## 1. Matrix multiplication: Let

 $A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ -4 & 1 \end{bmatrix}$ 

What is *AB*? What is *BA*?

 $AB = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -8 & 8 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 0 & 2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -1 & 6 \end{bmatrix}$ 

2. What is the dimension of a 4 × 2 matrix times a 2 3 matrix? Could you multiply them the other way around?

The "inner" dimensions (both 2 here) must match for matrix multiplication to be possible. The "outer" dimensions will be the dimensions of the multiplicatin result. So in this case, the result would be a  $4 \times 3$  matrix. We cannot multiply them the other way around.

3. **Rotate:** Let  $\vec{y} = \begin{bmatrix} 0 \\ y \end{bmatrix}$ 

What are the new coordinates if  $\vec{y}$  is rotated  $\theta$  (counter-clockwise)?

$$\vec{y}' = \begin{bmatrix} -y\sin\theta\\ y\cos\theta \end{bmatrix}$$

- 4. Scale: What scaling matrix could you use to get  $\begin{bmatrix} 2 & 3 \end{bmatrix}^T$  from  $\begin{bmatrix} -5 & 6 \end{bmatrix}^T$ ?  $\begin{bmatrix} -2/5 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
- 5. **Composition:** What transformations could we apply to the house below to rotate it 25 degrees about its <u>center</u>?

The idea: rotation matrices rotate the plane around the origin. So if we apply rotation directly, the position of the house will shift. The easiest way to get the desired effect is to move the center of the house to the origin, apply the rotation, and then move it back. This is a translation, followed by a rotation, followed by a second translation. (It is also possible to find solutions that involve a single translation followed by a rotation, or a rotation followed by a single translation.)



6. Composition II: Discuss how to use composition to build a matrix that reflects across an arbitrary line. Can you find a matrix that reflects across the line y = 2x + 4?

(0,0)

Start with a reflection around a known line, like y = 0. Apply a transformation that maps y = 2x + 4 onto it, then apply the reflection, then invert the original transformation. For the specified problem, here is the plan:

- 1. Translate the line so that it goes through the origin
- 2. Rotate the line so that it aligns with y = 0. Since the target line has slope 2, the rotation angle is  $\tan^{-1} 2 = 63.4^{\circ}$ .
- 3. Reflect the line
- 4. Reverse the rotation from step 2
- 5. Reverse the translation from step 1.

Here are the matrices (listed right to left in order of application):

[1 0 0	0 1 0	$ \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0.447 \\ 0.894 \\ 0 \end{bmatrix} $	$-0.894 \\ 0.447 \\ 0$	$\begin{bmatrix} 0\\0\\1\end{bmatrix} \cdot \begin{bmatrix} 1\\0\\0\end{bmatrix}$	0 -1 0	$ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0.447 \\ -0.894 \\ 0 \end{bmatrix} $	0.894 0.447 0	$\begin{bmatrix} 0\\0\\1\end{bmatrix} \cdot \begin{bmatrix} 1\\0\\0\end{bmatrix}$	0 1 0	$\begin{bmatrix} 0\\ -4\\ 1 \end{bmatrix}$
			=	[-0.599 0.799 0	0.79 0.59 0	$\begin{array}{cc} 9 & -3.197 \\ 9 & 1.602 \\ & 1 \end{array}$				

7. As you know, matrix transformations may be composed by multiplying two or more transformation matrices so as to form a new one. For this problem, your task is to construct a matrix that will transform the unit square with opposite corners at (0,0) and (1,1) into the shape shown. (Assume that the positive *y* axis points upwards in these figures.) The only catch is that you have to construct your transformation using only the limited set of building blocks shown below. You may use each of these as many times as you like, and multiply them in any order that you like. If you multiply by a matrix several times in a row, you may use exponents in your answer. For example,  $A^4B^2 = AAAABB$ 

Transformation matrices you may use to build your result:

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Target shapes:

