CSC 240 Computer Graphics
Lecture 8: Bézier Curves

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Some slides & content courtesy Sara Mathieson

Bézier Curves

- What is a curve?

  Typically, curve refers to a 1-dimensional shape with a smoothly $\underbrace{\text{changing tangent along its length}}_{\textit{No sharp angles!}}$

One way to represent a curve is to give the *x* and *y* coordinates as parameterized functions of a variable *t*: *x(t)* and *y(t)*

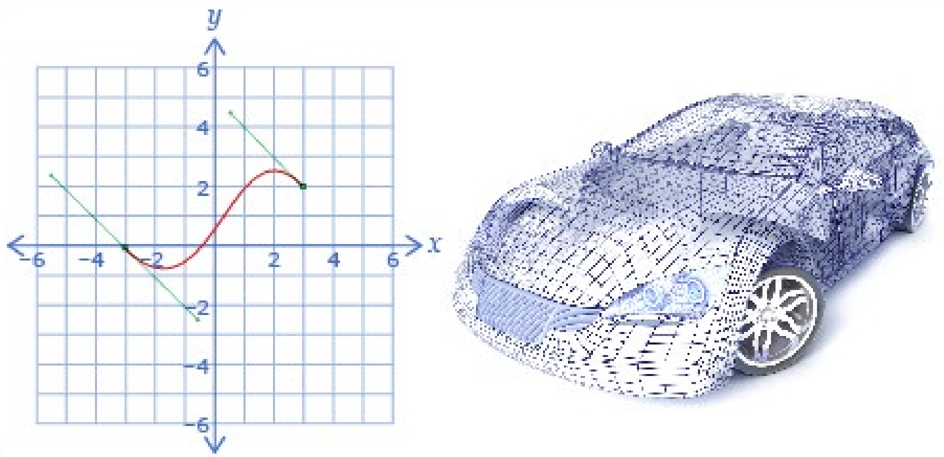
The derivatives of this function, $\frac{dx}{dt}$ and $\frac{dy}{dt}$, should vary continuously

https://www.carbuyer.co.uk/reviews/renault/captur/suv/review

https://pngtree.com/freepng/curved-road_1499869.html

Bézier Curves

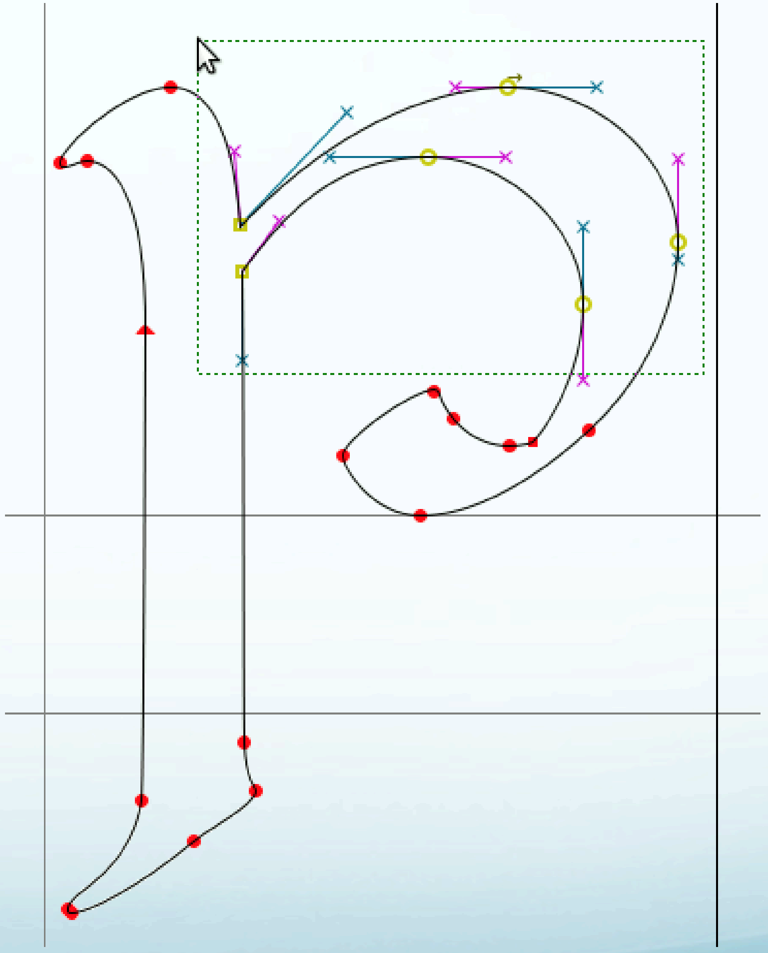
- Named for Renault auto body designer Pierre Bézier, who began using them c. 1960
- Paul de Casteljau also did early work at Citroën
- Mathematics (Bernstein polynomials) understood much earlier, c. 1910



iStockphoto/Thinkstock

Bézier Curves

- Most fonts are defined using Bézier curves.
- Also used in 3D modeling software (Blender, CAD, etc.)



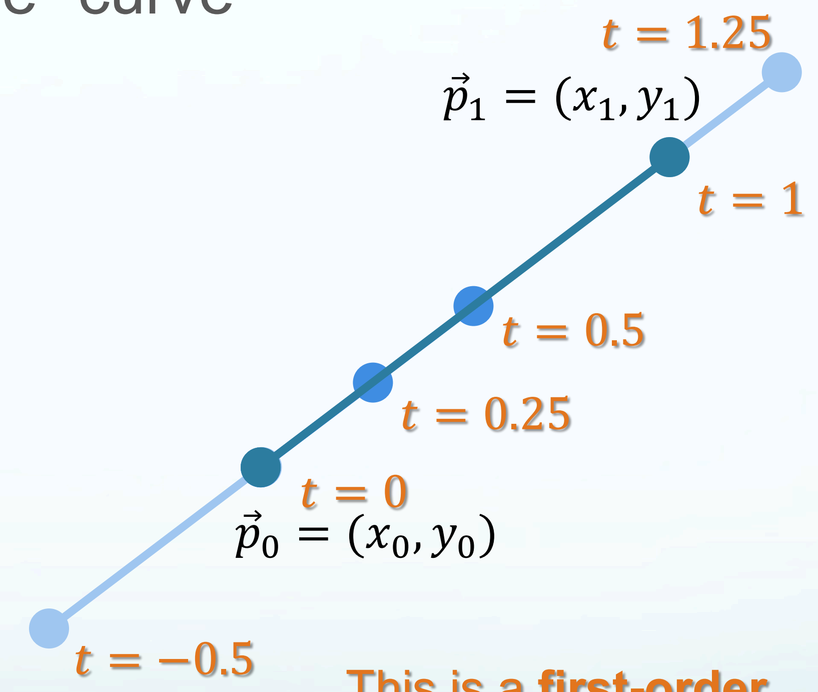
http://jamie-wong.com/post/bezier-curves/

https://tengtelc.wordpress.com/

Bézier Curves

- Straight line segment is simplest possible “curve”
  - Defined by endpoints p_0 and p_1
  - Parameterize on t
    - $t = 0$ at one end
    - $t = 1$ at the other
    - Linear interpolation between
  - Can write equations for x and y
    $x = (1 - t)x_0 + tx_1$
    $y = (1 - t)y_0 + ty_1$
    Or: $\vec{p} = (1 - t)\vec{p}_0 + t\vec{p}_1$
  - Equations allow extrapolation

$\vec{p}$ is a weighted sum of the two endpoints



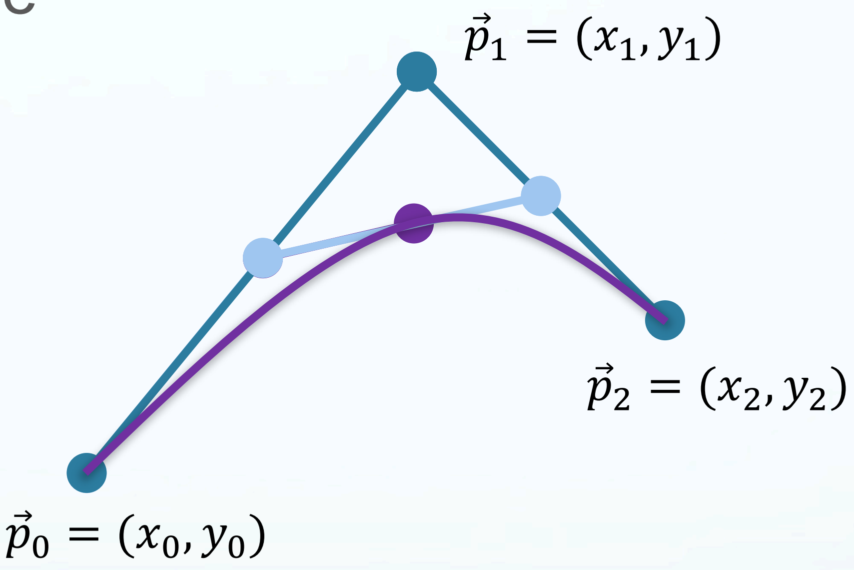
This is a **first-order** or **linear** Bézier curve

We call p_0 and p_1 the **control points**

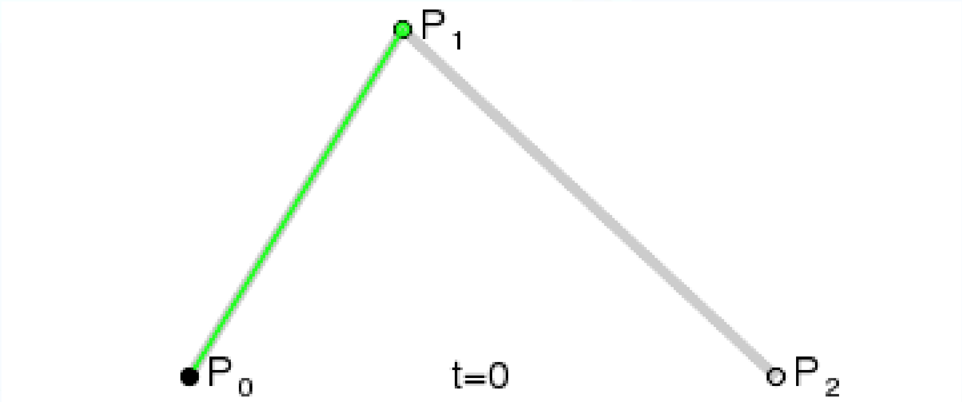
Bézier Curves

A **second order** or **quadratic** Bézier curve uses three control points

- Given *t*, interpolate secondary points between each pair
- Then locate point on curve by interpolating between the secondary points
- Evaluate for all *t* between 0 and 1 to complete curve



Bézier Curves



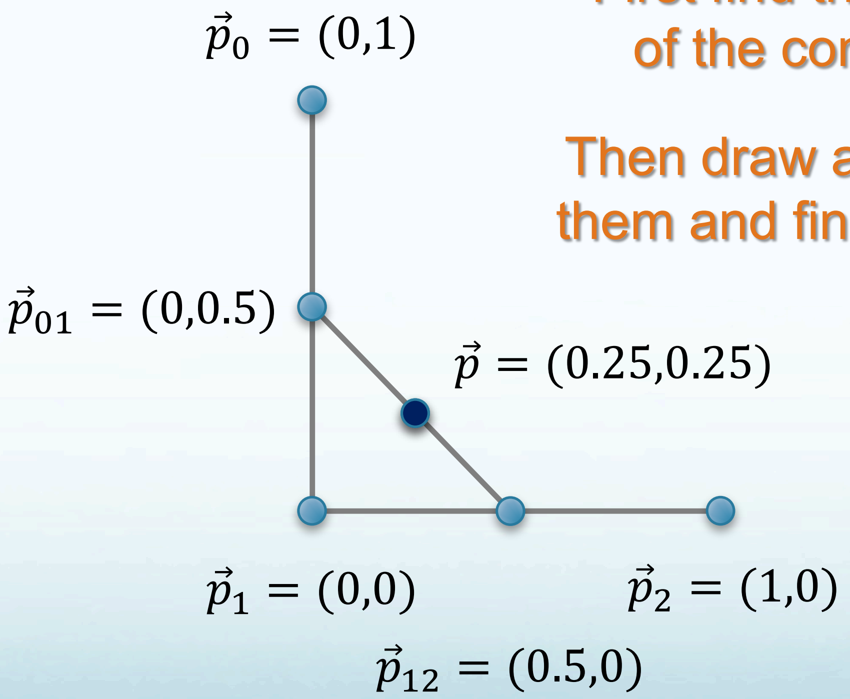
Animation of a quadratic Bézier curve, t in [0,1] by Phil Tregoning [Wikipedia]

Bézier Curves

- Where does the point at $t = 0.5$ fall for this quadratic curve?

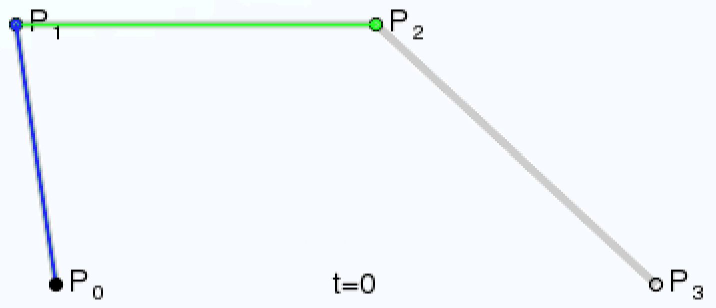
First find the midpoints (*t = 0.5*) of the connecting segments

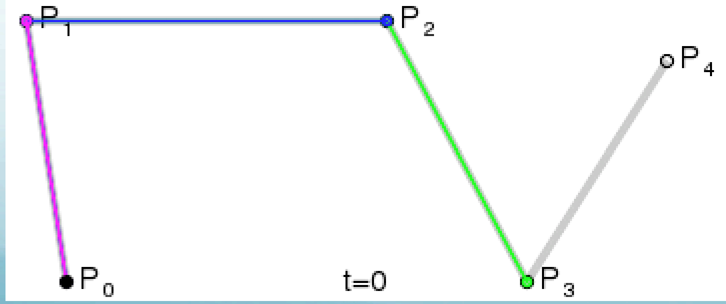
Then draw a segment connecting them and find its midpoint (*t = 0.5*)

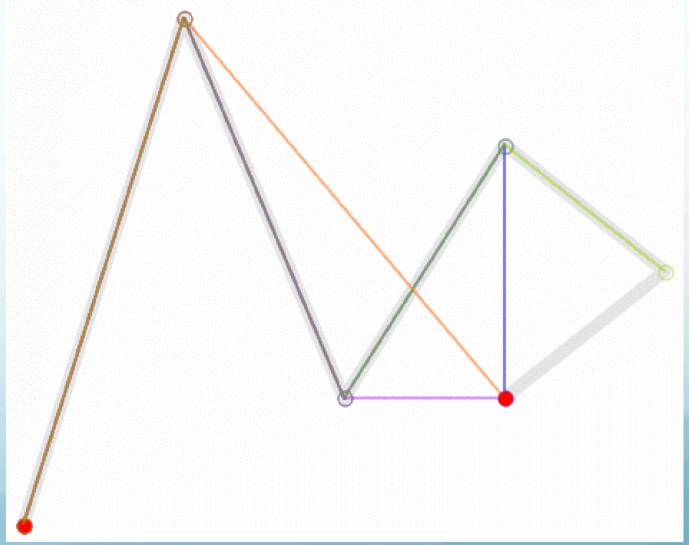


Bézier Curves

- **Third order** or **cubic** Bézier curves use four control points
- Interpolate on consecutive pairs to get three secondary points
- Treat these as a quadratic curve
- Similar process generates higher order curves: interpolate neighbor points to get lower order







All images by Phil Tregoning, Wikipedia

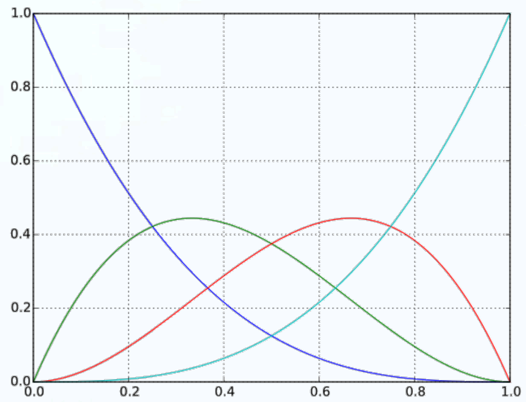
Bézier Curve Orders

Higher order curves allow greater complexity

- Order of curve = maximum number of **inflection points** (bends)
  - First order curve: no bends (straight line)
  - Second order (quadratic) curve: one bend (parabolic)
  - Third order (cubic) curve: two bends maximum
  - Fourth order (quartic) curve: three bends maximum
  - etc.
- # of parameters also rises
  - 2 x Number of control points

Order	Description	Maximum bends	Control Points
First	Linear	0 (straight)	2
Second	Quadratic	1	3
Third	Cubic	2	4
Fourth	Quartic	3	5
N^{th}	N^{th} degree	N-1	N+1

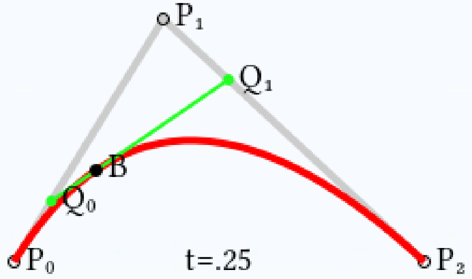
Bézier Curves



- Math of Bezier curves:

  First order: $\vec{p} = (1-t)\vec{p}_0 + t\vec{p}_1$

  Second order:



$$Q_0 \equiv \vec{p}_{01} = (1-t)\vec{p}_0 + t\vec{p}_1$$

$$Q_1 \equiv \vec{p}_{12} = (1-t)\vec{p}_1 + t\vec{p}_2$$

$$B \equiv \vec{p} = (1-t)\vec{p}_{01} + t\vec{p}_{12}$$

$$= (1-t)\big((1-t)\vec{p}_0 + t\vec{p}_1\big) + t\big((1-t)\vec{p}_1 + t\vec{p}_2\big)$$

$$= (1-t)^2\vec{p}_0 + 2t(1-t)\vec{p}_1 + t^2\vec{p}_2$$

  Third order:

$$\vec{p} = (1-t)^3\vec{p}_0 + 3t(1-t)^2\vec{p}_1 + 3t^2(1-t)\vec{p}_2 + t^3\vec{p}_3$$

- Note appearance of binomial coefficients

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Questions

PAUSE NOW & ANSWER

1. How many control points are needed for a cubic Bézier curve?

   Four

2. Consider a curve with control points $p_0 = (0,0)$, $p_1 = (5,0)$, $p_0 = (5,5)$. What is the point at $t = 0.2$? (Hint: figure all the interpolations.)

   $p_{01} = (1,0)$, $p_{12} = (5,1)$, *so* $p_{02} = (1.8,0.2)$

3. What is the point at $t = 2.0$ for the same curve?

   $p_{01} = (10,0)$, $p_{12} = (5,10)$, *so* $p_{02} = (0,20)$

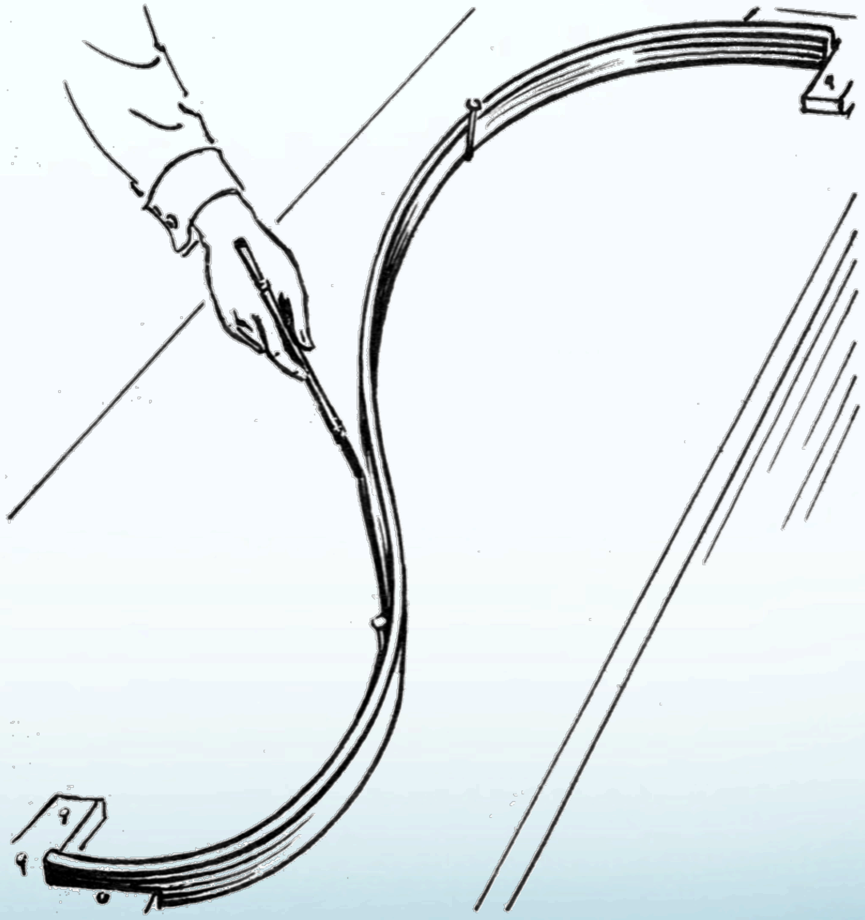
4. What is the minimum order of this Bézier curve?

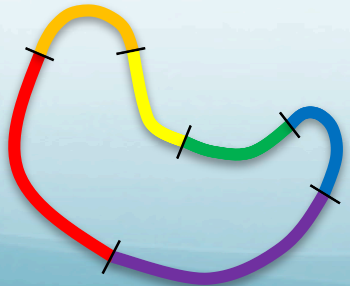
   

   Sixth (because it has five bends)

Splines

- Shipbuilders once formed curves using thin wooden strips constrained by pegs
- The thin strips of wood were called **splines**
- The term now applies to piecewise mathematical modeling of a curve

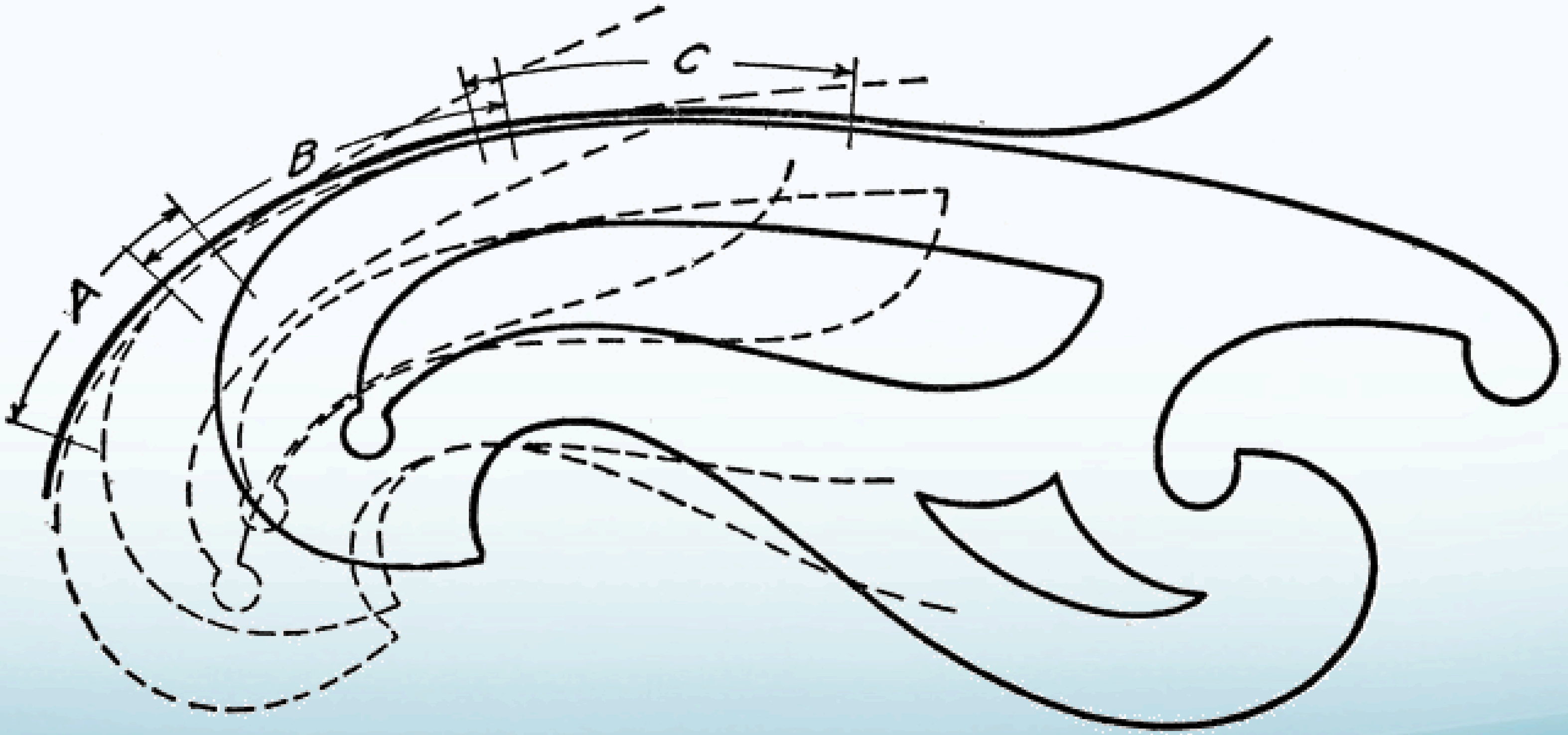




https://en.wikipedia.org/wiki/Non-uniform_rational_B-spline

Splines

When drafting blueprints, splines can be approximated using a **French Curve** applied between control points

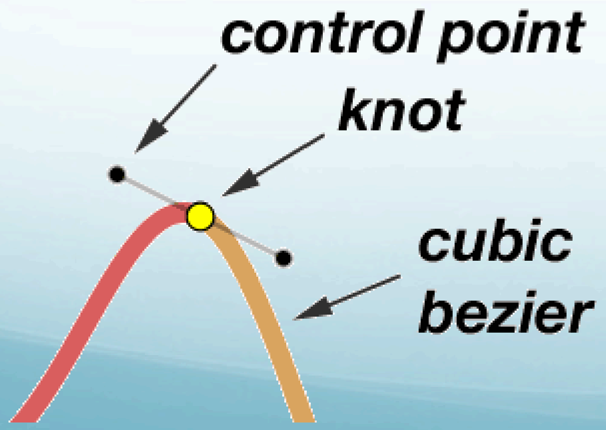


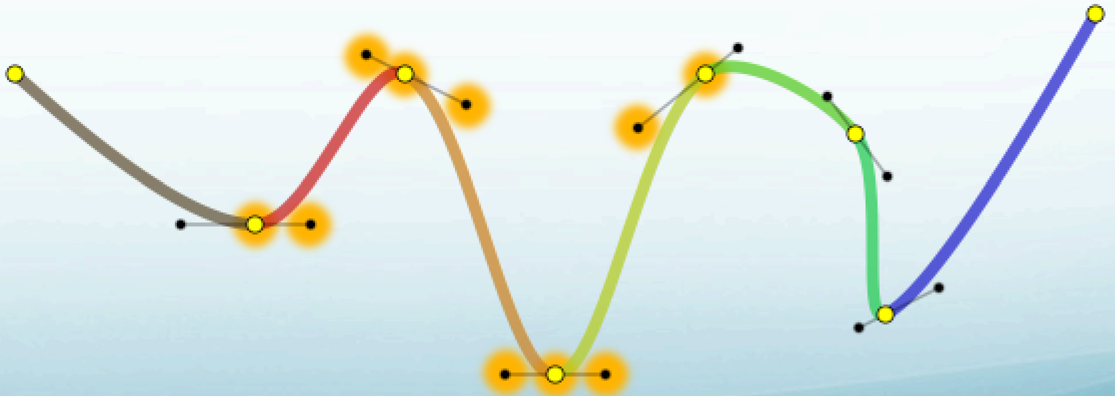
https://etc.usf.edu/clipart/76100/76174/76174_frnch_crvuse.htm

Splines

We can build a spline out of cubic Bézier curves

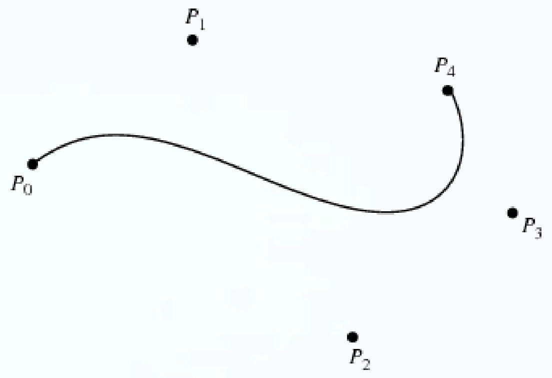
- Each segment has four control points
- Curve passes through two, shape governed by others
- Neighboring control points must be collinear with knot for smooth derivative





http://scaledinnovation.com/analytics/splines/aboutSplines.html

Splines

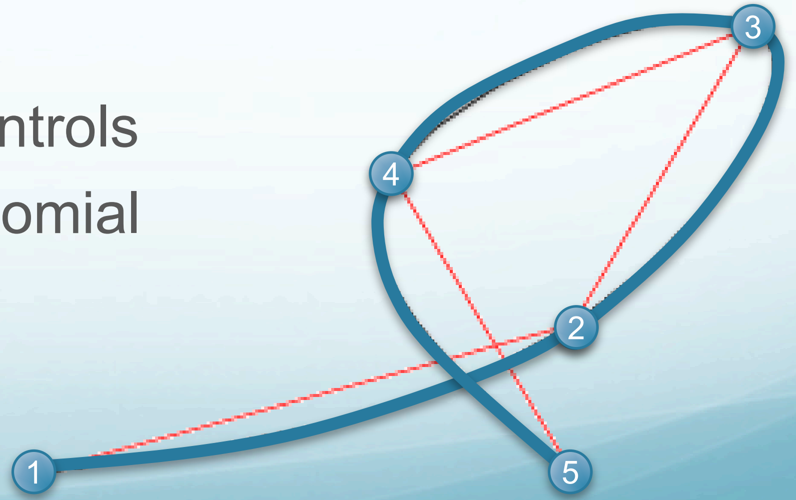


Cubic Bézier splines have some drawbacks

- Four control points per segment
- Curve does not pass through all control points

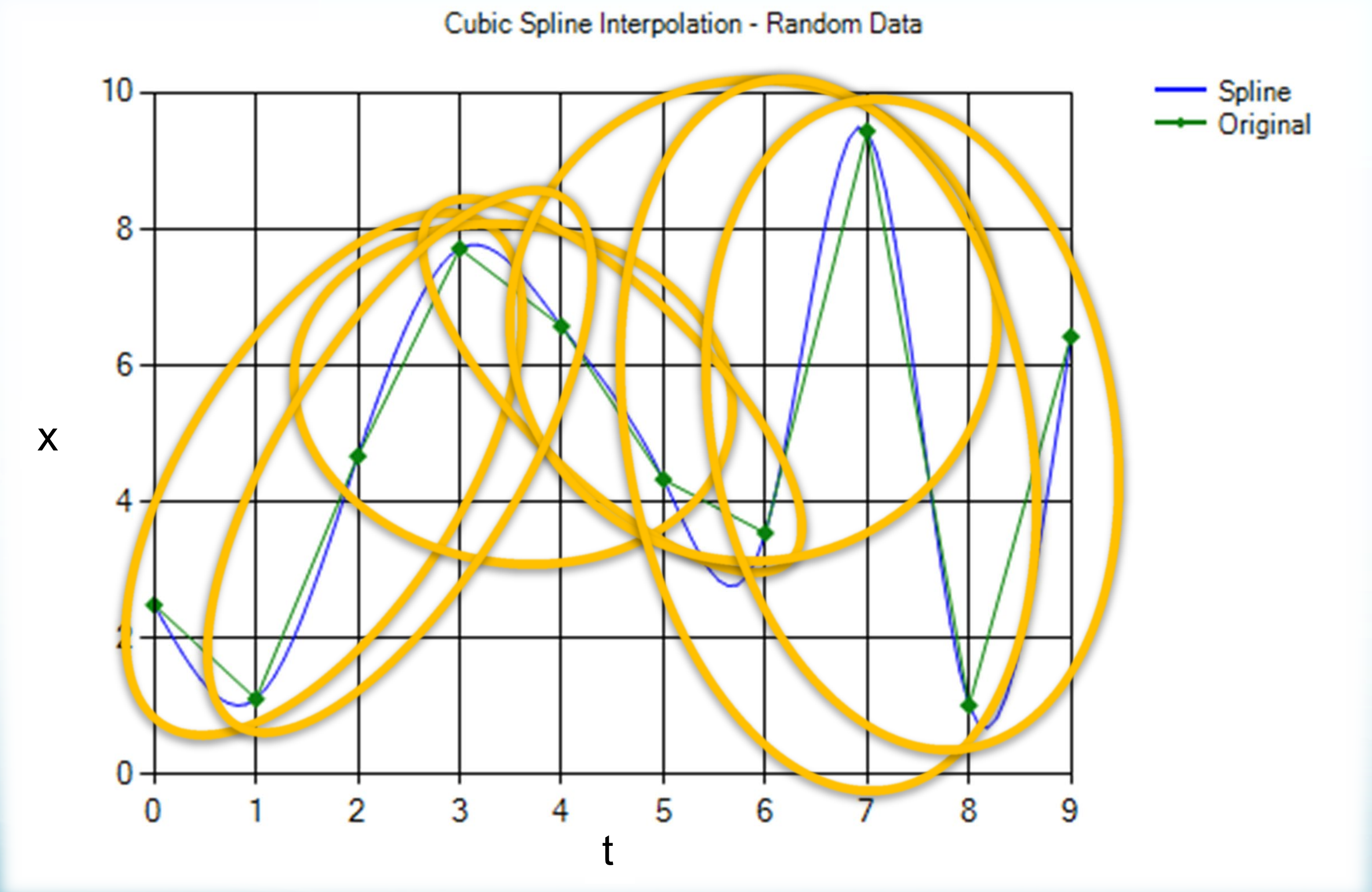
Alternative option is **cubic spline**

- Any number of control points
- Curve passes through each point in sequence
- Shape of section depends on four surrounding controls
- Each section modeled by 3rd degree (cubic) polynomial
- Adds just one point per extra segment
- Smooth derivatives at the joins



http://mathworld.wolfram.com/BezierCurve.html
http://mathworld.wolfram.com/CubicSpline.html

Example Cubic Spline (1D)



Cubic polynomial function:

$$x = at^3 + bt^2 + ct + d$$

- Four unknowns; values of a, b, c, and d can be solved given four points

Solve:

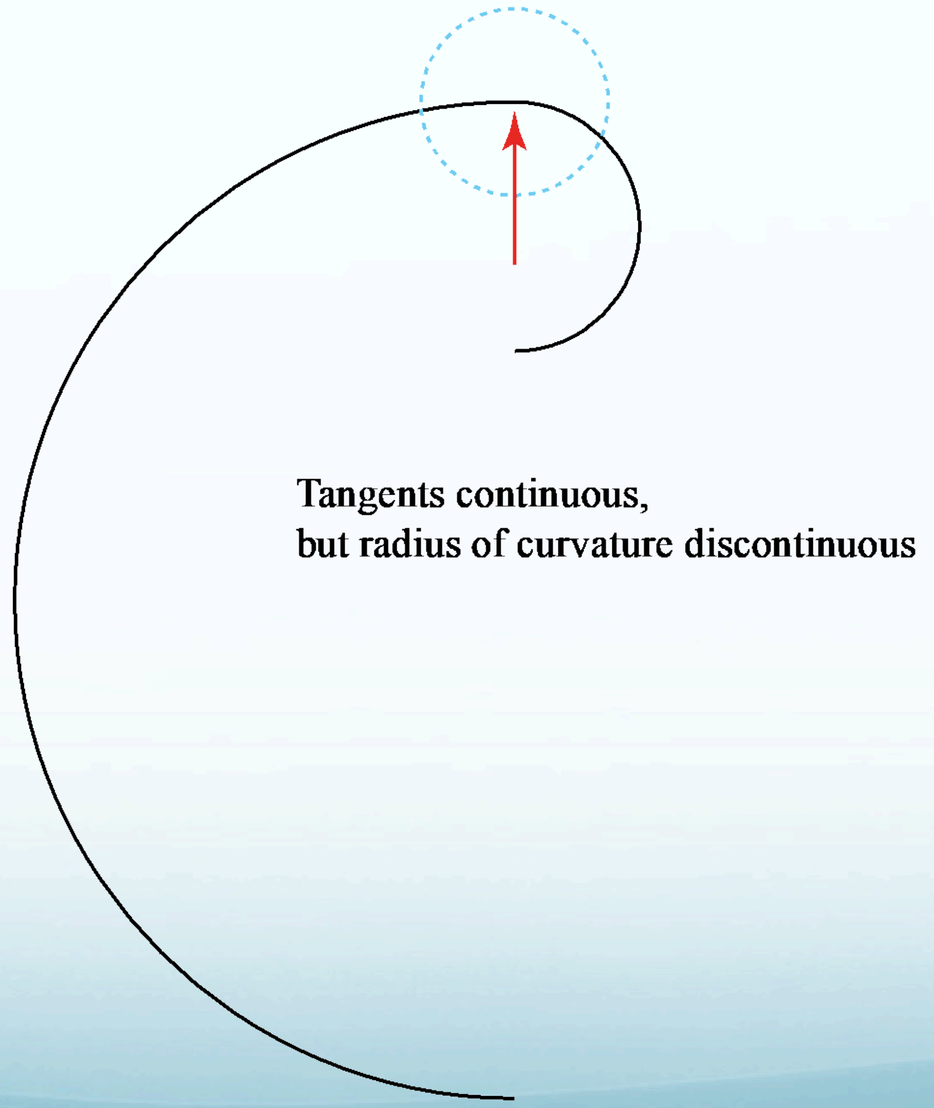
$$\begin{bmatrix} t_1^3 & t_1^2 & t_1 & 1 \\ t_2^3 & t_2^2 & t_2 & 1 \\ t_3^3 & t_3^2 & t_3 & 1 \\ t_4^3 & t_4^2 & t_4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2.50 \\ 1.23 \\ 4.74 \\ 7.81 \end{bmatrix}$$

Cubic Spline: Code project

$$\Rightarrow a = -0.87, b = 5, c = -5.4, d = 2.5$$

Why **cubic** spline?



Questions

PAUSE NOW & ANSWER

1. What advantages does a spline offer as compared to a high-order Bézier curve?

   It is hard to control a single curve with many parameters. Simple parts are easier.

2. Which of these passes through all of its control points?
   a) Bézier curve
   b) Bézier spline
   c) Cubic spline

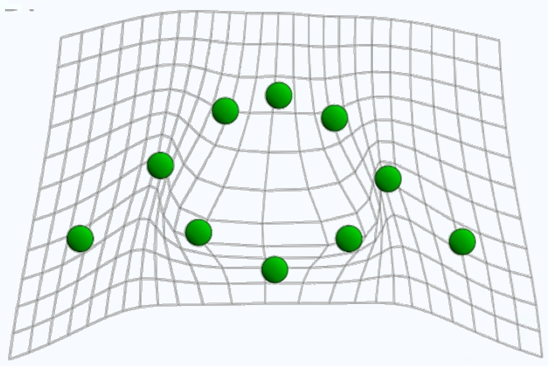
3. Under what conditions is a Bézier spline smooth?

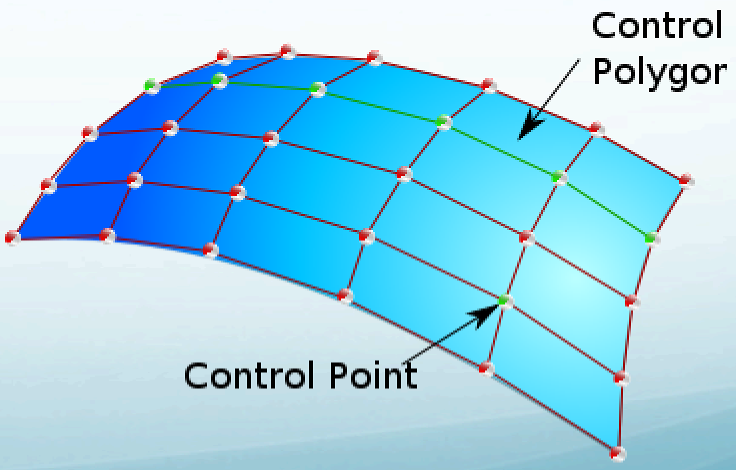
   The secondary control points on either side of a knot are collinear.

Other Splines

Spline concepts generalize to higher dimensions as well

- Two-dimensional deformations can be modeled using **thin plate splines**
- Three-dimensional curved surfaces can be modeled using **NURBS** (*Non-rational uniform basis splines*)
- This course probably won't have time to explore these more advanced topics





https://en.wikipedia.org/wiki/Non-uniform_rational_B-spline

Review

After watching this video, you should be able to…

- Define a Bézier curve of any order
- Compute the point on a Bézier curve at position t.
- Define a spline curve and give two example types
- Identify the necessary conditions for a Bézier spline to be smooth
- Construct a cubic spline given control points & a polynomial curve fitter
- Recognize other applications of splines

Music: https://www.bensound.com