

# Hypercube Unfoldings that Tile $\mathbb{R}^3$ and $\mathbb{R}^2$

Giovanna Diaz\*      Joseph O’Rourke†

December 9, 2015

## Abstract

We show that the hypercube has a face-unfolding that tiles space, and that unfolding has an edge-unfolding that tiles the plane. So the hypercube is a “dimension-descending tiler.” We also show that the hypercube cross unfolding made famous by Dali tiles space, but we leave open the question of whether or not it has an edge-unfolding that tiles the plane.

## 1 Introduction

The cube in  $\mathbb{R}^3$  has 11 distinct (incongruent) edge-unfoldings<sup>1</sup> to 6-square planar polyominoes, each of which tiles the plane [Kon15]. A single tile (a *prototile*) that tiles the plane with congruent copies of that tile (i.e., tiles via translations and rotations, but not reflections) is called a *monohedral* tile. The cube itself obviously tiles  $\mathbb{R}^3$ . So the cube has the pleasing property that it tiles  $\mathbb{R}^3$  and all of its edge-unfoldings tile  $\mathbb{R}^2$ . The latter property makes the cube a *semi-tile-maker* in Akiyama’s notation [Aki07], a property shared by the regular octahedron.

In this note we begin to address a higher-dimensional analog of these questions. The 4D hypercube (or *tesseract*) tiles  $\mathbb{R}^4$ . Do all of its face-unfoldings monohedrally tile  $\mathbb{R}^3$ ? The hypercube has 261 distinct face-unfoldings (cutting along 2-dimensional square faces) to 8-cube polycubes, first enumerated by Turney [Tur84] and recently constructed and confirmed by McClure [McC15] [O’R15a]. The second author posed the question of determining which of the 261 unfoldings tile space monohedrally [O’R15c].

Whether or not it is even decidable to determine if a given tile can tile the plane monohedrally is an open problem [O’R15b], and equally open for  $\mathbb{R}^3$ . The only general tool is Conway’s sufficiency criteria [Sch80] for planar prototiles, which seem too specialized to help much here. In the absence of an algorithm, this seems a daunting task.

Here we focus on two narrower questions, essentially replacing Akiyama’s “all” with “at least one”:

---

\*Depts of Computer Science, and Mathematics, Smith College. [gdiacz@smith.edu](mailto:gdiacz@smith.edu).

†Depts of Computer Science, and Mathematics, Smith College, Northampton, MA 01063, USA. [orourke@cs.smith.edu](mailto:orourke@cs.smith.edu).

<sup>1</sup> An *edge-unfolding* cuts along edges.

**Question 1** *Is there an unfolding of the hypercube that tiles  $\mathbb{R}^3$ , and which itself has an edge-unfolding that tiles  $\mathbb{R}^2$ ?*

Call a polytope that monohedrally tiles  $\mathbb{R}^d$  a *dimension-descending tiler* (DDT) if it has a facet-unfolding that tiles  $\mathbb{R}^{d-1}$ , and that  $\mathbb{R}^{d-1}$  polytope has a facet-unfolding that tiles  $\mathbb{R}^{d-2}$ , and so on down to an edge-unfolding that tiles  $\mathbb{R}^2$ . (Every polygon has a vertex-unfolding of its perimeter that trivially tiles  $\mathbb{R}^1$ .) Thus the cube is a DDT. We answer Question 1 positively by showing that the hypercube is a DDT, by finding one face-unfolding to an 8-cube polyform in  $\mathbb{R}^3$ , which itself has an edge-unfolding to a 34-square polyominoe that tiles  $\mathbb{R}^2$ .

It is natural to wonder about the other 260 face-unfoldings of the hypercube, and in particular, the most “famous” one, what we call the *Dali cross*, made famous in Salvadore Dali’s painting shown in Figure 1.

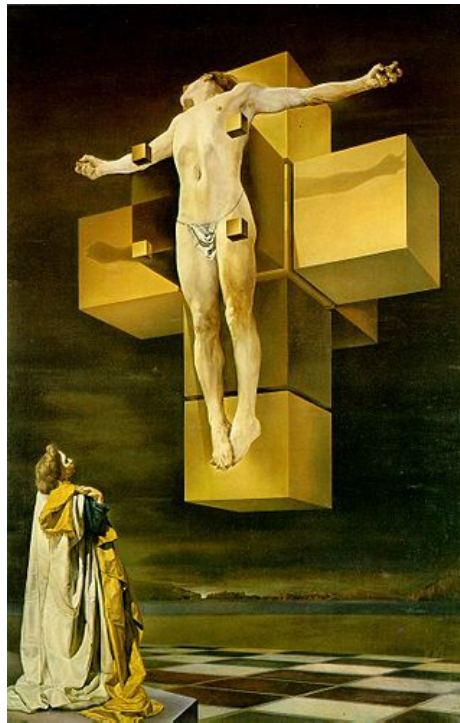


Figure 1: The 1954 Dali painting *Corpus Hypercubus*. (Image from Wikipedia).

**Question 2** *Does the Dali cross tile  $\mathbb{R}^3$ , and if so, does it have an edge-unfolding that tiles  $\mathbb{R}^2$ ?*

Here we are only partially successful: We show that the Dali cross does indeed tile space (Theorem 1), but we have not succeeded in finding an unfolding of this cross that tiles the plane.

## 2 Hypercube Unfoldings that Tile $\mathbb{R}^3$

So far as we are aware, there are now 4 hypercube unfoldings that are known to tile space. The first two were found by Steven Stadnicki [Sta15] in response to the question raised in [O'R15c]. We call the first of Stadnicki's unfoldings the  $L$ -unfolding. We describe this in detail for it is the unfolding we use to answer Question 1.

### 2.1 The Hypercube $L$ -unfolding tiles $\mathbb{R}^3$

The  $L$ -unfolding is shown in Figure 2. (The labels will not be used until Section 3.) Stadnicki showed this leads to a particularly simple tiling of space,

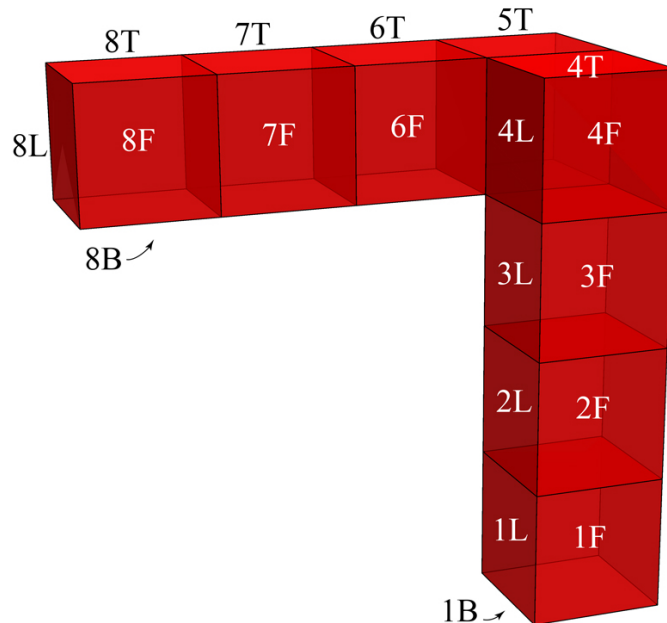


Figure 2: The  $L$ -unfolding of the hypercube. Some face labels are shown.

because nestling one  $L$  inside another as shown in Figure 3 leads to a 2-cube thick infinite slab, as illustrated in Figure 4. Then of course all of  $\mathbb{R}^3$  can be tiled by stacking the 2-cube thick slabs. We will return to edge-unfolding the  $L$  in Section .

Stadnicki showed that a second unfolding (Figure 5) also tiles space [Sta15], via a slightly more complicated but still simple structure. We will not describe that tiling.

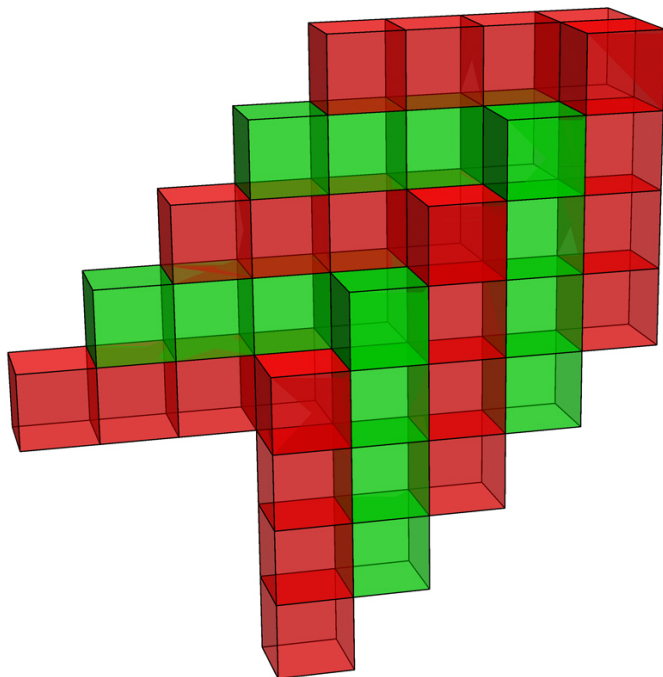


Figure 3: Five nested  $L$ 's.

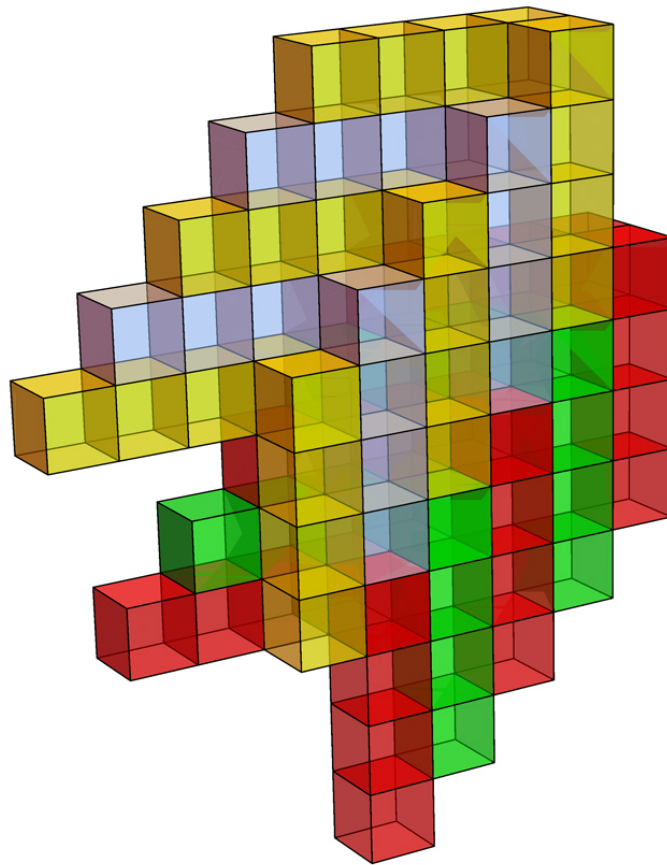


Figure 4: Ten nested  $L$ 's. Note the evolving structure is two-cubes thick in depth.

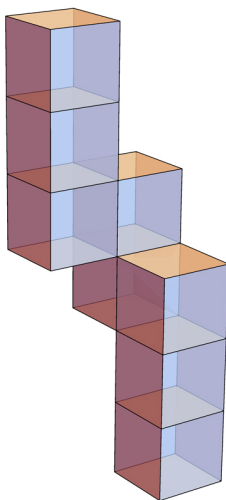


Figure 5: Another hypercube unfolding that tiles  $\mathbb{R}^3$  (Stadnicki).

## 2.2 The Dali Cross Unfolding tiles $\mathbb{R}^3$

Recall the Dali cross consists of four cubes in a tower, with the third tower-cube surrounded by four more; see Figure 6. (Again the labels will not be used until Section 3.)

Our proof that this shape tiles  $\mathbb{R}^3$  is in six steps:

1. 2-cross unit.
2. Cross-strip.
3. Cross-layer.
4. Two cross-layers.
5. Three cross-layers.
6. Four cross-layers.

### 2.3 2-Cross Unit

We first build a 2-cross unit with prone, opposing crosses, as illustrated in Figure 7. We will call planes of possible cube locations  $z$ -layers  $1, 2, 3, \dots$ , corresponding to  $z$ -height. The 2-cross unit has two cubes in  $z$ -layers 1 and 3, in the same  $xy$ -locations, and the remaining cubes in  $z$ -layer 2. It will be convenient to use *bump* to indicate a cube protruding above a particular layer of interest, and use *hole* to indicate a cube cell as-yet unoccupied by a cube.

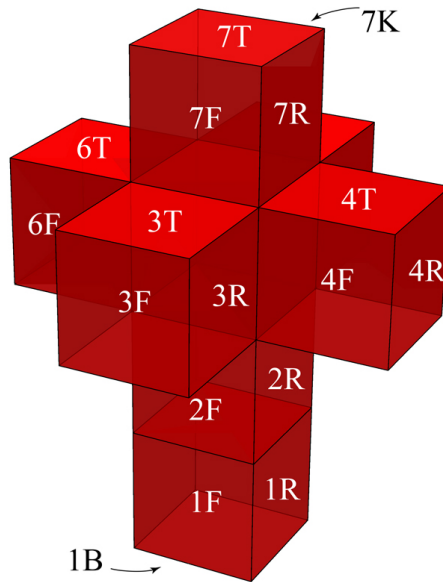


Figure 6: The Dali cross. Some face labels are shown.

## 2.4 Cross-strip

Now we form a vertical strip of 2-cross units as shown in Figure 8. Here we introduce a convention of displaying the construction by using colors and  $z$ -layer numbers. So the cubes in a cross-strip occupy  $z$ -layers 1, 2, 3, but only  $z$ -layers 2 and 3 are visible from above in an overhead view.

## 2.5 Cross-layer

Now we place cross-strips adjacent to one another horizontally, as shown in Figure 9. The remaining steps stack cross-layers one on top of the other. So the pattern of holes and bumps in each cross-layer will be important.

## 2.6 Two Cross-Layers

Henceforth we color all cubes in one cross-layer the same primary color, with the bumps slightly darker, as in Figure 10(a). Remember the bumps in one cross-layer align vertically. Now we place a second cross-layer on top of the first, with the bumps in the second cross-layer fitting into the holes of the first. Figure 10(b) shows the top view, which will be our focus. Note that now we see cubes at  $z$ -layers 2, 3, 4. That there are no holes all the way through; rather,  $z$ -layer-2 cells are dents and  $z$ -layer-4 cells bumps. We ask the reader to concentrate on the pattern depicted in Figure 11: in two adjacent columns, we see  $(4, 3, 3, 3, 4)$  and  $(2, 3, 3, 3, 2)$ , with the latter pattern shifted diagonally

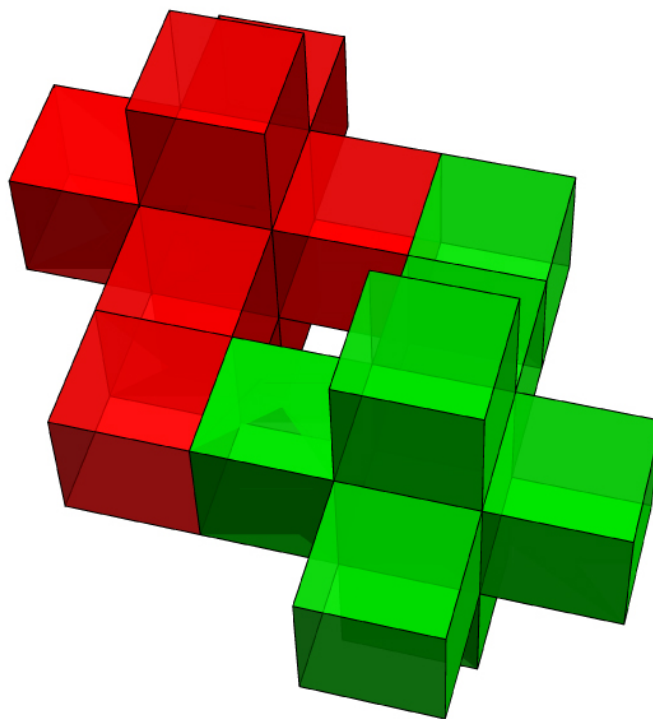


Figure 7: A 2-cross unit.



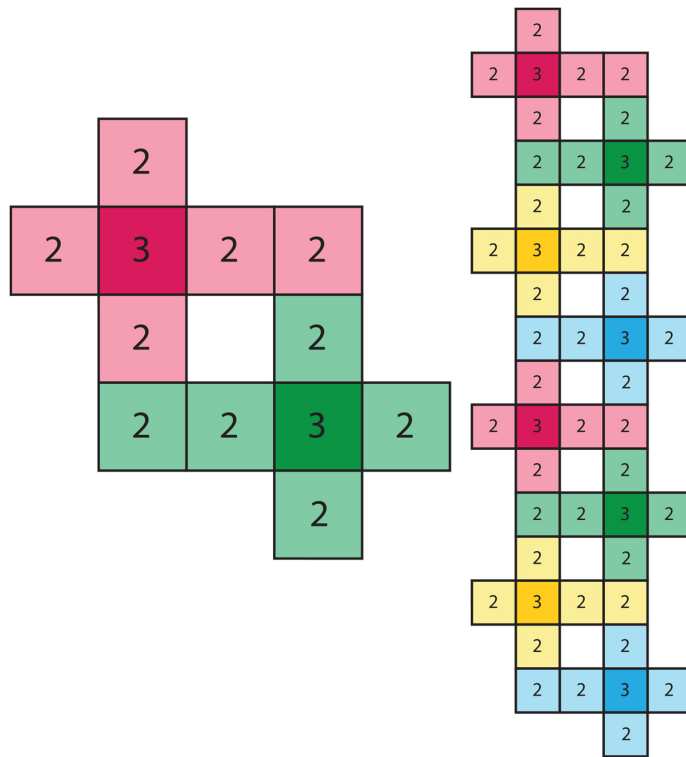


Figure 8: Cross-strip.

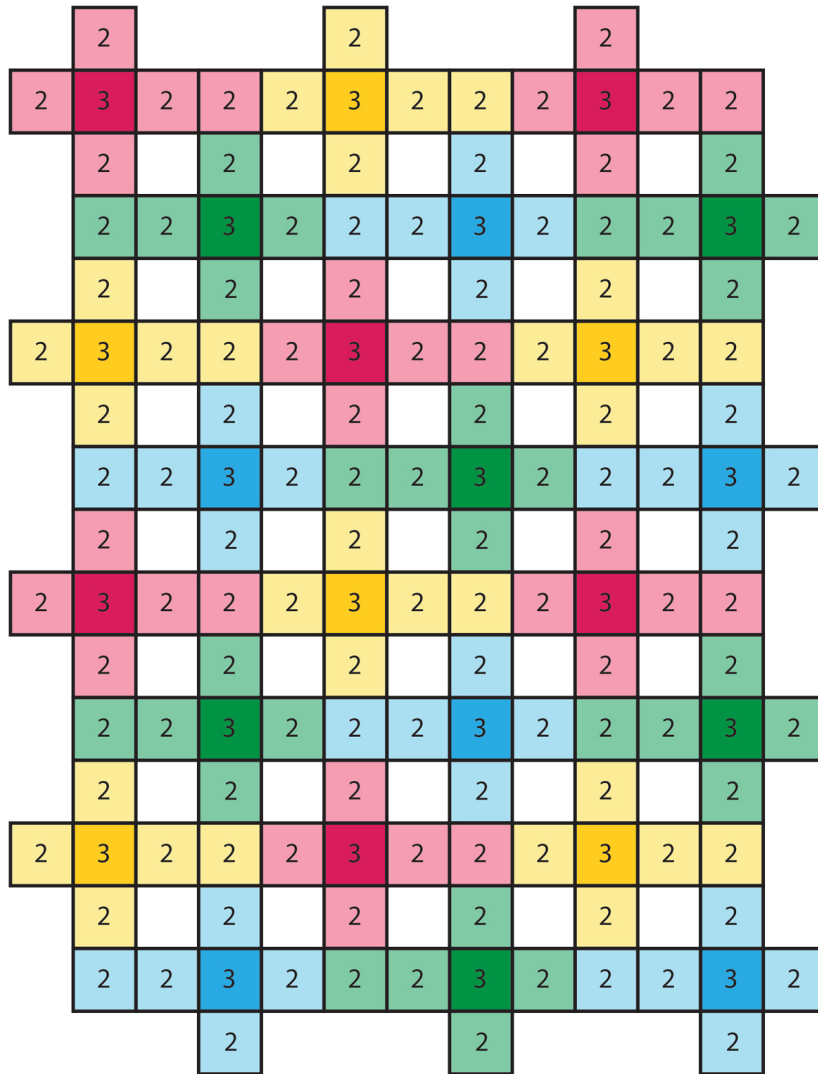


Figure 9: Cross-layer.

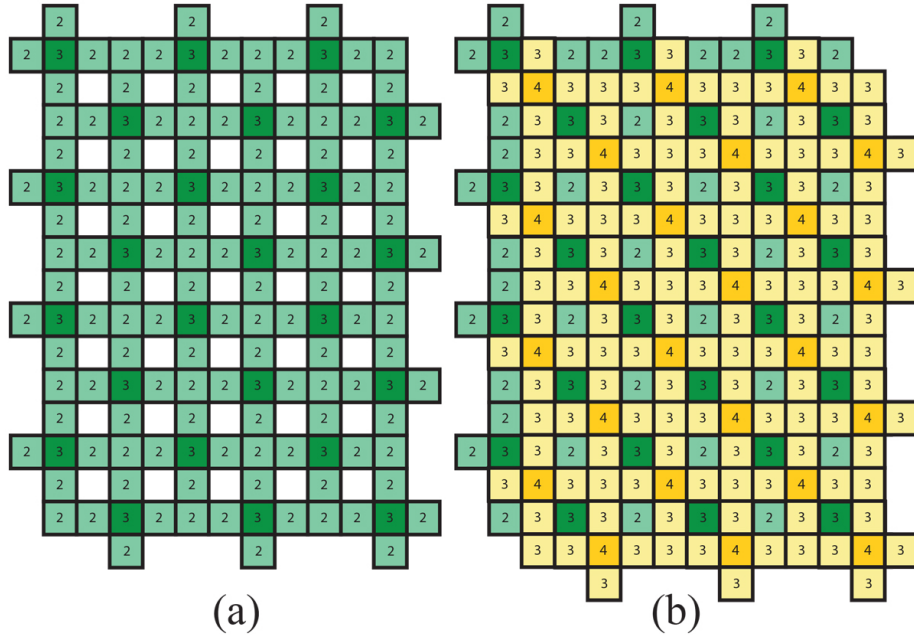


Figure 10: Two cross-layers. (a) One cross-layer. (b) Two cross-layers.

upward one unit. It should be clear that the entire overhead  $z$ -layer-view is composed of copies of this *fundamental layer-pattern*.

## 2.7 Three Cross-Layers

When we stack a third cross-layer on the construction, again inserting bumps into dents, we do not quite regain the fundamental layer-pattern. Instead we see that pattern shifted diagonally downward rather than upward; see Figure 12. Although we could argue that now we see a reflection (over a horizontal) of the full pattern of visible  $z$ -layer numbers, it seems easier and more convincing to us to add one more cross-layer.

## 2.8 Four Cross-Layers

With the addition of the fourth cross-layer (Figure 13), we regain the exact same pattern of  $z$ -layer numbers. Note the fundamental layer-pattern is now  $(6, 5, 5, 5, 6)$  and  $(4, 5, 5, 5, 4)$ , exactly  $+2$  of the pattern in two cross-layers, as emphasized in Figure 14.

It is now clear that because we have regained at four cross-layers the exact same “ $z$ -layer landscape” as we had at two cross-layers, the stacking can be continued indefinitely.

**Theorem 1** *The Dali cross unfolding of the hypercube tiles  $\mathbb{R}^3$  monohedrally.*



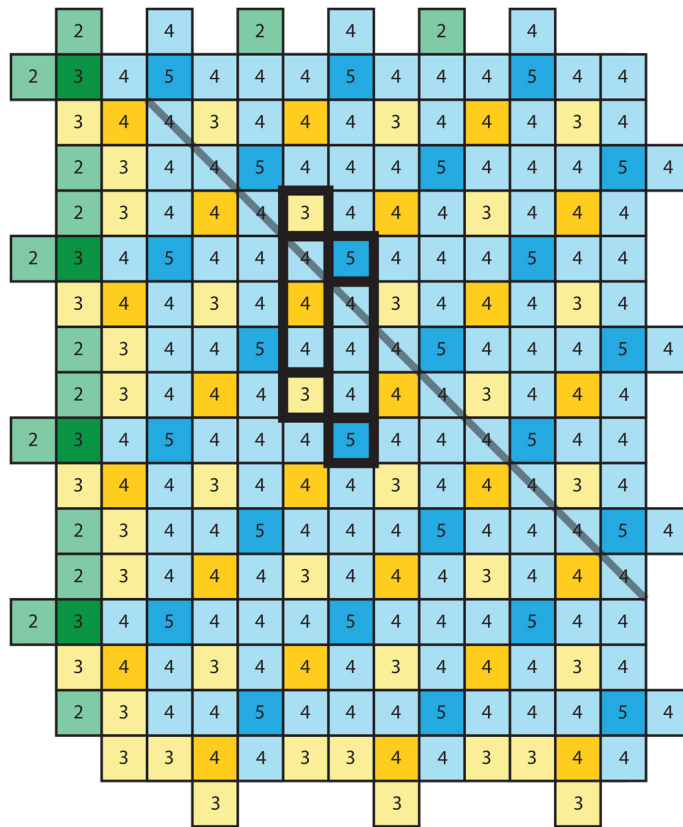


Figure 12: Three cross-layers and a reflected pattern.

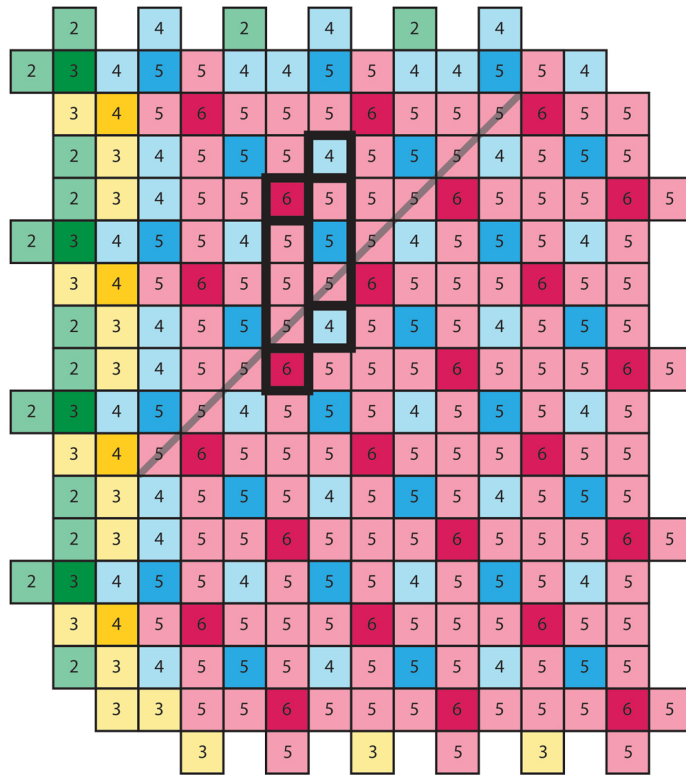


Figure 13: Four cross-layers.

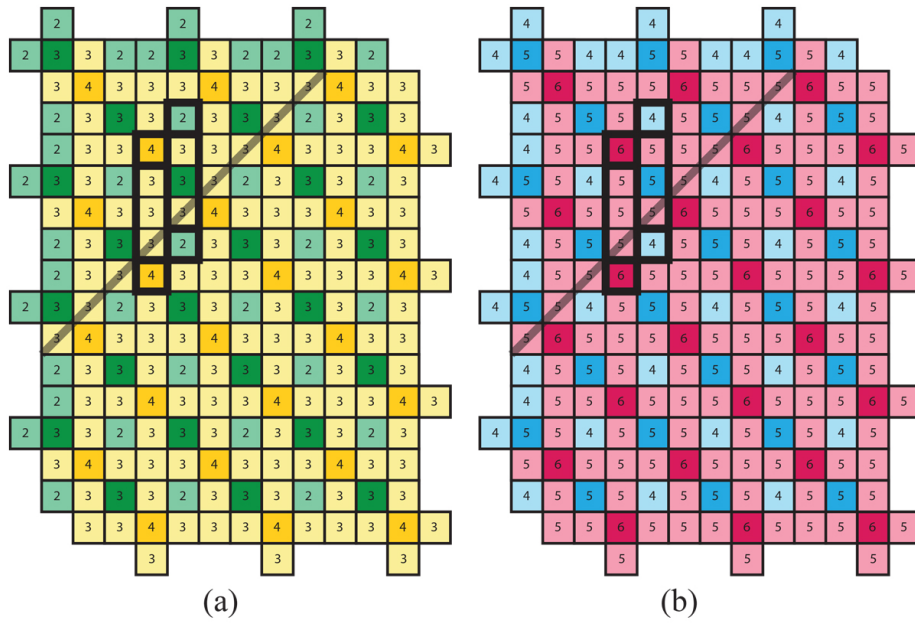


Figure 14: Two cross-layers (a) compared to four cross-layers (b), with the same fundamental pattern indicated.

We have found another hypercube unfolding, shown in Figure 15, that tiles  $\mathbb{R}^3$  in a similar manner, not described here.

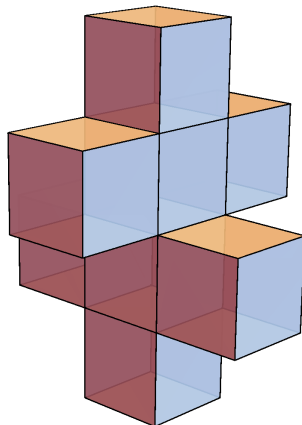


Figure 15: Another hypercube unfolding that tiles  $\mathbb{R}^3$ .

### 3 Edge-unfoldings to tile $\mathbb{R}^2$

Now we turn to unfolding the  $L$  to tile the plane. We label the cubes from 1 to 8, and the faces as  $\{F, L, K, R, B, T\}$  for  $\{\text{Front, Left, back, Right, Bottom, Top}\}$  respectively. Refer again to Figure 2. There are 34 exposed faces of the 8 cubes. Through a mixture of heuristic computer searches and hand tinkering, we found the unfolding shown in Figure 16.

That this tiles the plane (by translation only) is demonstrated in Figure 17. This then establishes our answer to Question 1:

**Theorem 2** *The  $L$ -unfolding of the hypercube has an edge-unfolding that tiles the plane, establishing that the hypercube is a dimension-descending tiler.*

#### 3.1 Edge-unfoldings of the Dali cross

There are a huge number of edge-unfoldings of each hypercube unfolding. Each edge-unfolding corresponds to a spanning tree of the dual graph, where each square face is a node, and arcs represent uncut edges. There are at most approximately  $5^n$  spanning trees [Rot05] of planar graphs with  $n$  nodes, and asymptotically that many for some graphs. It seems conservative to estimate that the dual graph of the Dali cross has at least  $2^n = 2^{34} \approx 10^{10}$  spanning trees, and more likely  $3^{34} \approx 10^{16}$ . (The square grid has  $3.2^n$  spanning trees, and each hypercube unfolding dual graph is also regular of degree 4.) Each of these trees leads to an unfolding, but many self-overlap in their planar layout, and even among those that avoid overlap, many delimit a region with holes, and so could



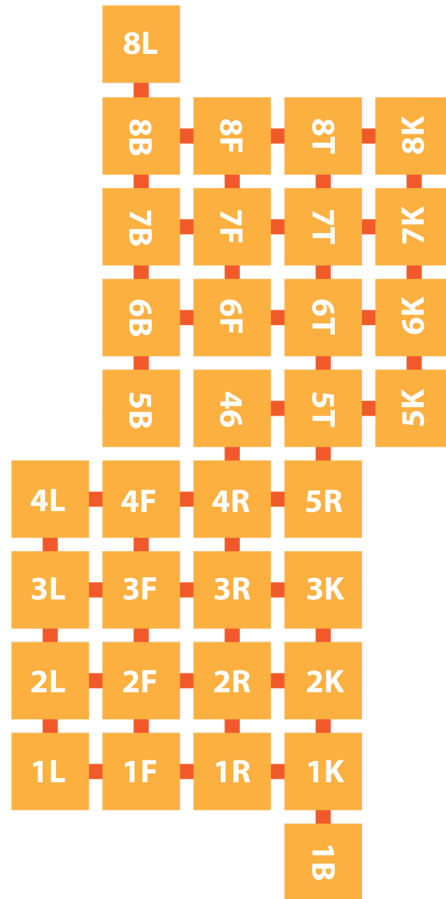


Figure 16: Unfolding of the  $L$  (Figure 2), with face labels and dual-tree (uncut) connections.

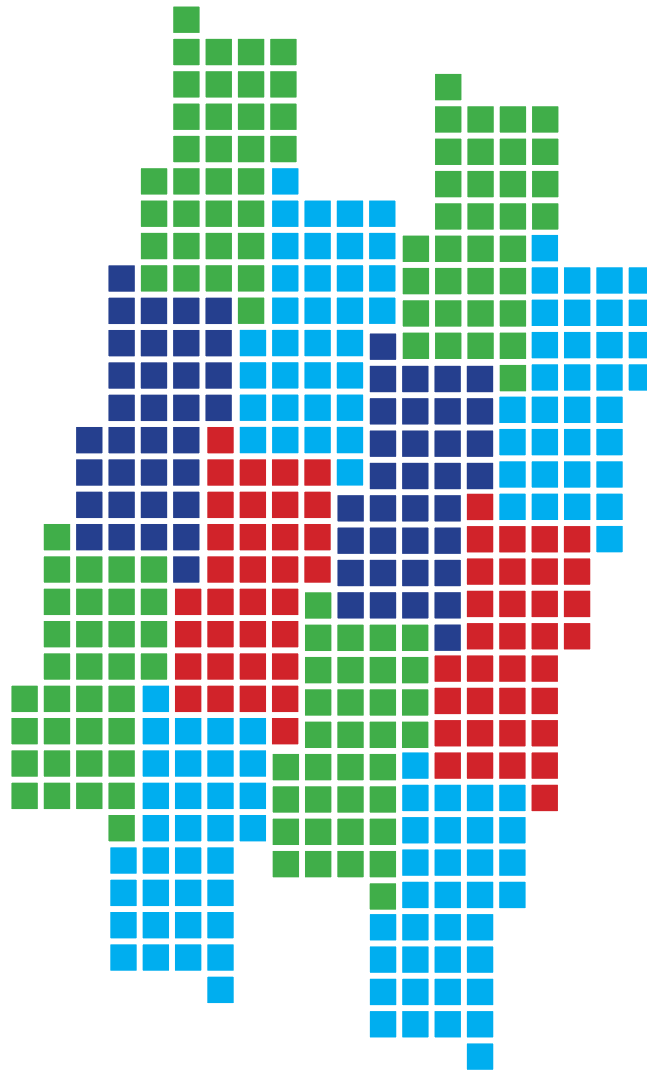


Figure 17: Tiling of the plane with the unfolding shown in Figure 16.

not form tilers. With brute-force search infeasible, and no algorithm available, we are left only with heuristics, with which we have not been successful.

Figure 18 shows the closest to a tiling unfolding of the Dali cross that we found.

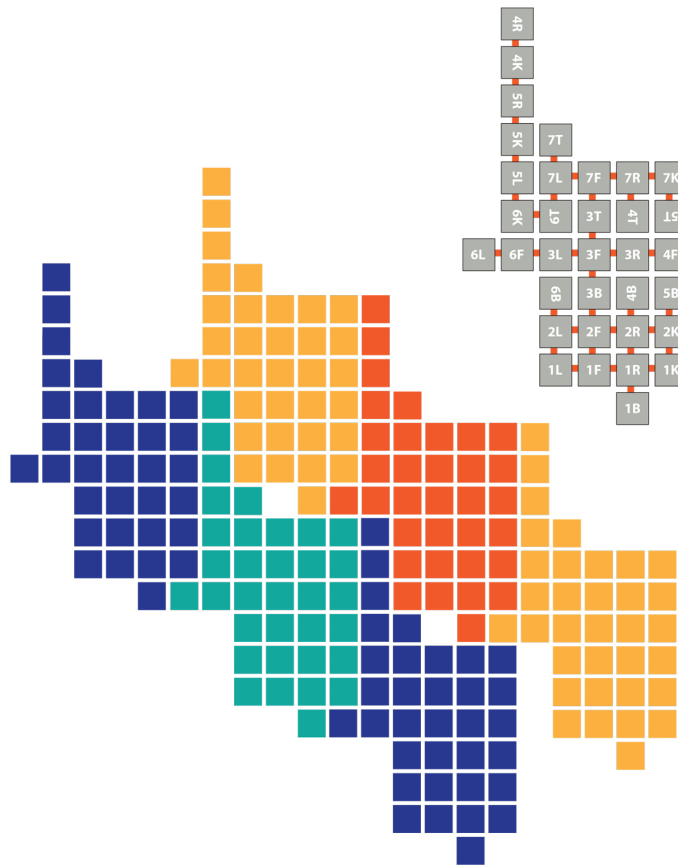


Figure 18: An edge-unfolding of the Dali cross that nearly tiles the plane. (See Figure 6 for labels.)

## 4 Open Problems

1. Is the 5-dimensional cube in  $\mathbb{R}^5$  a dimension-descending tiler?
2. What are good heuristics to test if the remaining  $257^2$  hypercube unfoldings tile  $\mathbb{R}^3$ ?
3. Can any of the hypercube unfoldings be proved not to tile  $\mathbb{R}^3$ ?

---

<sup>2</sup>  $261 - 4$ , because 4 are known to tile: Figures 2, 5, 6, 15.

4. Does the Dali cross have an unfolding that tiles  $\mathbb{R}^2$ ?

**Addendum.** We learned after posting this note that polyhedra that have an edge-unfolding that tiles the plane are called *tessellation polyhedra* in [AKL<sup>+</sup>11].

## References

- [Aki07] Jin Akiyama. Tile-makers and semi-tile-makers. *The American Mathematical Monthly*, 114(7):602, 2007.
- [AKL<sup>+</sup>11] Jin Akiyama, Takayasu Kuwata, Stefan Langerman, Kenji Okawa, Ikuro Sato, and Geoffrey C Shephard. Determination of all tessellation polyhedra with regular polygonal faces. In *Computational Geometry, Graphs and Applications*, pages 1–11. Springer, 2011.
- [Kon15] Sergei Petrovich Konovalov. Cubist parquet, 2015. <http://www.etudes.ru/ru/etudes/cubisme/>. In Russian.
- [McC15] Mark McClure. 3D models of the unfoldings of the hypercube? MathOverflow, March 2015. <http://mathoverflow.net/q/199003>.
- [O’R15a] Joseph O’Rourke. 3D models of the unfoldings of the hypercube? MathOverflow, March 2015. <http://mathoverflow.net/q/198722>.
- [O’R15b] Joseph O’Rourke. Is it decidable to determine if a given shape can tile the plane? Theoretical Computer Science Stack Exchange, September 2015. <http://cstheory.stackexchange.com/q/32538>.
- [O’R15c] Joseph O’Rourke. Which unfoldings of the hypercube tile 3-space: How to check for isometric space-fillers? MathOverflow, March 2015. <http://mathoverflow.net/q/199097>.
- [Rot05] Günter Rote. The number of spanning trees in a planar graph. *Oberwolfach Reports*, 2:969–973, 2005.
- [Sch80] Doris Schattschneider. Will it tile? Try the Conway criterion! *Mathematics Magazine*, pages 224–233, 1980.
- [Sta15] Steven Stadnicki. Which unfoldings of the hypercube tile 3-space: How to check for isometric space-fillers? MathOverflow, March 2015. <http://mathoverflow.net/q/199117>.
- [Tur84] Peter D Turney. Unfolding the tesseract. *J Recreational Math*, 17:1–16, 1984.