CSC 240 Computer Graphics
Bézier Curves

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Upcoming Schedule

- **Homework 5** assigned today, due in two weeks (18 October)
- **Midterm** (self-scheduled, 1 page of notes) weekend after break
- Should be time for everything plus a break!
Your Questions

Q. How do you find the interpolations? Like could you explain how you found question 2?

Q. Will you please go over how you got those points in questions 2 and 3 in class?

A. A. The fastest way is to use the equations on slide 11. You compute the \( x \) and \( y \) components separately.

Q. So is it correct that the curves come from the lines, and not the other way around?

A. We use the lines to find the points of the curve.
Interpolation Example

- Compute each coordinate separately
  \[ p_{01}^x = (1 - t)p_0^x + tp_1^x = 0.8 \times 0 + 0.2 \times 5 = 1.0 \]
  \[ p_{01}^y = (1 - t)p_0^y + tp_1^y = 0.8 \times 0 + 0.2 \times 0 = 0.0 \]
  \[ p_{12}^x = (1 - t)p_1^x + tp_2^x = 0.8 \times 5 + 0.2 \times 5 = 5.0 \]
  \[ p_{12}^y = (1 - t)p_1^y + tp_2^y = 0.8 \times 0 + 0.2 \times 5 = 1.0 \]

- Repeat the process with the second-order interpolations
  \[ p_{012}^x = (1 - t)p_{01}^x + tp_{12}^x = 0.8 \times 1 + 0.2 \times 5 = 1.8 \]
  \[ p_{012}^y = (1 - t)p_{01}^y + tp_{12}^y = 0.8 \times 0 + 0.2 \times 1 = 0.2 \]
Q. Could you please explain more about the math of Bezier curves? Specifically about why is it (1-t) and t.

A. The two numbers represent a proportion – how close we are to one end of the segment or the other. Some notes:

- They always sum to 1
- If we define \( u = 1 - t \), then \( t = 1 - u \)
- The math works just as well starting from the other end
Q. What do you mean by weighted sum of the endpoints? What does interpolate mean? Why is \( t=2.0 \) of \( p_0=(0,0), \ p_1=(5,0), \ p_0=(5,5) \) \( p_{02}=(0,20) \) I don't understand how we go from the value for \( p_{01} \) to the value for \( p_{12} \) and get 0 on the left hand side or x value.

A. Interpolation uses the formulae given together with \( t \) between 0 and 1. We can use \( t \) outside this range as well; this is called **extrapolation**.
Extrapolation uses the same procedure, except $t$ is outside the 0 to 1 range.

- **Ex:** $t = 2.0 \Rightarrow (1 - t) = -1.0$

  In this example we have a double helping of the first point. Thus we must compensate by subtracting 1x the second point so that our weights sum to 1.

  $$ p_{01}^x = (1 - t)p_0^x + tp_1^x = -1 \times 0 + 2 \times 5 = 10 $$
  $$ p_{01}^y = (1 - t)p_0^y + tp_1^y = -1 \times 0 + 2 \times 0 = 0 $$
  $$ p_{12}^x = (1 - t)p_1^x + tp_2^x = -1 \times 5 + 2 \times 5 = 5 $$
  $$ p_{12}^y = (1 - t)p_1^y + tp_2^y = -1 \times 0 + 2 \times 5 = 10 $$

  Our first order points are (10,0) and (5,10).

  $$ p_{012}^x = (1 - t)p_{01}^x + tp_{12}^x = -1 \times 10 + 2 \times 5 = 0 $$
  $$ p_{012}^y = (1 - t)p_{01}^y + tp_{12}^y = -1 \times 0 + 2 \times 10 = 20 $$
Your Questions

Q. When might you use a bezier curve instead of a spline?
A. Splines are useful for longer curves with more bends.

Q. What does abcd mean in the context of the cubic spline equation?
A. They are parameters in the cubic formula: \( x = at^3 + bt^2 + ct + d \)

Q. Why do the control points in bezier spline need to be colinear for the curve to be smooth?
A. We’ll explore this more in class today.
Your Questions

Q. Why is it a downside of splines that the curve does not pass through all control points? Is it less efficient?

A. I would not call it a downside. It’s just a difference.

Q. Can you explain the parameter/segment relationship for the cubic spline (8 param/segment - 2)?

A. The first segment needs 4 control points. Each additional adds just one more, since they overlap. So the number of points is (#segments)+3

Q. Can you please do more visual comparison of the different kinds of curves and how we might want to apply them in different situations?

A. There is no obvious pattern of difference between different curve types.
Other Questions?
1. 2nd order (quadratic) Bézier curve: Draw the Bézier curve with control points $\vec{p}_0, \vec{p}_1, \vec{p}_2$, using guiding points with $t = 0.25, 0.5, 0.75$. 
2. Here is the parametric equation of a quadratic Bézier curve

\[ Q(t) = (1 - t)^2 \vec{p}_0 + 2(1 - t)t \vec{p}_1 + t^2 \vec{p}_2 \]

(a) Rearrange this function to make it look more like a quadratic in \( t \) (i.e. \( Q(t) = at^2 + bt + c \)).

(b) Take the derivative of this rearranged function with respect to \( t \).

(c) What is the derivative at \( t = 0 \)? \( t = 1 \)? What can we say about the tangents at \( \vec{p}_0 \) and \( \vec{p}_2 \)?
3. 3rd order (cubic) Bézier curve: Draw the Bézier curve with control points \( \vec{p}_0, \vec{p}_1, \vec{p}_2, \vec{p}_3 \), using guiding points with \( t = 0.25, 0.5, 0.75 \).
1. Here is the parametric equation of a quadratic Bézier curve

\[ Q(t) = (1 - t)^2 p_0 + 2(1 - t)t p_1 + t^2 p_2 \]

(a) Rearrange this function to make it look more like a quadratic in \( t \) (i.e. \( Q(t) = at^2 + bt + c \)).

\[ Q(t) = p_0 - 2t p_0 + t^2 p_0 + 2t p_1 - 2t^2 p_1 + t^2 p_2 \]
\[ Q(t) = t^2 p_0 - 2t^2 p_1 + t^2 p_2 - 2tp_0 + 2tp_1 + p_0 \]
\[ Q(t) = (p_0 - 2p_1 + p_2)t^2 + (2p_1 - 2p_0)t + p_0 \]

(b) Take the derivative of this rearranged function with respect to \( t \).

\[ \frac{dQ}{dt} = 2(p_0 - 2p_1 + p_2)t + 2(p_1 - p_0) \]

(c) What is the derivative at \( t = 0 \) \( t = 1 \)? What can we say about the tangents at \( p_0 \) and \( p_2 \)?

At \( t = 0 \), \( \frac{dQ}{dt} = 2(p_1 - p_0) \), in other words it points from \( p_0 \) towards \( p_1 \).

At \( t = 1 \), \( \frac{dQ}{dt} = 2(p_0 - 2p_1 + p_2) + 2(p_1 - p_0) = 2(p_2 - p_1) \), so it points from \( p_1 \) towards \( p_2 \).