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On coloring box graphs

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ABSTRACT

We consider the chromatic number of a family of graphs we call box graphs, which arise from a box complex in n-space. It is straightforward to show that any box graph in the plane has an admissible coloring with three colors, and that any box graph in n-space has an admissible coloring with $n+1$ colors. We show that for box graphs in n-space, if the lengths of the boxes in the corresponding box complex take on no more than two values from the set $\{1, 2, 3\}$, then the box graph is 3-colorable, and for some graphs three colors are required. We also show that box graphs in 3-space which do not have cycles of length four (which we call "string complexes") are 3-colorable.

1. Introduction and results

There are many geometrically-defined graphs whose chromatic numbers have been studied. Perhaps the most famous such example is the Four Color Theorem, which states that any planar graph is 4-colorable [1]. Another famous example is the chromatic number of the plane. More specifically, a graph $G = (V, E)$ is defined where $V = \mathbb{R}^2$ and $(x, y) \in E$ precisely when $\|x - y\|_2 = 1$ (where $\|\cdot\|_2$ is the usual Euclidean norm in the plane). Through simple geometric constructions, one can show that $4 \le \chi(G) \le 7$ for this graph, although the precise value is still not known; see [8], for example.

In this article, we consider graphs that arise from box complexes. We first define what a box complex is:

Definition 1. An n-dimensional *box* is a set $B \subset \mathbb{R}^n$ that can be defined as:

$$B = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : a_i \le x_i \le b_i\}$$

where $a_i < b_i$ for $i = 1, 2, \dots, n$.

An n-dimensional *box complex* is a set of finitely many n-dimensional boxes $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ such that if the intersection of two boxes $B_i \cap B_j$ is nonempty, then $B_i \cap B_j$ is a face (of any dimension) of both B_i and B_j, for any i and j (see Fig. 1).

Now we can define a box graph:

Definition 2. An n-dimensional *box graph* is a graph defined on an n-dimensional box complex. The box graph $G(\mathcal{B}) = (V, E)$ defined on the box complex $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ is the undirected graph whose vertex set is the boxes:

$$V = \{B_1, B_2, \dots, B_m\}$$

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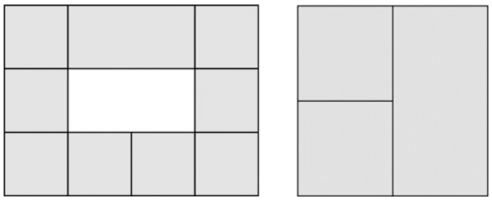
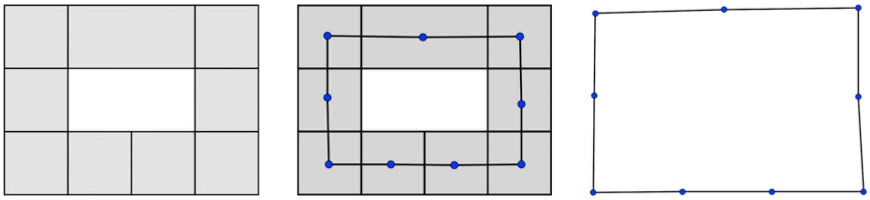


(a) A box complex. (b) Not a box complex.

Fig. 1. Examples in $\mathbb{R}^2$.



(a) A box complex. (b) The box graph inside the box complex. (c) Just the box graph.

Fig. 2. Defining a 2-dimensional box graph.

and whose edges $(B_i, B_j) \in E$ record when $B_i \cap B_j$ is an $(n-1)$-dimensional face of both B_i and B_j. In other words, the box graph is the dual graph of the box complex, and the colorings we are considering are in some sense "solid colorings."

When it eases understanding, we may use the terms box complex and box graph interchangeably. We also may use boxes and vertices interchangeably.

The following proposition shows that, as far as the corresponding box graphs are concerned, we may as well restrict ourselves to box complexes where each of the vertices of the boxes has integer coordinates (and thus all boxes have integer lengths).

Proposition 1. *Let $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ be a box complex and let $G(\mathcal{B}) = (V, E)$ be its corresponding box graph. There exists a box complex $\{C_1, C_2, \ldots, C_m\}$ where the vertices of each C_i $(i = 1, 2, \ldots, m)$ have all integer coordinates, such that the box graph corresponding to complex $\{C_1, C_2, \ldots, C_m\}$ is the same graph G.*

We will prove Proposition 1 in Section 2.

We ask the following natural question:

Question 1. What is the minimum number of colors k that are required so that every n-dimensional box graph has an admissible k-coloring?

From Fig. 2(c), we can see that three colors may be necessary to color a 2-dimensional box graph. In fact, as we will prove in Section 2, three colors are also sufficient:

Proposition 2. *Any box graph in n-space has an admissible coloring with $n+1$ colors.*

Our goal is to answer Question 1 in dimension 3, which is still open. In the case where the "boxes" are zonotopes (as opposed to right-angled bricks), sometimes 4 colors are needed [4], and in the case where the "boxes" are now touching spheres, the chromatic number is between 5 and 13 [2]. Analogously, for simplicial complexes in $\mathbb{R}^n$, $n+1$ colors suffice [6]. We suspect that any 3-dimensional box graph is 3-colorable, and we can show that this is true for a few families of 3-dimensional box graphs. The following are the main results of this paper:

Theorem 1. *Let G be an n-dimensional box graph such that the lengths of all of the boxes in the corresponding box complex take on no more than two values from the set $\{1, 2, 3\}$. That is, all the side lengths of the boxes are 1 or 2, or all the side lengths are 1 or 3, or all the side lengths are 2 or 3. Then G is 3-colorable.*

Theorem 2. *Let G be a 3-dimensional box graph that has no cycles on four vertices. Then G is 3-colorable.*

The rest of this paper is organized as follows: in Section 2 we will state and prove some straightforward results on box graphs. We will prove Theorem 1 in Section 3, and we will prove Theorem 2 in Section 4.

2. Straightforward results on box graphs

As promised, we will start with proofs of Propositions 1 and 2.

Proof of Proposition 1. Suppose $\{B_1, B_2, \ldots, B_m\}$ is a box complex in $\mathbb{R}^n$, so that each vertex of each box has n coordinates. Let $x_0, x_1, \ldots, x_k$ be the list of *all* of the different first coordinates of all of the vertices of the boxes in the box complex. Order them so that

$$x_0 < x_1 < \cdots < x_k.$$

Now make a new box complex $\{B_1^1, B_2^1, \ldots, B_m^1\}$ such that the vertices are all the same except the first coordinates. Specifically, if the first coordinate of a vertex in B_j is x_i, then the first coordinate of the corresponding vertex in B_j^1 is the integer i. Thus, the vertex $(x_i, y_2, y_3, \ldots, y_n)$ of B_j becomes the vertex $(i, y_2, y_3, \ldots, y_n)$ of B_j^1.

Note that each B_i^1 is still a box, and this does not change the intersection pattern of the boxes. That is, if $B_j \cap B_\ell$ is d-dimensional, then so is $B_j^1 \cap B_\ell^1$. (And if $B_j \cap B_\ell$ was empty, then so is $B_j^1 \cap B_\ell^1$.)

We continue with this process for the 2nd, 3rd, ..., nth coordinates. Finally, we get a box complex $\{B_1^n, B_2^n, \ldots, B_m^n\}$ with the same intersection pattern as $B_1, B_2, \ldots, B_m$ but with all integer coordinates for the vertices. Thus, the box graph for complex $\{B_1^n, B_2^n, \ldots, B_m^n\}$ is the same as the box graph for complex $\{B_1, B_2, \ldots, B_m\}$. □

In order to prove Proposition 2 we first give the definition of k-degenerate graphs, and show the well-known result that k-degenerate graphs are $k+1$-colorable [5].

Definition 3. A graph G is *k-degenerate* if each of its induced subgraphs has a vertex of degree k.

Lemma 1. *Every k-degenerate graph is $k+1$-colorable.*

Proof. Let $G = (V, E)$ be a k-degenerate graph. We will proceed by induction on $|V|$, the size of the vertex set. If $|V| = 1$ then certainly G is k-colorable for any $k \geq 1$. Now, suppose that $|V| = m \geq 2$, and assume as the induction hypothesis that any k-degenerate graph on $m-1$ vertices is $k+1$-colorable.

Then, since G is k-degenerate we know there exists a vertex $v \in V$ with $\deg(v) = k$. Consider the graph $G - v$, formed by removing vertex v and all of its incident edges, with $m-1$ vertices. This graph must be k-degenerate since it is an induced subgraph of G. Therefore, by the induction hypothesis we can color $G - v$ using $k+1$ colors. Now, when v and its edges are added back in to form G we must have at least one available color since v has only k neighbors and there are $k+1$ total colors. Therefore, by induction, any k-degenerate graph is $k+1$-colorable. □

We now prove Proposition 2 by showing that any box graph is n-degenerate.

Proof of Proposition 2. Let $G = (V, E)$ be a box graph, so that each $v \in V$ is a box in the corresponding box complex. We will label each box in V by its "right, forward, top" vertex. More precisely, each box can be defined as

$$\{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}$$

where $a_i < b_i$ for $i = 1, 2, \ldots, n$. We then label this box with $(b_1, b_2, \ldots, b_n)$.

Now find a "right, forward, top" box in the graph. That is, find a vertex $u \in V$ with corresponding label $(u_1, u_2, \ldots, u_n)$ such that for any other $v \in V$ with label $(v_1, v_2, \ldots, v_n)$ and $(u, v) \in E$, we have

$$u_1 \geq v_1,\ u_2 \geq v_2, \ldots, u_n \geq v_n.$$

(Such a box is guaranteed to exist because G is finite.) Note that, by our choice of u, u has at most n neighbors.

Since we began with an arbitrary box graph, the existence of a degree n vertex must be true for all induced subgraphs of G. Therefore, any box graph corresponding to a box complex in $\mathbb{R}^n$ is n-degenerate, and by Lemma 1 is $n+1$ colorable. □

We note that the above argument is the n-dimensional analogue to the "elbow" argument in [7].

We state the following result as a reminder to the reader:

Proposition 3. *Let $G = (V, E)$ be a graph. Then the following are equivalent:*

1. *The graph G contains no odd cycle.*
2. *The graph G is bipartite.*
3. *The graph G is 2-colorable.*

Proof. Proposition 3 is a well-known introductory graph theory result. See Section I.2 of [3], for example. □

The following proposition shows that if a box graph cannot be colored with just 2 colors, it must have some boxes with side lengths that are different from each other.

Proposition 4. *Suppose a box complex only contains boxes that are cubes; that is, boxes with all side lengths equal. Then the corresponding box graph is 2-colorable.*

Proof. Suppose a box complex contains only cubes, and let $G = (V, E)$ be the corresponding box graph. Without loss of generality, we may assume that G is connected. Thus, since all of the boxes in the corresponding box complex are cubes, they must all be cubes of the same size; let the side length of the cubes be k. By the proof of Proposition 1, we can assume that $k \in \mathbb{N}$ and the coordinates of all the vertices of the boxes in the box complex are integer multiples of k.

Just as we did in the proof of Proposition 2, label each $v \in V$ with the "right, forward, top" vertex. Let $(v_1, v_2, \ldots, v_n)$ be the label for vertex v. Color vertex v with color

$$\frac{1}{k}\left(v_1 + v_2 + \cdots v_n\right) \pmod 2.$$

Note that exactly two colors are used. If two vertices are adjacent: $(u, v) \in E$, then we know that their corresponding labels $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ must be the same in every coordinate except one, in which they differ by k. That is, there exists $i \in \{1, 2, \ldots, n\}$ such that

$$u_j = v_j \quad \text{if } j \in \{1, 2, \ldots, n\} \text{ and } j \neq i$$
$$u_i = v_i \pm k.$$

Thus, if two vertices are adjacent then their colors must be different. Thus, this is a valid 2-coloring of G. □

In [4] it was proved that any box complex in $\mathbb{R}^3$ that is homeomorphic to a ball is 2-colorable.

3. Proof of Theorem 1

We shall prove Theorem 1 in parts via a few lemmas. Here is the first of our lemmas:

Lemma 2. *Suppose that each side length of each box in a box complex is a positive integer which is congruent to either* 1 *or* 2 *mod* 3*. Then the corresponding box graph is* 3*-colorable.*

Proof. Consider an n-dimensional box complex $\{B_1, B_2, \ldots, B_m\}$, and label each box again by its "right, forward, top" vertex coordinates, $(b_1, b_2, \ldots, b_n)$. Now, color each box by $(b_1 + b_2 + \cdots + b_n)$ mod 3. We claim that this is a valid coloring.

If two boxes, B_i, B_j are adjacent then their right, forward, top vertices will differ in exactly one coordinate. Let $(b_{i,1}, b_{i,2}, \ldots, b_{i,n})$ be the label for B_i and $(b_{j,1}, b_{j,2}, \ldots, b_{j,n})$ the label for B_j. Then, WLOG, $b_{i,1} \neq b_{j,1}$ and $b_{i,k} = b_{j,k}$ for $k = 2, 3, \ldots, n$. These two boxes will have the same color iff $b_{i,1} - b_{j,1} \equiv 0$ mod 3. However, this value is the side length of one of these boxes which we have restricted to not equal any multiple of 3. Therefore neighboring boxes may not have the same color, so this 3-coloring is admissible. □

The following corollary follows directly from Lemma 2:

Corollary 1. *Suppose a box complex in* $\mathbb{R}^n$ *has boxes with side lengths only equal to* 1 *or* 2*. Then the corresponding box graph is* 3*-colorable.*

The next in our series of lemmas:

Lemma 3. *Suppose that each side length of each box in a box complex is an odd integer. Then the corresponding box graph is* 2*-colorable.*

Proof. We will prove this by showing that there can be no odd cycles in the graph (see Proposition 3).

Assume we have a box complex $\mathcal{B} = \{B_1, \ldots, B_k\}$. Consider any cycle within the corresponding box graph. Label the vertices of this cycle by the "right, forward, top" corner of the corresponding box, and label each of the edges of the cycle with the distances between those corners, mod 2. In other words, if the neighboring vertices are labeled $(1, 1, \ldots, 1)$ and $(4, 1, \ldots, 1)$ then we label the edge with $3 \bmod 2 = 1$. Moreover, we will choose a direction of travel around the cycle and sign the length of the edge positive if we are moving along that edge in the positive direction, and negative if we move along the edge in the negative direction. Thus, for example, if we move from vertex $(1, 1, \ldots, 1)$ to $(4, 1, \ldots, 1)$, the edge is labeled with 1 since moving from 1 to 4 is in the positive direction in the first coordinate, whereas if we move from vertex $(4, 1, \ldots, 1)$ to $(1, 1, \ldots, 1)$, the edge is labeled with -1.

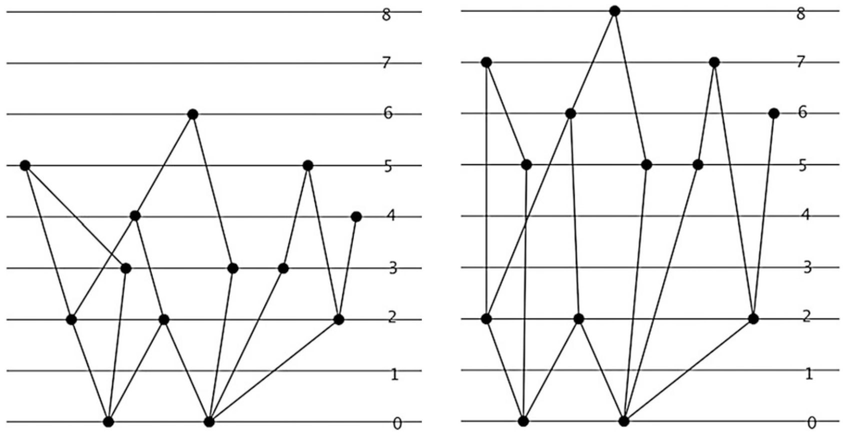
We now claim that the sum of the integers along the cycle must be 0. This is because in each dimension, any length we move in the positive direction must be traveled again in the negative direction, and therefore their parity must also be equal.

Finally, we note that, by assumption, all of the lengths are odd. Thus, all edge labels must be either 1 or -1. Since we have a list of edges labeled 1 or -1 and the sum of the labels is 0, there must be an even number of edges in the cycle. □

The following corollary follows directly from Lemma 3:

Corollary 2. *Suppose a box complex in* $\mathbb{R}^n$ *has boxes with side lengths only equal to* 1 *or* 3*. Then the corresponding box graph is* 3*-colorable.*

The proof for Theorem 1 when blocks have dimensions 2 or 3, given in the remainder of this section, relies on placing a partial order on the box graph corresponding to a given box complex. The elements of the partially ordered set (poset) are the vertices of the box graph, i.e., the individual boxes that comprise the box complex. As before, we label box $\{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}$ by its "right, forward, top" vertex coordinates, $(b_1, b_2, \ldots, b_n)$. The order relation for this poset is induced by the following cover relation: box B_i with label $(b_1, b_2, \ldots, b_n)$ covers box B_j with label



(a) A part of a poset corresponding to a box complex. (b) The poset stretched, according to T.

Fig. 3. All edges above the ones drawn do not change in length after T is applied.

$(c_1, c_2, \ldots, c_n)$ if and only if the two boxes are adjacent and $\sum_{k=1}^{n} b_k \geq \sum_{k=1}^{n} c_k$. Since these adjacent boxes must share an $(n-1)$-dimensional face, their labels will differ in exactly one coordinate, by a difference equal to the dimension of box B_i orthogonal to shared face $B_i \cap B_j$.

We note further that the sum $r(B_i) = \sum_{k=1}^{n} b_k$ of the entries of the label of a given box is a rank function for this poset. We will use the rank function and the poset structure to describe valid colorings of the box graph. This technique will consider an initial drawing of the poset (and subsequent re-drawings) with all nodes at integer heights. We then refer to the *length* of an edge in the poset as the positive vertical distance between its endpoints.

Here is the last of the lemmas that we will need for Theorem 1:

Lemma 4. *Suppose a box complex has boxes with side lengths only equal to* 2 *or* 3*. Then the corresponding box graph is* 3*-colorable.*

Proof. Consider now the case in which all dimensions of the boxes in a box complex $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ are 2 or 3. We produce the associated poset $\mathcal{P}$ described above, and make an initial drawing of P with nodes having heights corresponding to their ranks. Note that this implies that if two boxes B_i and B_j which are adjacent in the box graph are drawn with heights h_i and h_j respectively, then $r(B_i) - r(B_j) = h_i - h_j$, and $h_i - h_j$ is either 2 or 3 if $h_i > h_j$. In other words, all lengths of the edges in the poset are either 2 or 3. Without loss of generality, we can make this drawing so that all rank-minimal vertices have height h-value of 0. We now describe how to redraw the poset $\mathcal{P}$ in such a way that all adjacencies and cover relations are preserved, but all edges have lengths equivalent to 1 or 2 mod 3.

We now consider the lengths of edges in the poset, working our way in order of increasing height h of the terminal endpoints. Since the first nodes occur on the line $h = 0$ and all edges have length 2 or 3, no edges terminate on $h = 1$, and edges that terminate on $h = 2$ have length 2, which is among the desired values. Edges terminating on $h = 3$ or above may have length 2 or length 3. We perform the following transformation on the drawing of the poset. Let h_i denote the height of vertex B_i in the initial drawing of the poset. We perform transformation T below to the drawing of the poset:

$$T(h_i) = \begin{cases} h_i & \text{if } h_i \leq 2, \\ h_i + 2 & \text{if } h_i \geq 3. \end{cases}$$

Note that T has no effect on the length of edges terminating at or below $h = 2$, and no effect on the length of edges commencing at or above $h = 3$. For edges that include the interval [2, 3], two units are added to their length. In the new drawing of the poset, no edges will terminate on lines $h = 3$ or $h = 4$. Edges terminating on $h = 5$ were either originally of length 3 commencing from $t = 0$ or of length 2 commencing at $h = 1$. The former now have length 5, while the length of the latter is now 4. In either case, edges terminating on $h = 5$ have lengths equivalent to 1 or 2 mod 3. A similar argument shows that edges in the revised drawing that terminate on $h = 6$ or $h = 7$ are either of length 2, 4, or 5. (See Fig. 3.)

Any edges terminating on h-values of 8 or higher were not affected by the first stretch, and thus may have length 3. Continue the stretching/redrawing procedure as before, extending the interval [7, 8] by two units and redrawing the poset. This procedure only changes the lengths of edges which include the interval [7, 8], so in particular it does not change the lengths of any prior edges. Since our complex is finite, only finitely many re-drawings are needed to draw the poset with edges all having length equivalent to 1 or 2 mod 3. At that time, the nodes can be colored using the argument from Lemma 2. □

We can now finally prove Theorem 1:

Proof of Theorem 1. This is a direct consequence of Corollaries 1, 2, and Lemma 4. □

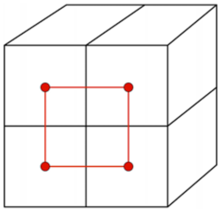


Fig. 4. This 2×2 pattern (a 4-cycle in the dual) is forbidden as part of a string complex.

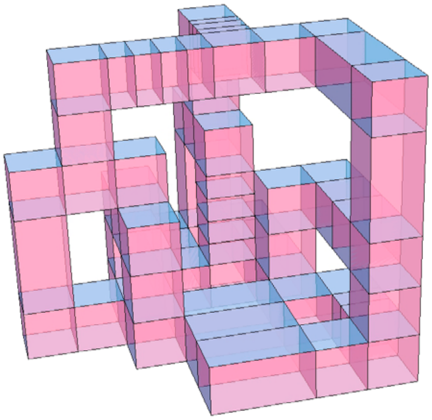


Fig. 5. An example of a string complex.

4. Proof of Theorem 2

First, a couple of definitions:

Definition 4. A *string complex* is any box complex in $\mathbb{R}^3$ that does not contain a 2×2 pattern of boxes shown in Fig. 4. The dual of the forbidden pattern is a 4-cycle, which is the shortest cycle possible in a box complex. So in other words, a string complex is a 3-dimensional box complex in whose corresponding graph has no 4-cycle (see Fig. 5).

We use the term "string complex" because, without the 2×2 pattern in Fig. 4, the box complex is forced to have lots of "holes" and be "stringy."

Definition 5. A 3-dimensional box complex $\{B_1, B_2, B_3, \dots, B_m\}$ is *reducible* to the 3-dimensional box complex $\{A_1, A_2, \dots, A_\ell\}$ ($\ell \le m$) if one can sequentially remove boxes from complex $\{B_1, B_2, \dots, B_m\}$ of degree ≤ 2 in order to obtain complex $\{A_1, A_2, \dots, A_\ell\}$. More specifically, there exists an ordering $B_1, B_2, \dots, B_m$ such that

$$B_i = A_i \quad \text{for } i = 1, 2, \dots, \ell$$

and for $j = 0, 1, 2, \dots, m - \ell - 1$, the box B_{m-j} has degree ≤ 2 in the box complex

$$\{B_1, B_2, \dots, B_{m-j}\}.$$

A box complex is *irreducible* if every vertex is of degree ≥ 3.

Note that a complex may be reducible to a smaller complex which is itself irreducible.
The following lemma is analogous to the tools we used in the proof of Proposition 2:

Lemma 5. *If a* 3*-dimensional box complex is reducible to the empty complex, then its corresponding box graph is* 3*-colorable.*

Proof. We prove by induction on m, the number of boxes in the box complex. Certainly if $m = 1$, the box graph is 3-colorable.
Suppose that $m \ge 2$, and that for any 3-dimensional box complex on $m - 1$ boxes which is reducible to the empty complex, the corresponding box graph is 3-colorable. Suppose that the box complex $\{B_1, B_2, \dots, B_m\}$ is reducible to the empty complex. That is, for $i = 1, 2, \dots, n$, the box B_i has degree ≤ 2 in the complex

$$\{B_1, B_2, \dots, B_i\}.$$

Note that the box complex $\{B_1, B_2, \dots, B_{m-1}\}$ is also reducible to the empty complex and has $m - 1$ boxes in it. Thus, by our inductive assumption, the corresponding graph is 3-colorable. Now, because B_m had degree ≤ 2 in the box complex $\{B_1, B_2, \dots, B_m\}$, we can choose to color B_m a color which is different from the colors of its neighbors. Thus, we have proven the lemma. $\square$

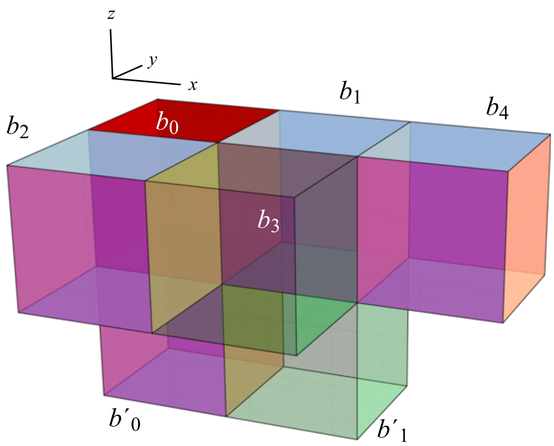


Fig. 6. b_0 is the topmost, leftmost box in the top layer T.

By Lemma 5, Theorem 2 is a direct corollary of the following theorem and its subsequent corollary:

Theorem 3. *Every string complex is reducible.*

Proof. Assume to the contrary. That is, let $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ be an irreducible string complex. We will show that irreducibility implies the complex must contain a 2×2 pattern of boxes, which contradicts the assumption that the complex is a string complex.

Let $T_1, T_2, \ldots, T_\ell$ be the top layer of boxes in $\mathcal{S}$; say the top faces lie in a plane parallel to the xy-plane, extreme in the $+z$ direction. We first claim that every box in $T_1, T_2, \ldots, T_\ell$ must have degree ≥ 2 *within the complex* $\mathcal{T} = \{T_1, T_2, \ldots, T_\ell\}$. Suppose otherwise. That is, suppose there is a box T_i with degree ≤ 1 within the box complex $\mathcal{T}$. Then T_i can have at most degree 2 in the complex $\mathcal{S}$ by joining to a box beneath it. But we know that every box in $\mathcal{S}$ must have degree ≥ 3, because the complex $\mathcal{S}$ was irreducible. Thus, it is indeed true that each T_i, $i = 1, 2, \ldots \ell$ has degree ≥ 2 in the complex $\mathcal{T}$.

Now we look at an extreme corner box of $T_1, T_2, \ldots, T_\ell$. Specifically, let b_0 be backmost (extreme in the $+y$ direction), and among the topmost boxes of $\mathcal{T}$, leftmost (extreme in the $-x$ direction). So b_0 is a type of "upper left corner". Because it is extreme in two directions, two of its faces in $\mathcal{T}$ are exposed, so it must have exactly degree 2 in $\mathcal{T}$. Because we assumed $\mathcal{S}$ is irreducible, b_0 (and indeed every box of $\mathcal{S}$) must have degree ≥ 3. So b_0 must be adjacent to a box b'_0 beneath it (beneath in the z-direction). See Fig. 6.

Let b_1 and b_2 be the boxes adjacent to b_0 in T, with b_1 adjacent to b_0 in the x-direction as in the figure. Again, by our previous arguments, b_1 must have degree ≥ 2 in $\mathcal{T}$. It is already adjacent to b_0 to its left, and it cannot be adjacent to a box above it, because it is topmost. So it must be adjacent to one or both of the boxes labeled b_3 and b_4 in the figure.

However, b_1 cannot be adjacent to b_3, for then $\{b_0, b_1, b_2, b_3\}$ forms a 2×2 pattern, contradicting the assumption that $\mathcal{S}$ is a string complex. Therefore b_1 must be adjacent to b_4 in Fig. 6. Now b_1 has degree exactly 2 in T. Because it must have degree ≥ 3 for $\mathcal{S}$ to be irreducible, it must be adjacent to box b'_1 underneath. But now $\{b_0, b_1, b'_0, b'_1\}$ forms a 2×2 pattern, again contradicting the assumption that $\mathcal{S}$ is a string complex.

We have now exhausted all possibilities, which have led to contradictions. So the assumption that $\mathcal{S}$ is irreducible is false, and $\mathcal{S}$ must be reducible. □

Corollary 3. *Every string complex can be reduced to the empty complex.*

Proof. Let $\mathcal{S}$ be a string complex. It cannot be irreducible by Theorem 3, and so it must have a box b of degree ≤ 2. Let $\mathcal{S}_1 = \mathcal{S} \setminus b$ be the complex with b removed. We claim that $\mathcal{S}_1$ is again a string complex. The reason is that the forbidden 2×2 pattern cannot be created by the removal of a box. Therefore, applying Theorem 3 again, $\mathcal{S}_1$ is reducible. Continuing in this manner, we can reduce $\mathcal{S}$ to the empty complex. □

5. Conclusion

That box complexes in $\mathbb{R}^2$ sometimes need 3 colors is a straightforward observation, but whether any box complex in $\mathbb{R}^3$ might need 4 colors is an open question. Although it is natural to expect that the chromatic number might be $n + 1$ for boxes in $\mathbb{R}^n$ as it is for simplices, we in fact have no example that requires more than 3 colors for any $n \geq 3$.

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