

Spiral Unfoldings of Convex Polyhedra

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October 20, 2015

Abstract

The notion of a spiral unfolding of a convex polyhedron, a special type of Hamiltonian cut-path, is explored. The Platonic and Archimedean solids all have nonoverlapping spiral unfoldings, although overlap is more the rule than the exception among generic polyhedra. The structure of spiral unfoldings is described, primarily through analyzing one particular class, the polyhedra of revolution.

1 Introduction

I define a *spiral* σ on the surface of a convex polyhedron P as a simple (non-self-intersecting) polygonal path $\sigma = (p_1, p_2, \dots, p_m)$ which includes every vertex v_j of P (so it is a Hamiltonian path), and so when cut permits the surface of P to be unfolded flat into \mathbb{R}^2 . (Other requirements defining a spiral will be discussed below.) The starting point for this investigation was Figure 2, an unfolding of a spiral cut-path on a tilted cube, Figure 1. The cut-path σ on P unfolds to two paths ρ and λ in the plane, with the surface to the right of ρ and to the left of λ . Folding the planar layout by joining (“gluing”) ρ to λ along their equal lengths results in the cube, uniquely results by Alexandrov’s theorem. Note that the external angle at the bottommost vertex (marked 1 in Figure 2) and the topmost vertex (marked 17, because σ has 16 segments and $m = 17$ corners¹) is 90° , which is the Gaussian curvature at those cube vertices. Note also that there are 8 vertices of P along both ρ and λ , with one shared at either end.

The notion of restricting unfoldings of convex polyhedra by following a Hamiltonian path along polyhedron edges was introduced by Shephard 40 years ago [She75]. Shephard noted that not every polyhedron has such a Hamiltonian edge-unfolding, because not every polyhedron 1-skeleton has a Hamiltonian path (e.g., the rhombic dodecahedron does not have such a path). Here we are not restricting cuts to polyhedron edges. A single Hamiltonian cut-path, not necessarily following polyhedron edges, leads to what was memorably christened as

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¹We reserve the term “vertices” to refer to P ’s vertices, and use “corners” for the turns of σ .

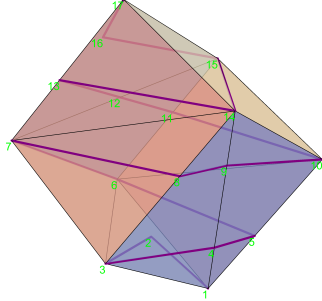


Figure 1: Cube with spiral cut-path σ .

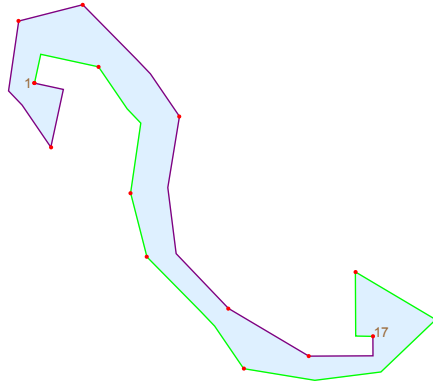


Figure 2: Spiral unfolding of cube. Vertices are marked with red dots. ρ is purple; λ is green.

a *zipper unfolding* in [LDD⁺10]. Those authors posed the still-open problem of whether or not every convex polyhedron has a nonoverlapping zipper unfolding. The investigation reported here does not make an advance on this question, as spiral unfoldings are a special subclass of zipper unfoldings, and nonoverlap is rare (as we will see in Section 5). Instead we pursue these unfoldings for their intrinsic, almost aesthetic, interest. In keeping with this attitude, we reach no grand conclusions, and do not offer formal proofs of claims. For terminology and background (e.g., Alexandrov’s theorem, Gaussian curvature), see [DO07].

The cube unfolding in Figure 2 suggests seeking spiral unfoldings of other convex polyhedra, seeing if they avoid overlap in the plane, i.e., if they unfold to simple (non-self-intersecting) polygons. First we add some detail to the definition of a spiral. We insist that a spiral σ satisfies these requirements:

1. p_{i+1} is vertically higher or the same height as p_i . Letting z_i be the z -coordinate of p_i , this condition is $z_{i+1} \geq z_i$. Therefore, p_1 is a bottommost vertex and p_m a topmost vertex of P .
2. Each segment $p_i p_{i+1}$ “advances” counterclockwise (ccw) around its band.

We defer the somewhat technical definition of what constitutes a ccw advance to Section 2.1 below.

2 Platonic Solids

All of the five Platonic solids have nonoverlapping spiral unfoldings: the tetrahedron (Figures 3 and 4), the octahedron (Figures 5 and 6), the dodecahedron (Figures 7 and 8), and the icosahedron (Figure 9 and 10).

Before proceeding further, a few remarks are in order. By no means are spirals uniquely defined. First, the polyhedron may be oriented with two degrees

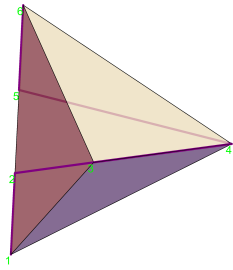


Figure 3: Tetrahedron with spiral cut path.

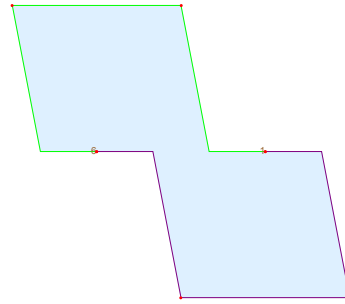


Figure 4: Spiral unfolding of tetrahedron.

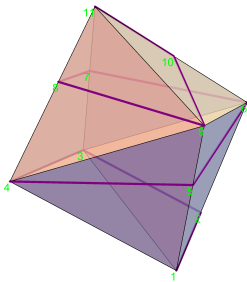


Figure 5: Octahedron with spiral cut path.

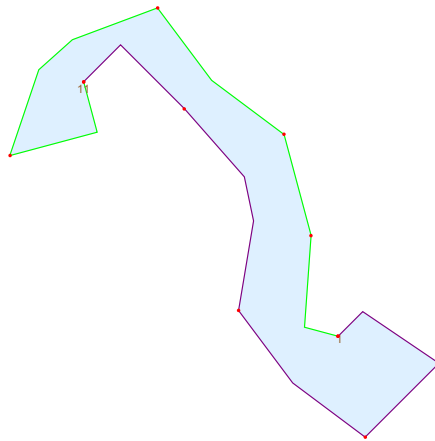


Figure 6: Spiral unfolding of octahedron.

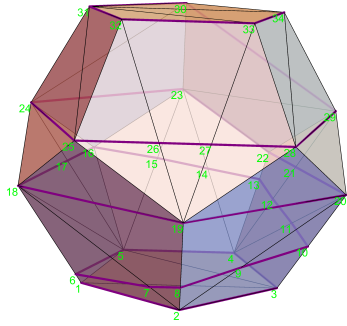


Figure 7: Dodecahedron with spiral cut path.

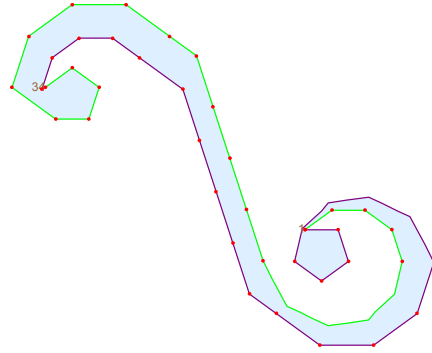


Figure 8: Spiral unfolding of dodecahedron.

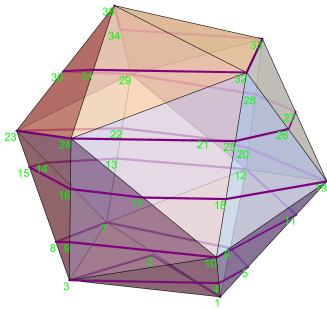


Figure 9: Icosahedron with spiral cut path.

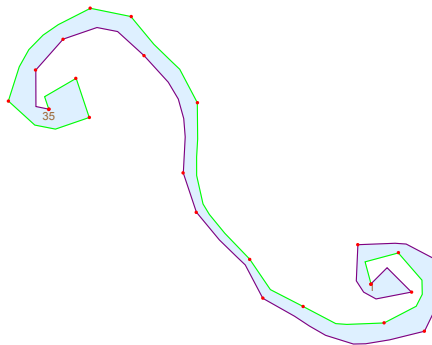


Figure 10: Spiral unfolding of icosahedron.

of continuous freedom. The orientation of the dodecahedron above was carefully selected to avoid overlap: see Figure 11. Second, for a fixed orientation, even

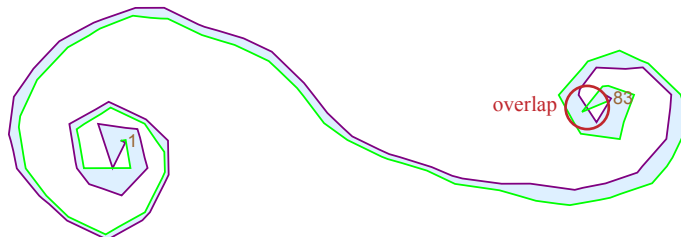


Figure 11: Overlapping spiral unfolding of a tilted dodecahedron.

though σ must pass through every vertex in sorted vertical order, there are still an infinite number of choices for spirals. For example, one may wind around several times between each vertically adjacent pair of vertices of P ; see Figure 12.

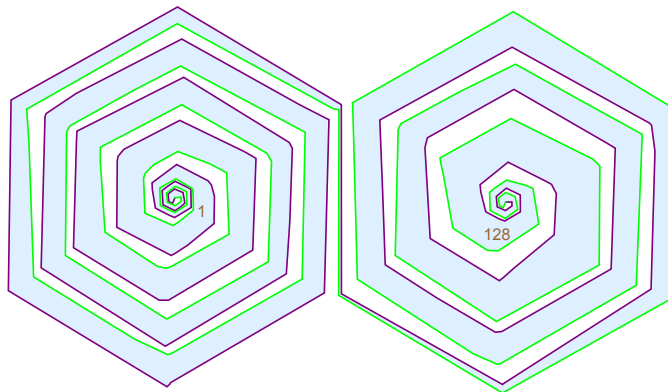


Figure 12: Upright octahedron with densely wound spiral cut-path.

Further intuition may be gained from an animation of the previously shown (Figure 9 and 10) icosahedron unfolding: see Figure 13.

2.1 Spiral Definition: Bands

We return to the definition of a spiral, to make precise the sense in which the spiral must always advance ccw around the polyhedron P . Let a *band* be the portion of P between two horizontal ($z = \text{constant}$) planes, such that there is no vertex of P strictly between the planes (although vertices may lie on those planes). See Figure 14. The edges of P crossing the band provide a natural combinatorial sense of ccw. If a segment $p_i p_{i+1}$ of σ connects two edges of a band, then it is ccw if the second edge is ccw of the first edge (recall our

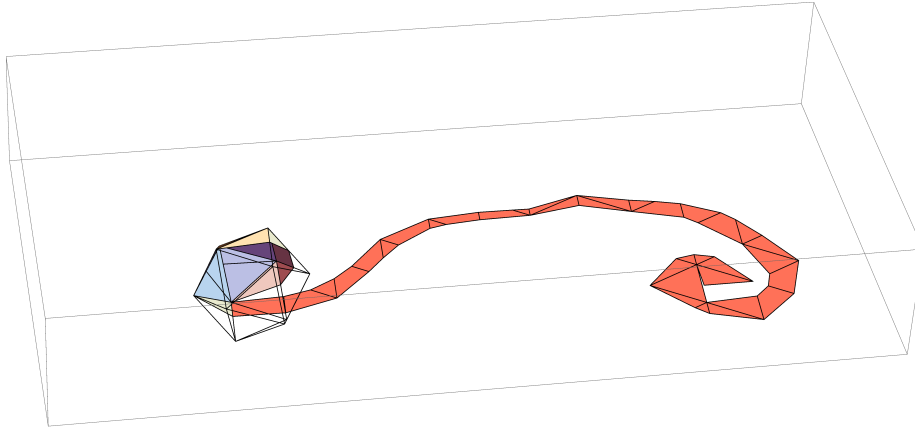


Figure 13: Snapshot of animation of icosahedron rolling out its spiral (35th out of 50 frames). <http://cs.smith.edu/~orourke/SpiralUnf/>.

convention that the spiral rises from a bottom to a top vertex). In general, and in all examples in this paper, each p_i does in fact lie on an edge (or several, where they meet at a vertex). However, this is not a necessary condition: p_i could lie interior to a face.

If $p_i p_{i+1}$ does not connect two band edges (e.g., both p_i and p_{i+1} might be vertices connected by an edge of P), then the following rule is used to determine whether it represents a ccw advance. Let ρ_i^h be the total surface angle to the right from $p_i p_{i+1}$ down to the horizontal plane through p_i , and similarly let λ_i^h be the surface angle to the left. Here the superscript h indicates an angle to the horizontal plane. For p_{i+1} the analogous angles measure up to the horizontal plane through p_{i+1} . Then $p_i p_{i+1}$ is ccw iff (a) $\rho_i^h < \lambda_i^h$, and (b) $\rho_{i+1}^h > \lambda_{i+1}^h$; see Figure 15. This condition requires appropriate “slant” at each end of $p_i p_{i+1}$, and ensures vertical symmetry: reversing z -coordinates of a path renders it a spiral iff the original is a spiral.

Before proceeding further, we describe the implementation that produced the unfoldings displayed in this paper.

3 Implementation

The spiral cut-paths illustrated were created by a particular implementation, selecting a specific σ among the infinite number of choices for a given, fixed orientation of P . We first describe the algorithm in the case when no two vertices of P lie at the same height. At any one time, the portion of σ below and up to a vertex v_i of P has been constructed. P is sliced with a horizontal plane through v_i , and again sliced through the next vertex in the z -direction, v_{i+1} . See Figure 16. These two planes define a vertex-free band B_i . The algorithm takes a parameter w indicating how many complete times σ should wind around

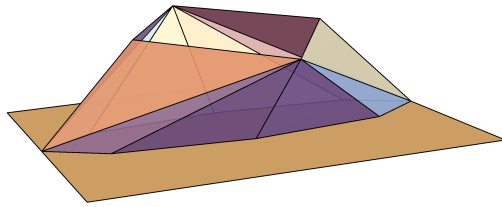


Figure 14: A band between two horizontal planes; bottom plane shown. Both planes pass through vertices in this example.

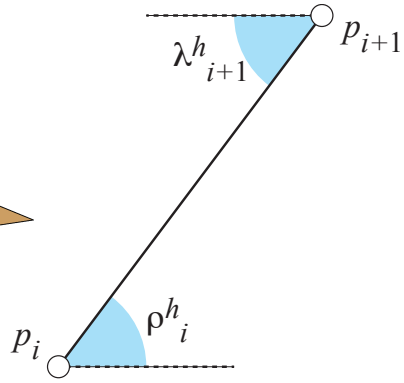


Figure 15: Surface angles above and below horizontal planes through p_i and p_{i+1} .

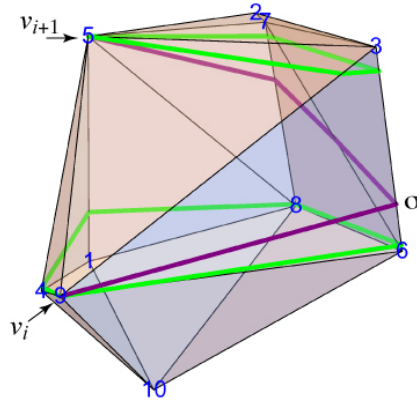


Figure 16: The portion of σ between v_i (9) and v_{i+1} (5). The green polygons outline the intersection of P with parallel horizontal planes through v_i and v_{i+1} .

B_i before connecting to v_{i+1} . In the figure, $w = 0$, so the connection is as direct as possible. In this instance, $v_i v_{i+1}$ is an edge of P and could be followed, but it is not slanting ccw (Section 2.1). Therefore σ spirals across several edges of P before reaching v_{i+1} . Note that, in general, σ cannot follow a geodesic as it winds around the band, as the shape of a band would only allow that in special circumstances.

The top and bottom vertices are handled specially. Because our goal is to avoid overlap, and overlap usually occurs at these apexes when acute angles are forced, the first segment from the bottom vertex v_1 , to v_2 , is selected to make a nonacute turn to connect to v_2 . This can be seen clearly in Figure 1, where a 90° turn is selected, and in other figures. I proved that such a nonacute turn is always possible. Of course the turn at v_2 to connect to v_3 might be highly acute, often leading to overlap (e.g., in Figure 11).

Because the algorithm progresses from the bottom vertex to the top without look-ahead, it is possible that the details of the connection to the top vertex are not identical to the details at the bottom vertex, even when P is symmetric. This is again evident in Figure 1.

When P is oriented so that more than one vertex lies at a particular z -height, it is necessary, by the definition of a spiral, for σ to cycle around the slice polygon at that height until all the vertices are included, before angling off to the next vertex vertically. This is evident for the dodecahedron (Figure 7), and for all the Archimedean solid unfoldings in Section 4.

4 Archimedean

All 13 of the Archimedean solids have nonoverlapping spiral unfoldings. Six unfoldings are shown in Figure 17, six more in Figure 18, and the most complex 13th, the great rhombicosidodecahedron, is shown in Figure 19. The σ used in this last case has 452 corners; the polyhedron itself has 60 vertices.

As shown in [LDD⁺10], all the Platonic and Archimedean solids have Hamiltonian edge-unfoldings, and in general they can be chosen to be “S-shaped,” visually not unlike the spiral unfoldings above. The exception is the great rhombicosidodecahedron, whose Hamiltonian edge-unfolding leads to a rather differently shaped planar polygon.

5 Overlapping spiral unfoldings

5.1 Random polyhedra

Despite the nonoverlapping spiral unfoldings of the Platonic and Archimedean solids, avoiding overlap is actually rare. Figure 20 shows data from polyhedra constructed as the convex hull of random points uniformly distributed on a sphere. By the time P has 25 vertices, essentially no random polyhedron, in a random orientation, lead to a nonoverlapping unfolding using the spiral cut-path generated by the algorithm discussed in the previous section. The intuition for

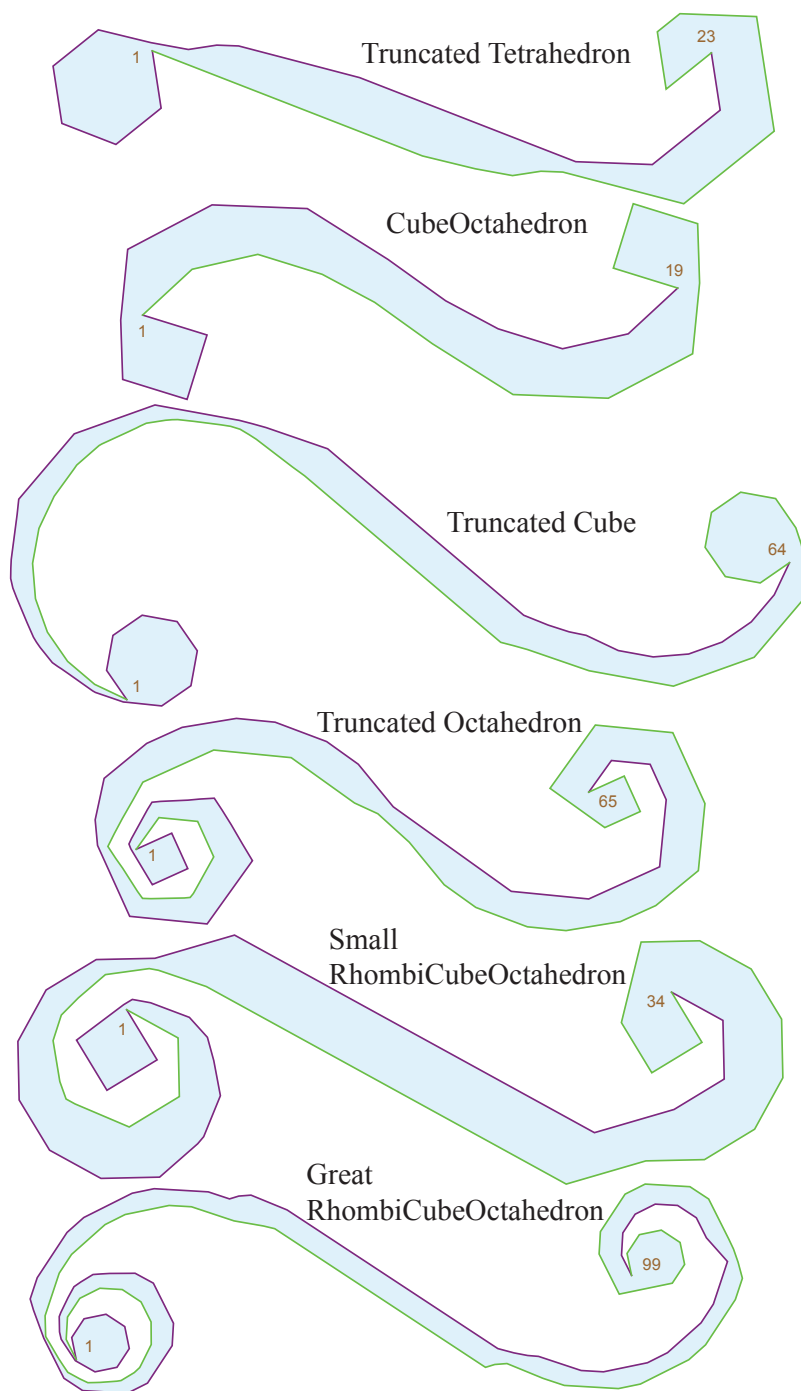


Figure 17: Spiral unfoldings of six Archimedean solids.

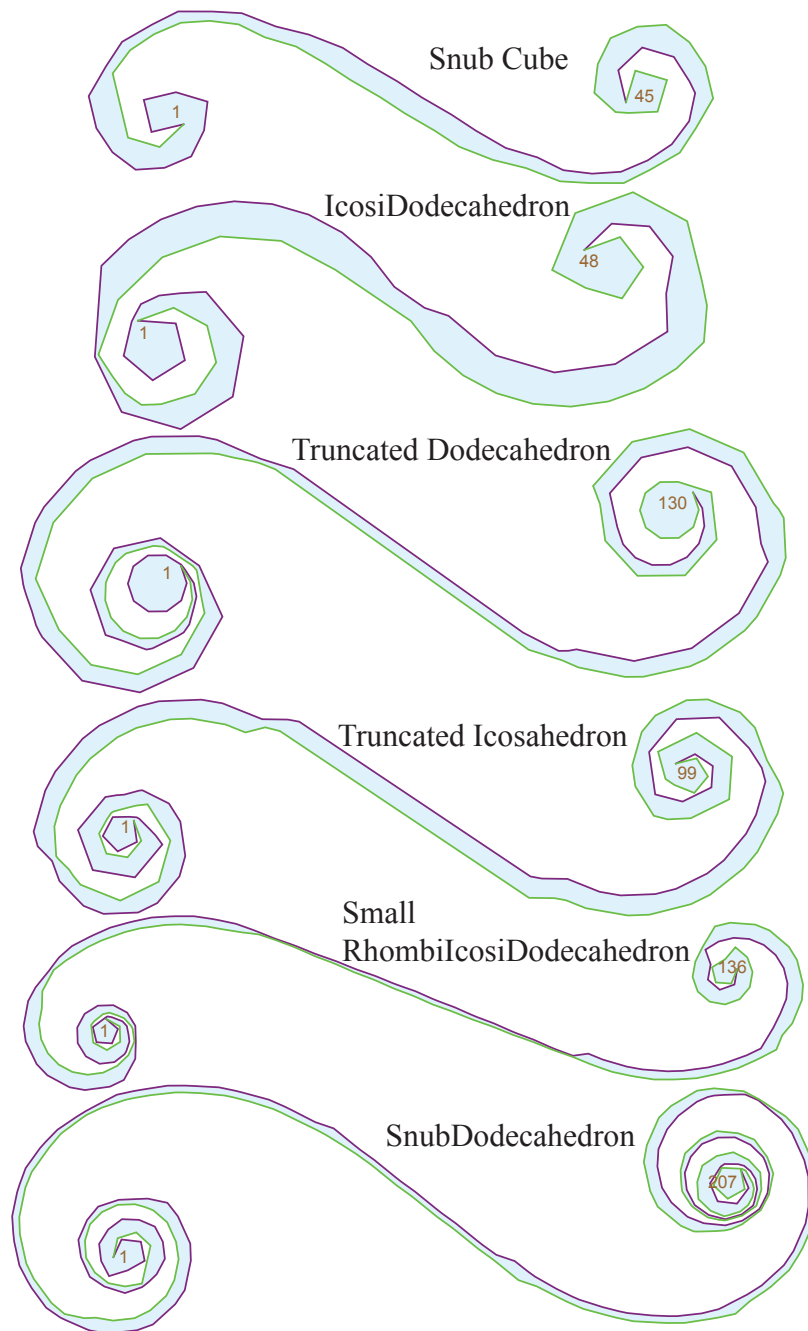


Figure 18: Spiral unfoldings of six more Archimedean solids

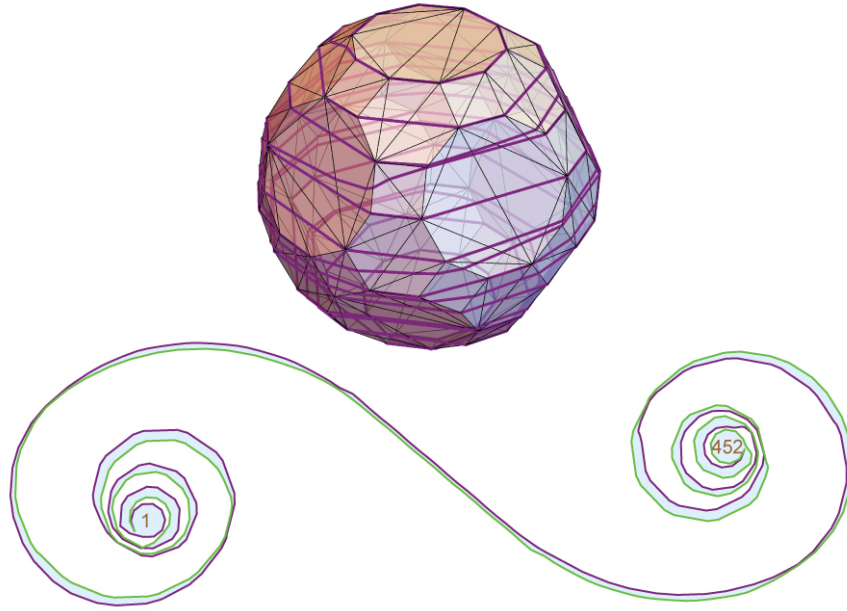


Figure 19: Great RhombIcosiDodecahedron.

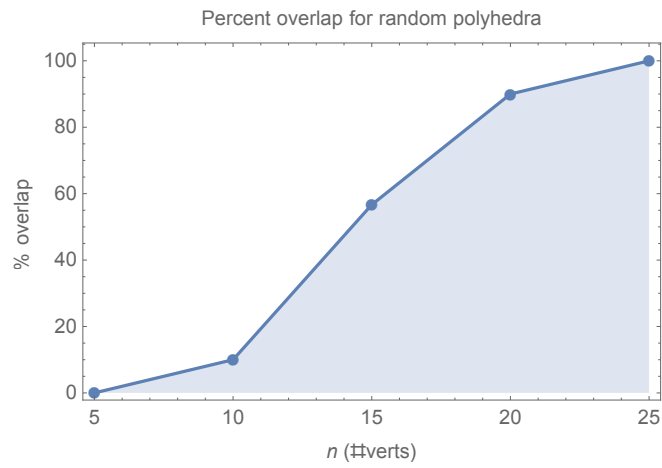


Figure 20: The percentage of overlapping spiral unfoldings for random polyhedron with n vertices. Each point plotted is the mean of 50 random trials.

why overlap is common can be seen in the nonoverlapping spiral unfolding of the 60-vertex great rhombicosidodecahedron in Figure 19: with many vertices, the spiral turns many times near the apex, and those must unfold to a precisely nested, tightly wound spiral in the plane to avoid overlap.

Of course, this random data just suggests that overlap is common, not that it cannot be avoided.

5.2 Polyhedra without nonoverlapping spiral unfoldings

Despite the rarity of nonoverlapping spiral unfoldings, it is not straightforward to identify a particular polyhedron P that has no nonoverlapping spiral unfolding, for two reasons: (1) For a given orientation of P , there are an infinite number of spirals compatible with that orientation. (2) All orientations must be blocked. Here I propose a P which almost certainly has no spiral unfolding, but my argument for this falls short of a formal proof.

Define a *hemiball* H as the convex hull of a circle and a semicircle of equal radii, as illustrated in Figure 21. Here the full (red) circle C lies in the xy -plane, and the (green) semicircle C^+ lies in the xz -plane. Let n be the number of points equally spaced around C , with $n/2$ around C^+ . Figure 22 shows the convex hull H_n . The lateral hull edges ab connect $a(\theta)$ on C and $b(\theta)$ on C^+ ,

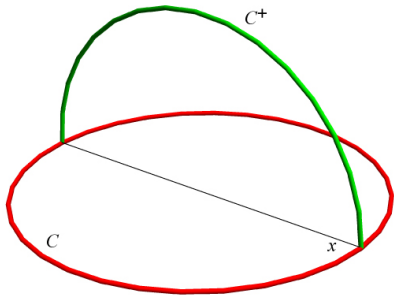


Figure 21: HemiBall rims.

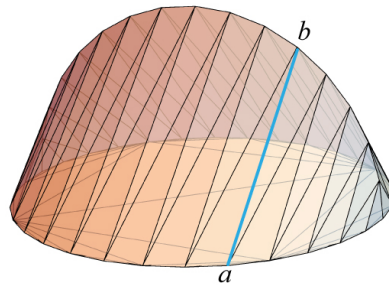


Figure 22: HemiBall H_n , $n = 32$.

where

$$a(\theta) = (\cos \theta, \sin \theta, 0), \quad (1)$$

$$b(\theta) = (\cos \theta, 0, |\sin \theta|). \quad (2)$$

Conjecture 1 *For sufficiently large n , the hemiball H_n has no nonoverlapping spiral unfoldings.*

I now present evidence for this conjecture, for $H = H_{16}$. There are two special orientations of H . The first is when the flat base is horizontal; see Figure 23. Then spiral unfoldings overlap near the top of H , as in Figure 24. This illustrates the logic of H : planes slicing through the rims cut highly non-circular bands, which tend to lead to acute angles and overlapping unfoldings.

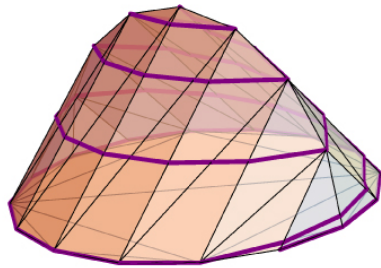


Figure 23: H oriented with $(0, 0, 1)$ vertical.

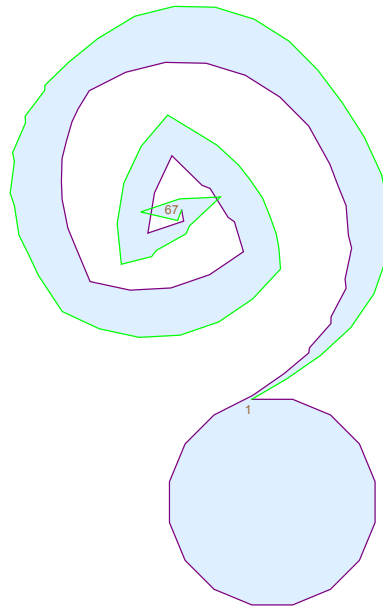


Figure 24: Overlapping unfolding.

A second special orientation, tilting H 90° , is shown in Figure 25, which leads to overlap at both ends: Figure 26.

For other orientations of H , either the top or the bottom (or both) plane slices cut H in eccentric bands, which result in overlaps. A typical example is shown in Figure 27. I have verified overlap occurs in hundreds of random orientations of H , which of course only implies that overlap is common, not necessary. I suggest a proof of necessity might be formulated around the following approach.

Figure 28 illustrates the normal vectors to H in its standard position. Now consider orienting H so that v is normal to the horizontal slicing planes. We need only consider v in the northern hemisphere of the figure. If v lies within the 90° wedge containing the northpole, then these nearly horizontal slicing planes will cut the upper rim C^+ in a manner similar to that seen in Figure 23. In particular, the top vertex will be unique. If v lies outside of that wedge, then the nearly vertical slicing planes will cut the rim C , and the bottom vertex will be unique. In either case, a rim is sliced at one end or the other, and such slices will be elongated by design, leading to overlapping unfoldings.

I believe this argument could be formalized. Incidentally, it is easy to find a nonoverlapping unzipping of the hemiball: see Figure 29.

Another candidate for a polyhedron with no nonoverlapping spiral unfoldings is a distorted dodecahedron, distorted so that no two pentagonal faces are parallel. A flat doubly covered regular polygon with large n has just one orientation whose spirals avoid overlap: that with the disk horizontal, in which case

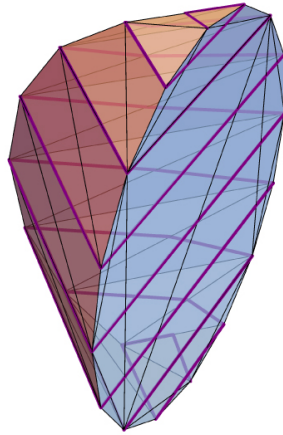


Figure 25: H oriented so $(1,0,0)$ is vertical.

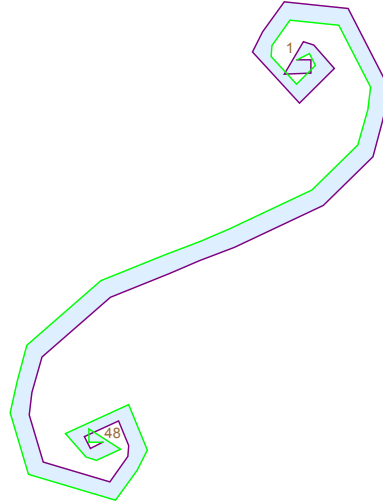


Figure 26: Overlapping unfolding.

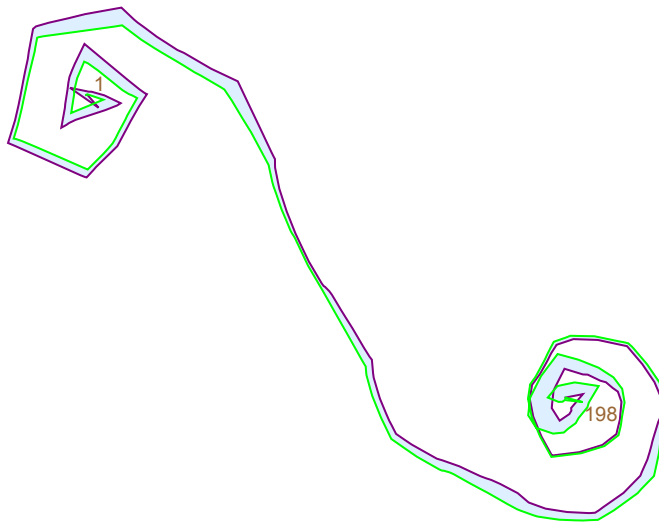


Figure 27: Random spiral unfolding of H .

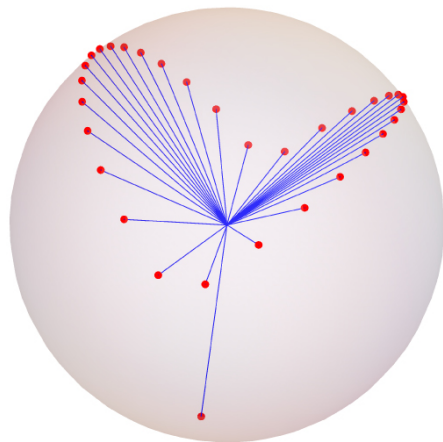


Figure 28: Normal vectors to H faces (Figure 22). The single vector in the southern hemisphere derives from the horizontal base.

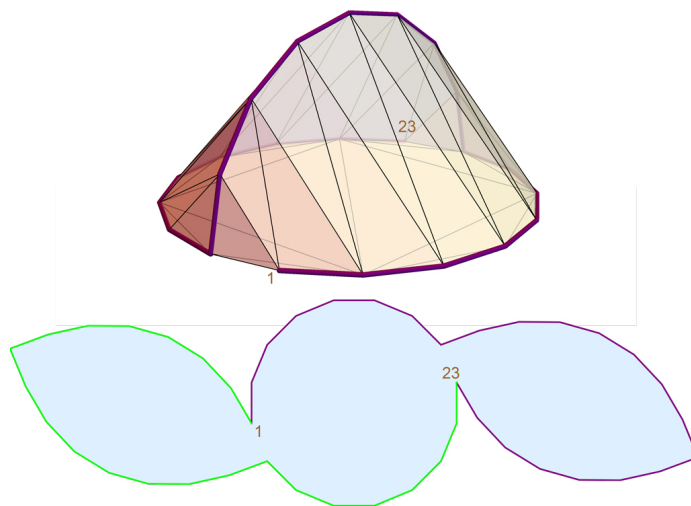


Figure 29: A zipper path that unfolds the hemiball H .

it unfolds to two connected regular polygons.

6 Polyhedra of Revolution

In this section we study a narrow class of convex polyhedra whose spiral unfoldings are easily understood. I believe this sheds light on the general case. The class is *polyhedra of revolution*, formed as follows. Let C be a convex curve in the yz -halfplane with $x \geq 0$. Spin this curve around the z -axis, forming n_{spin} discrete copies. Then take the convex hull to form P .

Two examples for the same C are shown Figure 30. Note the faces of such

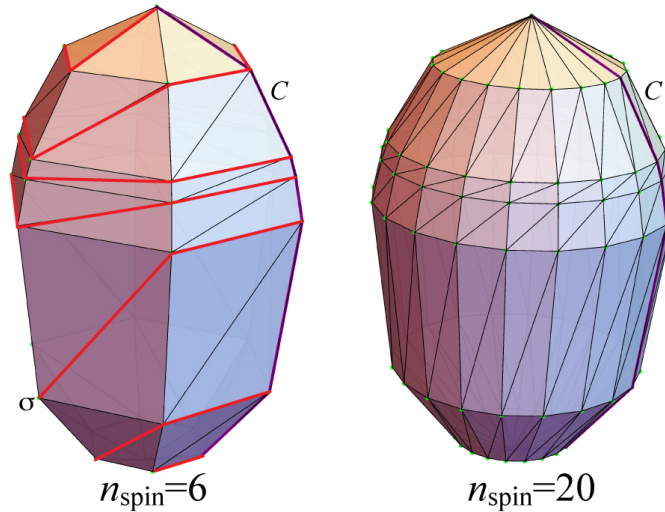


Figure 30: Polyhedra of revolution, with C shown, and $n_{\text{spin}} = 4$ and 20. The spiral cut-path σ is shown on the left polyhedron.

a P are trapezoids, but in the triangulated figure, each trapezoid is cut by a diagonal. Let v be a vertex of C . The natural spiral cut-path σ we explore circles around the horizontal regular polygon of n_{spin} sides formed by each v , following a trapezoid diagonal to the next ring upward.²

Typical spiral unfoldings are shown in Figure 31. Here we see overlap for $n_{\text{spin}} \leq 4$, just barely nonoverlap for $n_{\text{spin}} = 6$, and nonoverlap for all larger n_{spin} . The claim of this section is that this is the general situation:

Proposition 2 *For any convex curve C , there is some³ $n_{\text{spin}} \geq 2$ such that the spiral unfolding of the polyhedron of revolution determined by C and all $\geq n_{\text{spin}}$ does not overlap.*

²If σ follows a vertical trapezoid edge rather than the coplanar diagonal, σ would constitute a Hamiltonian edge-unfolding.

³ $n_{\text{spin}} = 2$ produces a flat, doubly covered convex polygon, which is usually considered a convex polyhedron in the context of unfolding.

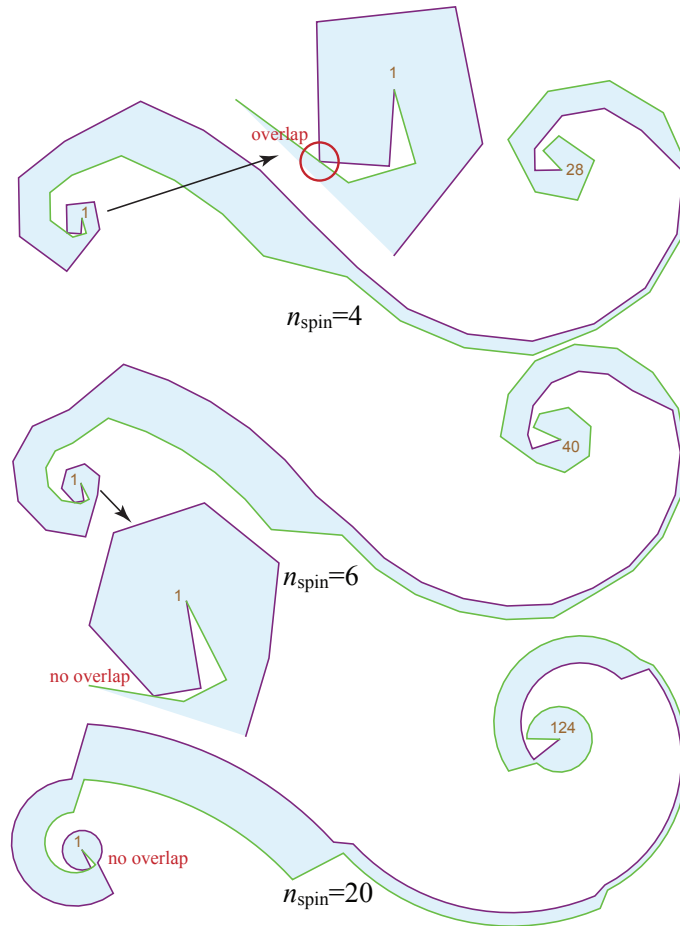


Figure 31: Spiral unfoldings of the shape in Figure 30, for $n_{\text{spin}} = 4, 6, 20$.

We now argue for this proposition, somewhat informally. First, it helps to imagine $n_{\text{spin}} \rightarrow \infty$. Then σ follows horizontal circles around P , and the vertical diagonal cuts join adjacent circles orthogonal to both.

Second, view adjacent bands as deriving from nested cones, as depicted in Figure 32. Let β be the half-angle of a cone in \mathbb{R}^3 , and let α be the total surface

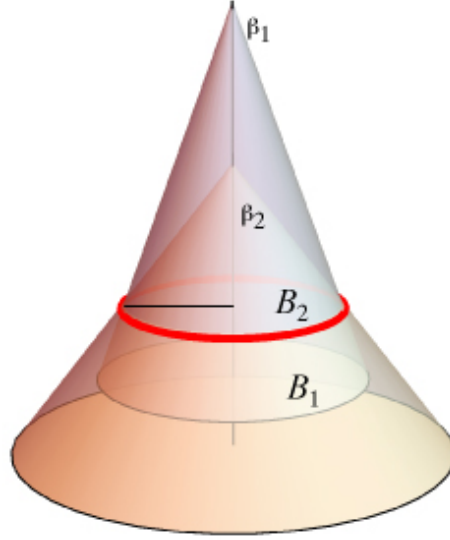


Figure 32: Nested cones determined by adjacent bands.

angle at the cone apex, i.e., the angle of the wedge when the cone is cut open along a generator and flattened to \mathbb{R}^2 . Then $\alpha = 2\pi \sin \beta$, which implies that these angles grow and shrink together monotonically. Let B_1 be the lower band and B_2 the upper band, sharing a circle of radius r . Then $\beta_1 < \beta_2$ and so $\alpha_1 < \alpha_2$.

Let ρ be the planar layout curve with surface to the right, and λ the curve with surface to the left. Each unfolded band B is bounded by ρ and λ , each of which is an arc of a circle centered on the image of the apex of the cone containing B .

Consider again two adjacent bands B_1 and B_2 sharing a horizontal circle of σ . B_1 is right of σ and B_2 left of σ on P . B_1 unfolds to a strip bounded by circle arcs of radius r_1 strictly larger than the radius r_2 . The shared seam between B_1 and B_2 therefore unfolds to ρ_1 and λ_2 as illustrated in Figure 33. The arcs ρ_1 and λ_2 share a common tangent at the orthogonal cuts to adjacent bands, when $n_{\text{spin}} \rightarrow \infty$. The centers of the circle arcs are aligned so that c_1 and c_2 are collinear with that point of common tangency.

So now it is clear that each band unfolding sits inside an annulus, with each annulus nested inside and tangent to its z -lower mate. Thus the bands cannot intersect one another except where they join. Of course the same holds at each

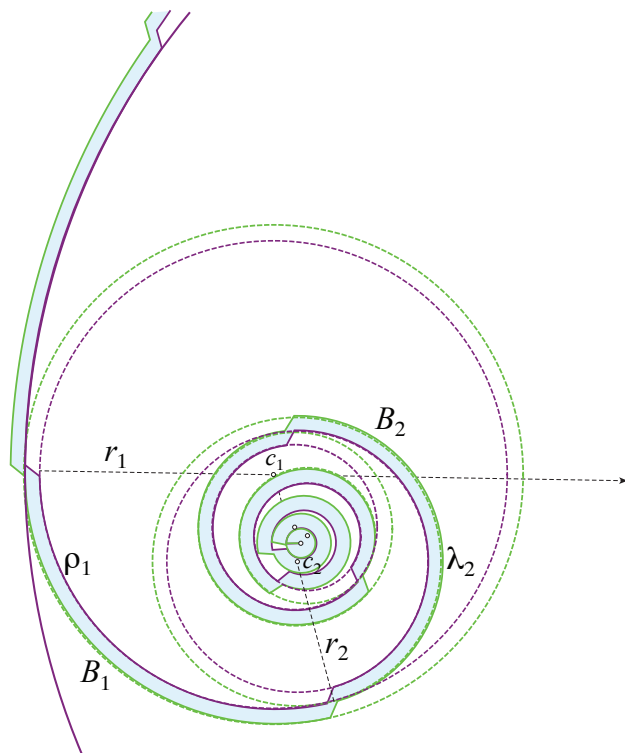


Figure 33: Portion of a spiral unfolding with band circles illustrated (polyhedron of revolution not shown). We must have $r_1 > r_2$, and ρ_1 joining smoothly with λ_2 , and so the B_2 annulus is tangent to and nested inside the B_1 annulus.

“end” of any polyhedron of revolution P , with the nesting occurring to the other side of the unfolding.

The only possible overlap occurs for small n_{spin} , when two regular polygons are separated near the unfolding of the apex of P , as we saw in Figure 31. But for any apex curvature ω , regardless of how small, there is some n_{spin} that avoids overlap, as do all larger n_{spin} values.

6.1 Relationship to arbitrary P .

For arbitrary P , the bands are not as circular, and do not necessarily lie on cones. But one can see an analogous structure to spiral unfoldings: bands are unfolded and attached, increasing in “radii” away from the apexes of P . The more vertices of P , and the more circular each cross-section, the closer will a spiral unfolding resemble an unfolding of a polyhedron of revolution. This incidentally reinforces the intuition of why nonoverlap is rare: even with polyhedra of revolution, the nesting to avoid overlap is delicate.

6.2 Nonconvex Polyhedra of Revolution

We end this section with a curiosity: some spiral unfoldings of nonconvex polyhedra avoid overlap. Figure 34 shows a nonconvex polyhedron of revolution and a nonoverlapping unfolding. But this cannot be pushed too far, as the right overlapping unfolding demonstrates.

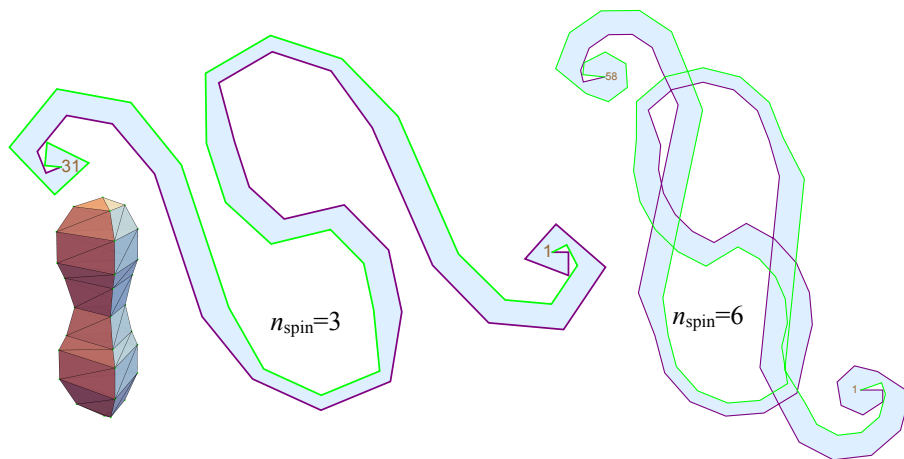


Figure 34: A peanut-shaped polyhedron of revolution, for two different n_{spin} values (sharing the same curve C).

7 Conjecture

We mentioned in Section 1 that it was posed as an open question in [LDD⁺10] whether or not every convex polyhedron has a nonoverlapping zipper unfolding. The spiral unfoldings we explored here are a narrow class of zipper unfoldings, and shed no direct light on the open problem. Rather than require a spiral to pass through the vertices in vertical order, one could imagine loosening the definition to allow spiral-like paths that rely more on intrinsic surface features rather than on an extrinsic vertical sorting. I have explored this direction enough to know that some overlaps can be avoided with more general spirals. Nevertheless, this effort has led me to conjecture that the answer to the Lubiw et al. open problem is *No*: there are polyhedra whose every zipper unfolding overlaps.

I reach this conclusion through two hunches. First, although there are a vast number of possible zipper unfoldings for polyhedra with many vertices n , if the vertices are sprinkled uniformly but unorganized, spiral-like paths are the only real options. Second, for polyhedra that are close to spheres with many unorganized vertices, any spiral will overlap. So I suggest two similar candidates for counterexamples: (1) P is the convex hull of a large number of random points on a sphere; (2) P is geodesic dome, but with the vertices perturbed slightly to break all symmetries.

Figure 35 shows the second example, and Figure 36 shows a spiral unfolding.

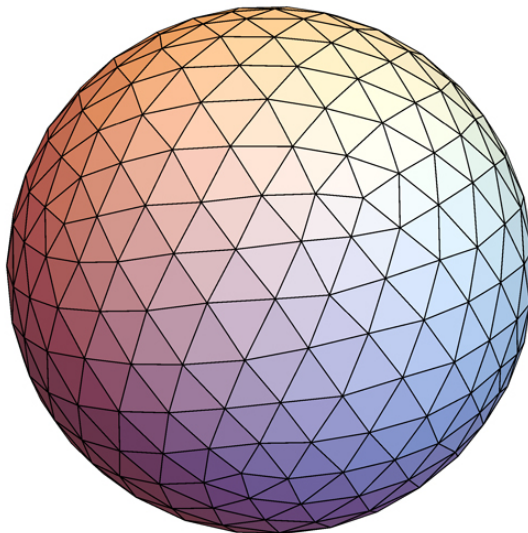


Figure 35: Geodesic dome with perturbed vertices.

The irregularity of the vertex positions causes significant overlap along many turns of the spiral.

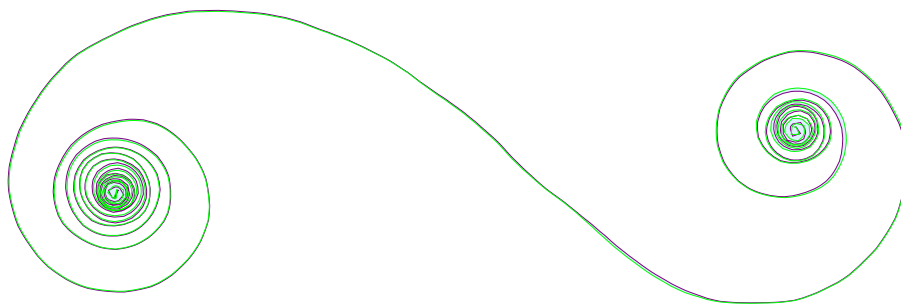


Figure 36: Overlapping spiral unfolding of a geodesic dome.

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