(V_H - 1) - l$. We refer to this as the subgraph condition.

Lemma 7 *Let G be a graph satisfying $1'$. If H_1, H_2 are subgraphs of G and $E_{H_i} = k(V_{H_i} - 1) - l$ then $E_{H_1 \cap H_2} = k(V_{H_1 \cap H_2} - 1) - l$.*

Proof of Lemma 7 Calculate the number of edges in the union:

$$E_{H_1 \cup H_2} = k(V_{H_1 \cup H_2} + V_{H_1 \cap H_2} - 2) - 2l - E_{H_1 \cap H_2}.$$

Since $H_1 \cup H_2$ is a subgraph of G,

$$E_{H_1 \cup H_2} \leq k(V_{H_1 \cup H_2} - 1) - l.$$

Combining these gives $E_{H_1 \cap H_2} \geq k(V_{H_1 \cap H_2} - 1) - l$ and since $H_1 \cap H_2$ is a subgraph of G equality must hold.

Proof that $1'$ implies $4'$ continued. There are $c = \binom{r+k}{2}$ potential edges and we must add r of these. Suppose only $0 \leq s < r$ edges can be added without violating the subgraph condition. Add these s edges to G to obtain G'. By lemma 7 if an edge cannot be added then there is a unique smallest subgraph which prevents it and any subgraph which prevents it will contain that smallest subgraph. Let $H_i \subset G' - v$ be the smallest subgraph which prevents the edge i $(i = 1, \ldots c)$ from being added, perhaps because it was already added. Consider the union of all of these subgraphs.

$$\begin{aligned} E_{\cup H_i} &= \sum E_{H_i} - \sum (E_{H_i} \cap E_{H_j}) + \cdots + (-1)^c \left(\bigcap_{i=1}^c E_{H_i}\right) \\ &= \sum \left(k(V_{H_i} - 1) - l\right) - \sum \left(k(V_{H_i \cap H_j} - 1) - l\right) + \cdots + (-1)^c \left(k\left(V_{\cap_{i=1}^c H_i} - 1\right) - l\right) \\ &= k\left(V_{\cup_{i=1}^c H_i}\right) + (k + l) \sum_{j=1}^c (-1)^j \binom{c}{j} \\ &= k\left(V_{\cup_{i=1}^c H_i}\right) + (k + l)(-1) \end{aligned}$$

However, $\cup_{i=1}^c H_i \cup \{v\}$ is a subgraph of G with s additional edges added to it. Thus $E_{\cup H_i} - s + (k + r) \leq k\left(V_{\cup_{i=1}^c H_i} + 1 - 1\right) - l$. Which is a contradiction, unless $r = s$. Thus r edges can be added. Call this set of edges F.

The proof is completed by observing that $G - \{v\} \cup F$ is a graph on $n - 1$ vertices which satisfies $1'$ and which becomes G when the vertex v is added and edges F removed following the operation $\mathcal{O}'$.

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