Toward Unfolding Doubly Covered n-Stars

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Abstract. We present nonoverlapping general unfoldings of two infinite families of nonconvex polyhedra, or more specifically, zero-volume polyhedra formed by double-covering an n-pointed star polygon whose triangular points have base angle α. Specifically, we construct general unfoldings when $n \in \{3, 4, 5, 6, 8, 9, 10, 12\}$ (no matter the value of α), and we construct general unfoldings when $\alpha < 60°(1 + 1/n)$ (i.e., when the points are shorter than equilateral, no matter the value of n, or slightly larger than equilateral, especially when n is small). It remains open whether all doubly covered star polygons, or more broadly arbitrary nonconvex polyhedra, have general unfoldings.

Keywords: polyhedra · nonconvex · nets · nonoverlapping

1 Introduction

Unfolding a polyhedron P refers to the process of cutting and flattening its surface into a connected planar piece without overlap [13, Part III]. Finding unfoldings is a classical problem with applications ranging from origami robots to sheet-metal manufacturing. Assuming P is genus 0, the cuts on the surface of P must form a forest to ensure a connected unfolding. To enable flattening into the

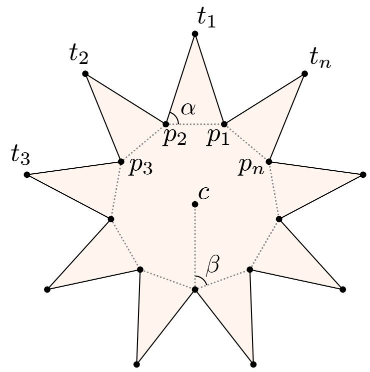

Fig. 1. An n-star polygon with base angle α, convex vertices (point tips) $t_1, t_2, \ldots, t_n$, and reflex vertices $p_1, p_2, \ldots, p_n$.

plane, the cuts must span all (non-zero curvature) vertices of P, with at least one cut at each positive curvature vertex and two at each negative curvature vertex [13, Sec. 22.1.3]. A polyhedron admits many different unfoldings, depending on the choice of cuts.

Edge unfoldings restrict cuts to lie along the polyhedron's edges. A famous open problem (dating back to 1975 or even 1525) is whether every convex polyhedron has an edge unfolding. Recent progress solves this problem for "nearly flat convex caps" [19]. On the other hand, there are several examples of nonconvex polyhedra without edge unfoldings [4,3,14].

We focus in this paper on *general unfoldings*, which allow cuts anywhere on the polyhedron's surface. All convex polyhedra have general unfoldings by a variety of methods [2,20,16,12]. Thus we focus on general unfoldings of nonconvex polyhedra. There are nonconvex polyhedra with boundary with no general unfoldings, but it remains open whether all nonconvex polyhedra without boundary have a general unfolding [3].

The main progress on this problem has been for *orthogonal* polyhedra, whose edges and faces meet at right angles. All orthogonal polyhedra of genus ≤ 2 have general unfoldings [10,6,5,7]. For orthogonal polyhedra, we can quantify the simplicity of an unfolding by how close the cuts stick to the natural *grid* of the polyhedron, defined by extending planes through every face and taking all intersections with orthogonal faces. A *grid unfolding* sticks to cuts along this grid, while an $(a \times b)$-*grid unfolding* allows cuts on a *refined grid* defined by subdividing each grid face into an $a \times b$ subgrid for positive integers a, b. Grid unfoldings are known for *orthotubes* [4], well-separated *orthotrees* [9], and one-layer block structures [17], but it remains open whether they exist for all orthogonal polyhedra; see the survey [18]. The original method for unfolding all orthogonal polyhedra of genus 0 [10] uses exponential refinement. The level of refinement was later reduced to quadratic [6] and then linear [5], and finally generalized to genus-2 polyhedra (with linear refinement) [7]. Grid unfoldings that use sublin-

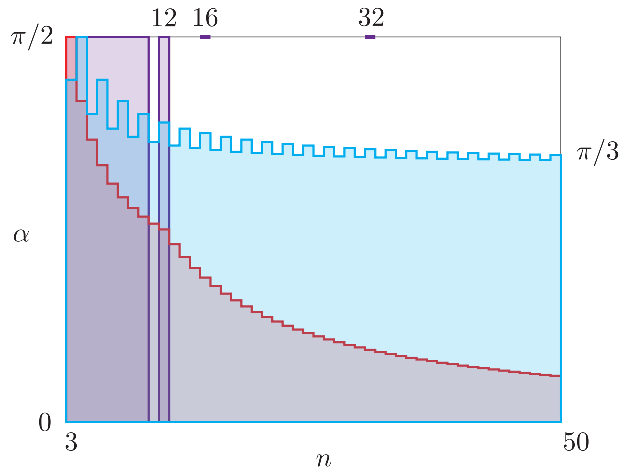

Fig. 2. Summary of (n, α)-stars which this paper shows how to unfold without overlap. Red is the naive unfolding, blue is the crown unfolding, purple shows comprehensive unfoldings.

ear refinement have been developed only for specialized orthogonal shape classes. For example, there exist (1×2)-grid unfoldings of orthostacks [4], (4×5)-grid unfoldings of Manhattan Towers [11], (2×1)-grid unfoldings of well-separated orthographs [15], and (4×4)-grid unfoldings of low-degree orthotrees [8].

Our results. Apart from orthogonal polyhedra, we are not aware of any study of general unfoldings of infinite classes of nonconvex polyhedra. In this paper, we consider the seemingly simple class of *doubly covered polygons* formed by joining two copies of a simple polygon "back to back" by gluing corresponding edges, thereby forming a zero-volume genus-0 polyhedron. Specifically, we study *doubly covered* (n, α)*-stars*: for positive integer n and positive angle $\alpha < \pi/2$, an (n, α)-star is a simple polygon having $2n$ vertices $\{p_1, t_1, p_2, t_2, \ldots, p_n, t_n\}$ where points $p_1, p_2, \ldots, p_n$ form a regular n-gon, all points t_i lie outside of it, and for all $1 \leq i \leq n$ triangle $\triangle p_i t_i p_{i+1}$ is isosceles with $|p_i t_i| = |t_i p_{i+1}|$ and base angle $\angle p_i p_{i+1} t_i = \alpha$; see Fig. 1. (Throughout, we assume all vertex indices are computed modulo n.) We also let $\beta = \frac{\pi}{2}\left(1 - \frac{2}{n}\right)$ be half the angle of a regular n-gon. This polyhedron has exactly two faces, and does not admit an edge unfolding.

In this paper, we explore the space of doubly covered stars in search of families of general unfoldings, and to search for counterexamples of polyhedra that do not admit a general unfolding. We show that general unfoldings of doubly covered (n, α)-stars exist:

- for any n when the base angle $\alpha < 60°(1 + 1/n)$, and
- for any base angle $\alpha \in (0, \pi/2)$ when $n \in \{3, 4, 5, 6, 7, 8, 9, 10, 12\}$.

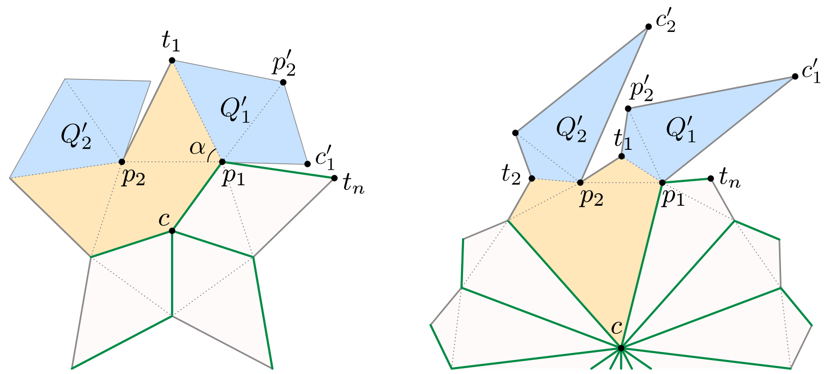

Fig. 3. Naive unfolding. The cuts and raw edges are shown in green; top layer pieces are shaded darker (blue) and bottom are lighter (yellow). Left: Two unfolded spikes for a star with $\alpha \geq \pi/4$. Right: two unfolded spikes for a star with $\alpha < \pi/4$.

These results are summarized in the plot in Fig. 2. We prove existence by construction, providing families of general unfoldings within specific subdomains of n and α.

2 Naive unfolding

When the base angle α of a doubly covered (n, α)-star is small, we can produce a *naive unfolding* that is non-overlapping. For each i, we cut the top layer of the star along segments cp_i (center to spike base) and $p_i t_{i-1}$ (left side of spike). This cuts the top layer of the star into n quadrilaterals $Q_1, Q_2, \ldots, Q_n$, where $Q_i = \diamond p_i c p_{i+1} t_i$. The star can be unfolded along the right side $p_i t_i$ of each spike: each quadrilateral Q_i is reflected along the line $p_i t_i$. See Fig. 3 (left). We now prove:

Lemma 1. *The naive unfolding of an (n, α)-star is non-overlapping when*

$$\alpha \leq \begin{cases} \frac{\pi}{6}\left(1 + \frac{6}{n}\right), & \textit{for } n \leq 12\,, \\ \frac{3\pi}{n}, & \textit{otherwise.} \end{cases}$$

Proof. We compute the values of n and α for which the naive unfolding is non-overlapping. Note that, due to symmetry, it is enough to show that one quadrilateral, say Q_1, unfolds without overlapping. Let c'_i and p'_{i+1} be the reflections of the points c and p_{i+1} across the line $p_i t_i$ respectively. The image of Q_i after unfolding is a quadrilateral $Q'_i = \diamond p_i c'_i p'_{i+1} t_i$. For the unfolding to be non-overlapping it is necessary that the total sum of the angles around point p_1 is not greater than 2π, that is $3\alpha + 3\beta \leq 2\pi$, which reduces to:

$$\alpha \leq \frac{\pi}{6}\left(1 + \frac{6}{n}\right)\,.$$

Assume that the above inequality holds, and note that Q'_1 lies on one side of the line p_1t_n. Now consider two cases, when the angle $\angle p_2t_1p'_2$ is non-reflex, and when it is reflex. In the first case, when $\angle p_2t_1p'_2 \leq \pi$, or equivalently $\alpha \geq \frac{\pi}{4}$, quadrilateral Q'_1 lies on one side of the line p_2t_1 (refer to Fig. 3). Therefore none of the images of the quadrilaterals Q_i overlap with each other, as they all lie inside the non-overlapping cones defined by the lines p_it_{i-1} and $p_{i+1}t_i$. That is, the unfolding is flat when

$$\frac{\pi}{4} \leq \alpha \leq \frac{\pi}{6}\left(1+\frac{6}{n}\right) .$$

This pair of inequalities has a solution when $0 < n \leq 12$.

In the second case, when $\angle p_2t_1p'_2 > \pi$, or equivalently $\alpha < \frac{\pi}{4}$, quadrilateral Q'_1 may overlap with Q'_2, the image of the quadrilateral Q_2, if the exterior angle $\angle t_1p_2c'_2$ is too small. Specifically, Q'_1 and Q'_2 will overlap if and only if p'_2 and t_1 lie on opposite sides of $p_2c'_2$, which happens when $\angle t_1p_2p'_2 = \frac{\pi-4\alpha}{2} > t_1p_2c'_2$. Thus, for the unfolding to be non-overlapping the total sum of the interior angles around p_2 has to be at most $2\pi - \frac{\pi-4\alpha}{2}$, which reduces to

$$\alpha \leq \frac{3\pi}{n} .$$

Putting the two cases together proves the lemma. □

3 Crown unfolding

For doubly-covered (n, α)-stars with larger base angles, we can produce a non-overlapping *crown unfolding, so named for its four crown-like pieces shown in Fig. 4 (right).* To produce this unfolding, first make one cut across the top layer of the star along segment p_1p_{m+1}, and two cuts across the bottom layer along p_2p_{m+1} and p_1p_{m+2}, where $m = \lceil n/2 \rceil$. Furthermore, cut along both edges of every spike, except for edges p_1t_1, p_1t_{2m}, $p_{m+1}t_m$, and $p_{m+1}t_{m+1}$; see Fig. 4 and Fig. 5 for even and odd n respectively. The top layer is cut into two crown-shaped pieces, while the bottom layer is cut into three pieces: a hexagon and two more crown-shaped pieces. The star can then be unfolded along the four preserved spike edges to form its crown unfolding. We now prove:

Theorem 1. *The crown unfolding of an (n, α)-star is non-overlapping when*

$$\alpha \leq \begin{cases} \frac{\pi}{3}\left(1+\frac{2}{n}\right), & \textit{for even } n\,, \\ \frac{\pi}{3}\left(1+\frac{1}{n}\right), & \textit{for odd } n\,. \end{cases}$$

Proof. We compute the values of n and α for which the crown unfolding is non-overlapping. Let T_1 and T_2 denote the two crown-shaped pieces from the top layer, where $T_1 = p_1t_1 \ldots t_mp_{m+1}$ and $T_2 = p_{m+1}t_{m+1} \ldots t_{2m}p_1$; let B_1 and B_2 denote the two crown-shaped pieces in the bottom layer, where $B_1 =$

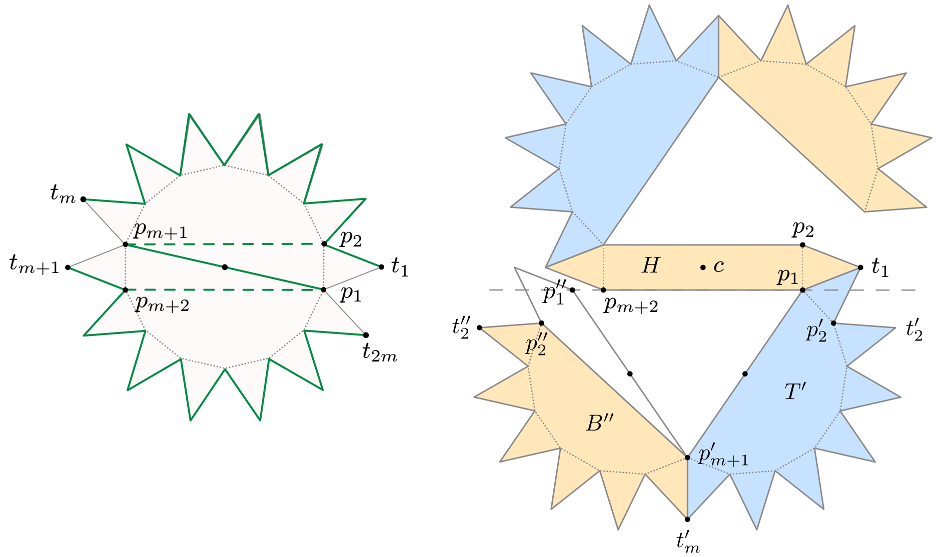

Fig. 4. Left: A doubly-covered $(2m, \alpha)$-star. Top layer cuts are shown as solid green lines and bottom layer cuts are dashed. Right: The resulting unfolding. Top layer pieces are shaded darker (blue) and bottom are lighter (yellow). The white quadrilateral is not a piece but is used to simplify the proof.

$p_2 t_2 \dots t_m p_{m+1}$ and $B_2 = p_{m+2} t_{m+2} \dots t_{2m} p_1$; and let $H = p_1 t_1 p_2 p_{m+1} t_{m+1} p_{m+2}$ be the hexagonal bottom piece. We will consider the images of these pieces after unfolding, and will prove bounds on α for when they do not overlap.

Denote the reflections of the points p_i and t_i across edge $p_1 t_1$ as p_i' and t_i' respectively, for all $1 \leq i \leq m+1$; and let $T' = p_1' t_1' \dots t_m' p_{m+1}'$ denote the image of the top layer piece T_1 after unfolding across edge $p_1 t_1$. Then, let p_i'' and t_i'' be the reflections of the points p_i' and t_i' across $t_m' p_{m+1}'$, for all $1 \leq i \leq m+1$. The bottom layer piece B_1 (that was attached to the uncut spike edge $t_m p_{m+1}$) then unfolds into the polygon $B'' = p_2'' t_2'' \dots t_m'' p_{m+1}''$.

Consider the images T' of T_1 and B'' of B_1, which unfold on the same side relative to the hexagonal bottom piece H. Polygons H and T' do not overlap as they share edge $p_1 t_1$ which lies on the convex hulls of both polygons. Similarly, polygons T' and B'' do not overlap because they share edge $t_m' p_{m+1}'$. It is not hard to see that the crown unfolding does not intersect for sufficiently small α (say, $\alpha < \frac{\pi}{4}$) as T' and B'' exist on one side of the extension of $p_1 t_1$, with the rest of the unfolding on the other side. Thus we focus our attention on the upper bound, for $\alpha \geq \frac{\pi}{4}$.

First we consider the case when n is even and prove that the crown unfolding is non-overlapping when $\alpha \leq \alpha_{even}^*$ where $\alpha_{even}^* = \frac{\pi}{3}\left(1 + \frac{2}{n}\right)$; see Fig. 4. Observe that the reflection of T' across $t_m' p_{m+1}'$ contains B'' and point p_1''. Then, when p_1'' is in the half-plane Π bounded by p_1 and p_{m+2} not containing p_2, then T' and B'' will both exist in the half-plane Π' bounded by t_1 and t_{m+1} and will

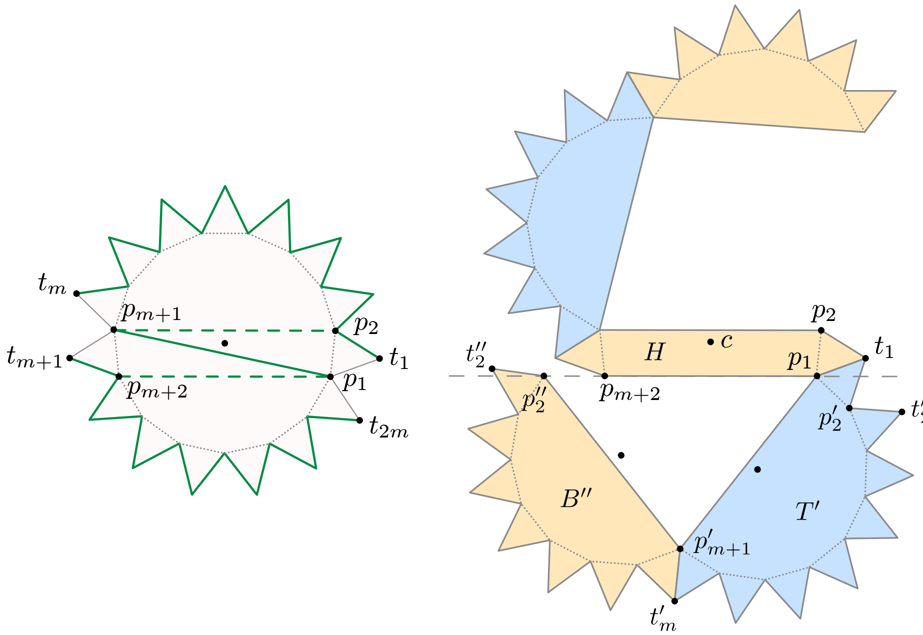

Fig. 5. Left: A doubly-covered $(2m+1, \alpha)$-star. Top layer cuts are shown as solid green lines and bottom layer cuts are dashed. Right: The resulting unfolding. Top layer pieces are shaded darker (blue) and bottom are lighter (yellow).

not overlap with the rest of the unfolding. Define angles:

$$\theta_{even} = \angle p_{m+2}p_1p'_{m+1} = \frac{3\pi}{2} - 2\alpha - \beta \text{ and } \phi_{even} = \angle p_1p'_{m+1}p''_1 = 2\pi - 2\alpha - 2\beta.$$

Observe that p''_1 is in Π when $\phi_{even} + 2\theta_{even} \geq \pi$, which holds exactly when $\alpha \leq \alpha^*_{even}$; so T' and B'' both stay in P' when $\alpha \leq \alpha^*_{even}$. Further, define distances $d_1 = |p_1p_{m+2}|$, $d_2 = |p_1p'_{m+1}|$, and $d_3 = |p_1p''_1|$. Clearly $d_1 < d_2$ since d_2 is a diameter of the n-gon and d_1 is a shorter diagonal. And when $\alpha \leq \alpha^*_{even}$, $d_2 < d_3$ since $\phi_{even} = \pi + \frac{2\pi}{n} - 2\alpha \geq \frac{\pi}{3} + \frac{2\pi}{3n} > \frac{\pi}{3}$. So $d_1 < d_3$, and B'' also does not intersect H.

By symmetry, the same argument also shows that the images of T_2 and B_2 do not overlap with H after unfolding, and lie on the other side of the line through t_1 and t_{m+1}. Thus, the crown unfolding of a doubly-covered (n, α)-star is non-overlapping when n is even and $\alpha \leq \alpha^*_{even}$.

Now we consider the case when n is odd and prove that the crown unfolding is non-overlapping when $\alpha \leq \alpha^*_{odd}$ where $\alpha^*_{odd} = \frac{\pi}{3}\left(1 + \frac{1}{n}\right)$; see Fig. 5. First we observe that, when α increases, the unfolded images T' and B'' of T_1 and B_1 approach H faster than the unfolded images of T_2 and B_2, so we focus our attention on T' and B''. We first observe that distance $|p_1p'_{m+1}|$ is the same as $|p'_{m+1}p''_2|$. Then, when p''_2 is in the half-plane Π bounded by p_1 and p_{m+2} not containing p_2, then T' and B'' will both exist in the half-plane Π' bounded by t_1 and t_{m+1} and will not overlap with the rest of the unfolding. Define angles:

$$\theta_{odd} = \angle p_{m+2}p_1p'_{m+1} = \frac{3\pi}{2} - 2\alpha - \beta - \frac{\pi}{n} \text{ and } \phi_{odd} = \angle p_1p'_{m+1}p''_1 = 2\pi - 2\alpha - 2\beta.$$

Observe that p''_2 is in P when $\phi_{odd} + 2\theta_{odd} \geq \pi$, which holds exactly when $\alpha \leq \alpha^*_{odd}$; so T' and B'' both stay in P' when $\alpha \leq \alpha^*_{odd}$. Further, define distances $d_1 = |p_1p_{m+2}|$, $d_2 = |p_1p'_{m+1}|$, and $d_3 = |p_1p''_2|$. Clearly $d_1 < d_2$ since d_2 is a longer diagonal of the n-gon than d_1. And when $\alpha \leq \alpha^*_{odd}$, $d_2 < d_3$ since $\phi_{odd} = \pi + \frac{2\pi}{n} - 2\alpha \geq \frac{\pi}{3} + \frac{4\pi}{3n} > \frac{\pi}{3}$. So $d_1 < d_3$, and B'' does not intersect H. Thus, the crown unfolding of a doubly-covered (n, α)-star is also non-overlapping when n is odd and $\alpha \leq \alpha^*_{odd}$, completing the proof. □

4 Comprehensive unfoldings

In this section, we provide unfoldings of (n, α)-stars for arbitrary α for some small choices of n. We call such unfoldings *comprehensive unfoldings.*

Theorem 2. *There exist non-overlapping unfoldings of (n, α)-stars for any positive $\alpha < \pi/2$ and any $n \in \{3, 4, 5, 6, 7, 8, 9, 10, 12\}$.*

Proof (Case $n = 3$).
The naive unfolding suffices, since $\alpha \leq \frac{\pi}{6}\left(1 + \frac{6}{3}\right) = \frac{\pi}{2}$. □

Proof (Case $n = 4$).
The crown unfolding suffices, since $\alpha \leq \frac{\pi}{3}\left(1 + \frac{2}{4}\right) = \frac{\pi}{2}$. □

Proof (Case $n = 5$). The naive unfolding suffices for small $\alpha \leq \frac{\pi}{3} < \frac{\pi}{6}\left(1 + \frac{6}{5}\right)$. It fails for larger α because the unfolded image of a top layer's spike intersects one of the bottom layer's spikes. Here we adopt an approach similar to the naive unfolding, but resolve the overlap by moving spike material from one side to the other; see Fig. 6.

To produce a comprehensive unfolding of a $(5, \alpha)$-star, first cut the top layer of the star along segments cp_i for all i; and also along the right side of the spike from t_i to q_i and along the bottom face to p_{i+1}, where q_i is the point on p_it_i such that $\angle p_ip_{i+1}q_i = \frac{\pi}{10}$. The bottom layer is cut into a central 10-gon B and five triangles T_i where $B = p_1q_1p_2 \dots p_nq_n$ and $T_i = \triangle t_ip_{i+1}q_i$; while the top layer is cut into five quadrilaterals $Q_i = \diamond p_icp_{i+1}t_i$. Note that Q_i remains attached to B alongside p_iq_i, and T_i remains attached to Q_i alongside t_ip_{i+1}.

Due to symmetry, it is enough to show that one spike unfolds without overlap. Let c'_i and p'_{i+1} be the reflections of c and p_{i+1} across line p_it_i respectively, and let $Q'_i = \diamond p_ic'_ip'_{i+1}t_i$ be the reflection of Q_i across line p_it_i. Further, let q'_i be the reflection of q_i across line $p'_{i+1}t_i$, and let $T'_i = t_ip'_{i+1}q'_i$. It suffices to show that the total sum of angles around point p_i is not greater than 2π and that $p_ic'_i$ does not intersect $q_{i-1}t_{i-1}$ for any $\frac{\pi}{3} \leq \alpha \leq \frac{\pi}{2}$. The first condition is satisfied by

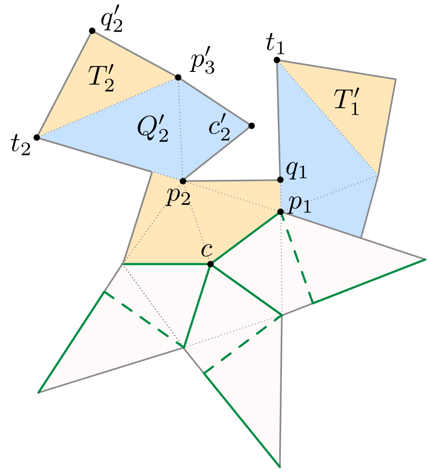

Fig. 6. General unfolding for $n = 5$. Top layer cuts are shown as solid green lines and bottom layer cuts are dashed. Top layer pieces are shaded darker (blue) and bottom are lighter (yellow). Only the first and second spikes are shown unfolded. Note that c'_1 is under the fifth spike, but it will be uncovered when the fifth spike is unfolded.

construction, as $2\alpha+3\beta+\frac{\pi}{10} \leq 2\pi$ implies $\alpha \leq \frac{\pi}{2}$. To prove the second condition, the closest any point on $q_{i-1}t_{i-1}$ gets to p_i, for $\frac{\pi}{3} \leq \alpha \leq \frac{\pi}{2}$, is $|p_ip_{i+1}| \sin\alpha \geq |p_ip_{i+1}| \sin(\frac{\pi}{3}) = |p_ip_{i+1}|\frac{\sqrt{3}}{2} \approx 0.866|p_ip_{i+1}|$. Simple calculations show that the distance from c'_i to p_i is $|p_ic'_i| = \frac{1}{2\cos\beta}|p_ip_{i+1}| = \sqrt{\frac{2}{5-\sqrt{5}}}|p_ip_{i+1}| \approx 0.851|p_ip_{i+1}|$. These together show that p_i and c'_i lie on the same side of $q_{i-1}t_{i-1}$, so the unfolding does not overlap. □

Proof (Case $n = 6$). The naive unfolding suffices for small $\alpha \leq \frac{\pi}{6}\left(1+\frac{6}{6}\right) = \frac{\pi}{3}$, so here we focus on larger angles $\alpha > \frac{\pi}{3}$. We cannot solve $n = 6$ in the same way as $n = 5$ because the center point c'_i would approach p_{i-1} as α approaches $\frac{\pi}{2}$, and would cut off the next spike. We solve this problem by unfolding two spikes at once; see the left of Fig. 7. While the $n \in \{3, 5\}$ comprehensive unfoldings each had n-fold rotational symmetry, the $n \in \{6, 9\}$ comprehensive unfolding constructions we present each have 3-fold rotational symmetry, and the $n \in \{8, 12\}$ each have 4-fold rotational symmetry.

To produce a comprehensive unfolding of a $(6, \alpha)$-star, first cut the top layer of the star along segments cp_i, but only for even i; and also along the right side of every odd spike, from t_i to p_i for odd i, and along the left side of every even spike, from t_i to p_{i+1} for even i. The bottom layer is cut into a central 9-gon B and three triangles T_i, where $B = p_1p_2t_2p_3p_4t_4p_5p_6t_1$ and $T_i = \triangle t_ip_{i+1}p_i$ for odd i; while the top layer is cut into three hexagons $H_i = \hexagon t_ip_icp_{i+2}t_{i+1}p_{i+1}$ for even i. Note that H_i remains attached to B alongside t_ip_i, and T_{i+1} remains attached to H_i alongside $t_{i+1}p_{i+2}$.

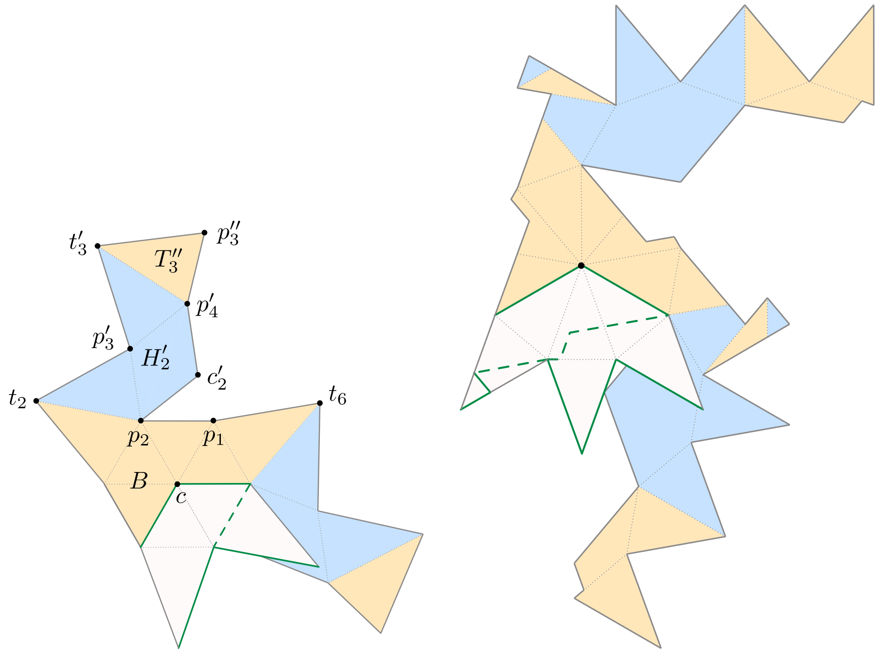

Fig. 7. General unfoldings for $n = 6$ [Left] and $n = 9$ [Right] . Top layer cuts are shown as solid green lines and bottom layer cuts are dashed. Top layer pieces are shaded darker (blue) and bottom are lighter (yellow).

Due to symmetry, it is enough to show that one spike unfolds without overlap. Let c'_i, p'_{i+1}, t'_{i+1}, and p'_{i+2} be the reflections of c, p_{i+1}, t_{i+1}, and p_{i+2} across line $p_i t_i$ respectively for even i, and let $H'_i = \hexagon t_i p_i c'_i p'_{i+2} t'_{i+1} p'_{i+1}$ be the reflection of H_i across line $p_i t_i$. Further, let p''_i be the reflection of p'_i across line $p'_{i+1} t'_i$ for odd i, and let $T''_i = t'_i p'_{i+1} p''_i$ also for odd i. It suffices to show that the total sum of angles around point p_i for even i is not greater than 2π, and that points c'_i and p''_{i+1} lie on the counterclockwise side of $p_{i-1} t_{i-2}$ for even i, for any $\frac{\pi}{3} \leq \alpha \leq \frac{\pi}{2}$. The first condition is satisfied by construction, as $2\alpha + 3\beta \leq 2\pi$ implies $\alpha \leq \frac{\pi}{2}$. To prove the second condition, we observe that points c'_i and p''_{i+1} always stay on the counterclockwise side of the ray R extending from p_{i-1} away from the center in the direction perpendicular to $p_{i-2} p_{i-1}$, while t_{i-2} always stays on the clockwise side of R, for any $\frac{\pi}{3} \leq \alpha \leq \frac{\pi}{2}$. We have found a dividing line separating the leaves of the unfolding, thus the unfolding does not overlap. □

For the remaining cases, we omit full details of the construction, and instead provide a sketch of how they were discovered and adapted.

Proof (Sketch for case $n = 9$).

Just as with $n = 6$, for $n = 9$ we target 3-fold symmetry, so as to allow the top layer center point to move farther away than the naive unfolding would achieve. The center n-gon of the top face is divided into three equal pieces and, as with

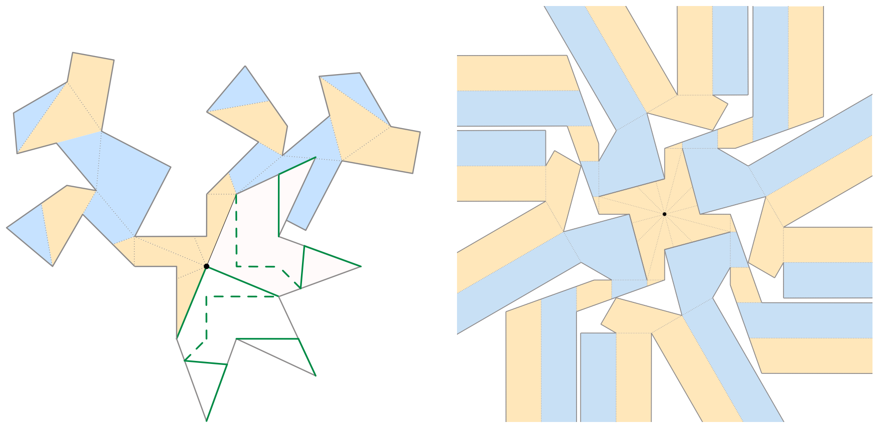

Fig. 8. General unfoldings for $n = 8$ [Left] and $n = 12$ [Right]. Top layer cuts are shown as solid green lines and bottom layer cuts are dashed. Top layer pieces are shaded darker (blue) and bottom are lighter (yellow).

the $n = 5$ comprehensive unfolding, additional cuts are made and propagated to avoid overlap for larger α; see the right side of Fig. 7. The proof of correctness is tedious, but follows the same proof techniques as used earlier in this paper, ensuring that the unfolding of spike i (and everything attached to it) never intersects the counterclockwise side of spike $i - 1$. □

Proof (Sketch for case $n = 8$).
For $n = 8$, we choose to divide the top face, this time with 4-fold rotational symmetry; see the left of Fig. 8. The authors also have another comprehensive unfolding construction for $n = 8$ based on 2-fold rotational symmetry, but have found that 4-fold approaches tend to be easier to implement than 2-fold approaches. As before, additional cuts are made to avoid overlap for larger α. □

Proof (Sketch for case $n = 12$).
As with $n = 8$, we provide a comprehensive unfolding for $n = 12$ that has 4-fold symmetry, shown on the right of Fig. 8. We also have a comprehensive unfolding for $n = 12$ that has 2-fold symmetry, but it is substantially more complex. The figure shows an unfolding for an α angle close to $\frac{\pi}{2}$. □

Proof (Sketch for case $n = 7$).
Unlike $n = 5$ for which we could adapt the naive unfolding fairly directly, $n = 7$ is the first prime number for which the naive unfolding maps c_i' to a point in the interior of the original n-gon bottom layer face. Thus, we attempt to generalize the 3-fold symmetry approach of $n \in \{6, 9\}$ to unfold $n = 7$ in three pieces, two of which unfold two sectors of the original 7-gon while the third piece unfolds three sectors; see left of Fig. 9. □

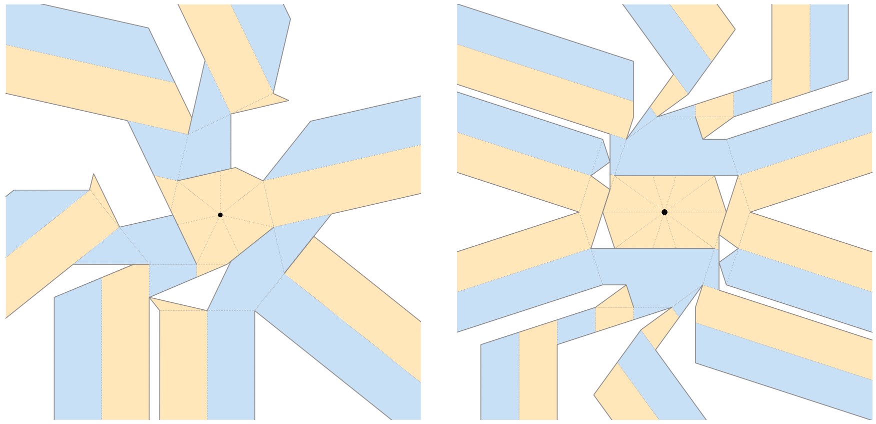

Fig. 9. General unfolding for $n = 10$. The cuts and raw edges are shown in green; top layer pieces are shaded darker (blue) and bottom are lighter (yellow).

Proof (Sketch for case $n = 10$).

While not prime, we were unable to find an comprehensive unfolding for $n = 10$ having 5-fold symmetry. Instead, we attempted to generalize the crown unfolding having 2-fold rotational symmetry, cutting away material that would cause the unfolding to intersect; see right of Fig. 9. Note that, while the crown unfolding retains exactly two whole sectors of the original n-gon from the bottom layer at the center of the unfolding, this unfolding retains four sectors in the center. We have also used this construction approach as the basis for various 2-fold rotational symmetry comprehensive unfoldings, specifically for $n \in \{6, 8, 10, 12\}$. □

5 Future Work

Our initial goal in studying doubly covered stars was to find a counterexample to the proposition that every polyhedron admits a general unfolding. While we have not yet found a counterexample, our search resulted in a number of interesting unfoldings for a new generalized family of nonconvex polyhedra. Our exploration suggests that (n, α)-stars with large n and α close to $\frac{\pi}{2}$ may be potential counterexample candidates. We have attempted non-overlapping unfoldings for n larger than 12, for example for $n \in \{16, 32\}$ for α close to $\frac{\pi}{2}$ (as shown in Fig. 10), but we are not at all convinced that these unfoldings can be generalized for arbitrary α. We hope that future exploration of doubly-covered star unfoldings will either provide unfoldings for the remaining space over n and α, or find a polyhedron that does not admit a general unfolding.

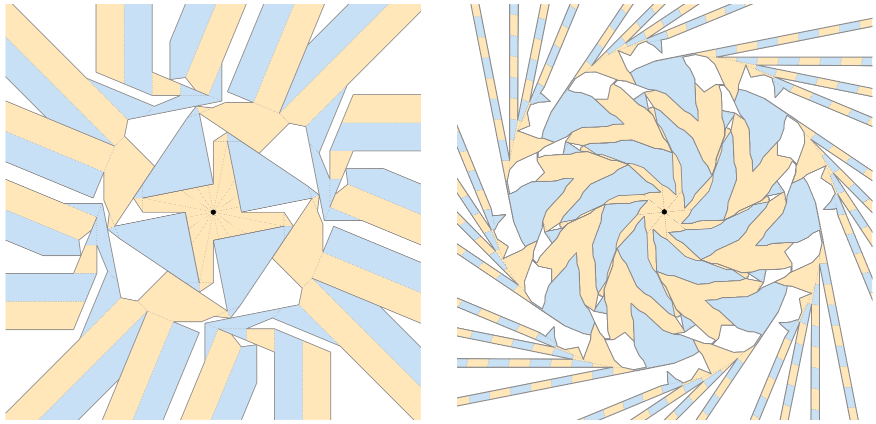

Fig. 10. Possible unfoldings for $n = 16$ [Left] and $n = 32$ [Right] for α close to $\frac{\pi}{2}$.

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