CSC 240 Computer Graphics
Lecture 6: Transformations

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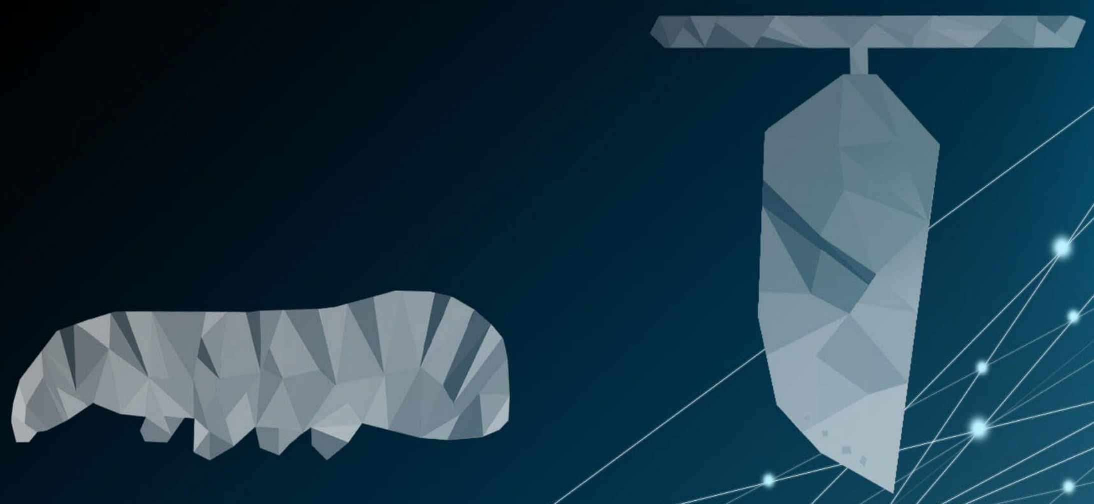
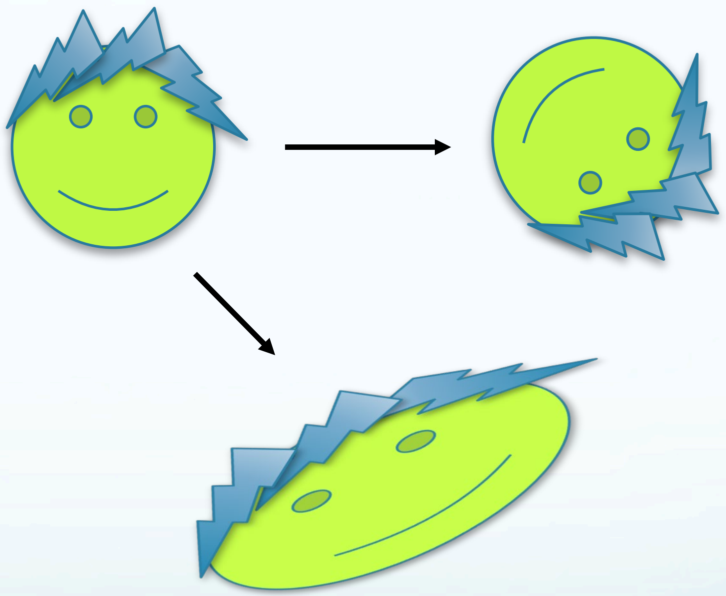


https://www.hitachi-systems-security.com/blog/an-outlook-on-digital-transformation/

Transformations

What is a 2D transformation? Any $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

- Special categories of transformation:
  - **Rigid translation**: preserves distance
  - **Affine transformation**: preserves collinearity & distance ratios
  - **Linear transformation**: Expressed via matrix & vector operations



$$\vec{u} = T\vec{v} + \vec{b}$$

Key point here: any 2x2 matrix represents a 2D transformation!

Transformations

Why does a house look different depending on where you stand?

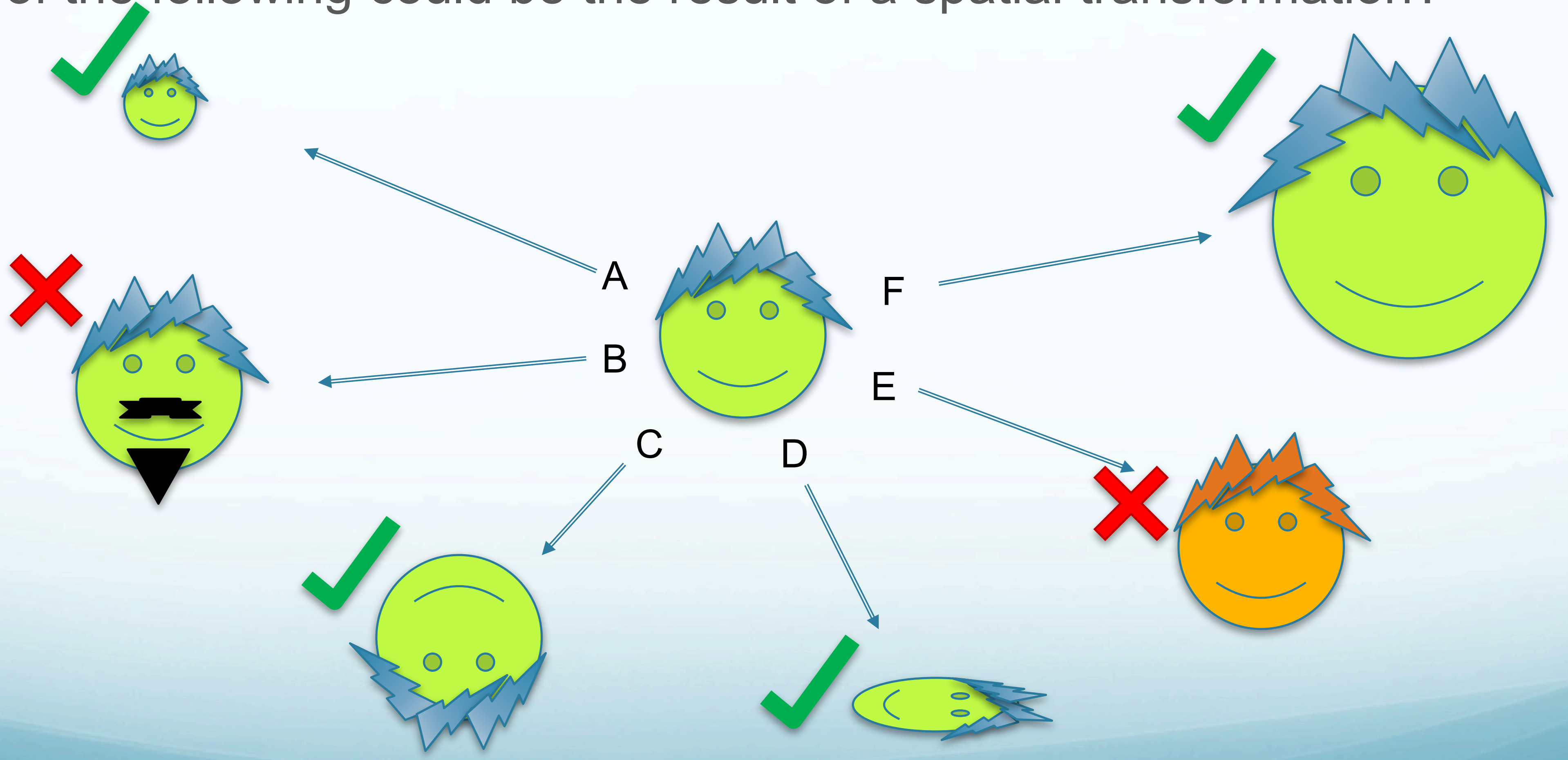
- Object is the same; arrangement shifts with viewpoint
  - Transformations allow the viewpoint to be treated separately from the object
  - *Define geometry once; change viewpoint via transformations*
  - Simplifies thinking and organizes mathematics

Images: https://carlcolsonarchitect.com/adu-cottage-creating-the-design-drawings/

Questions

PAUSE NOW & ANSWER

Which of the following could be the result of a spatial transformation?



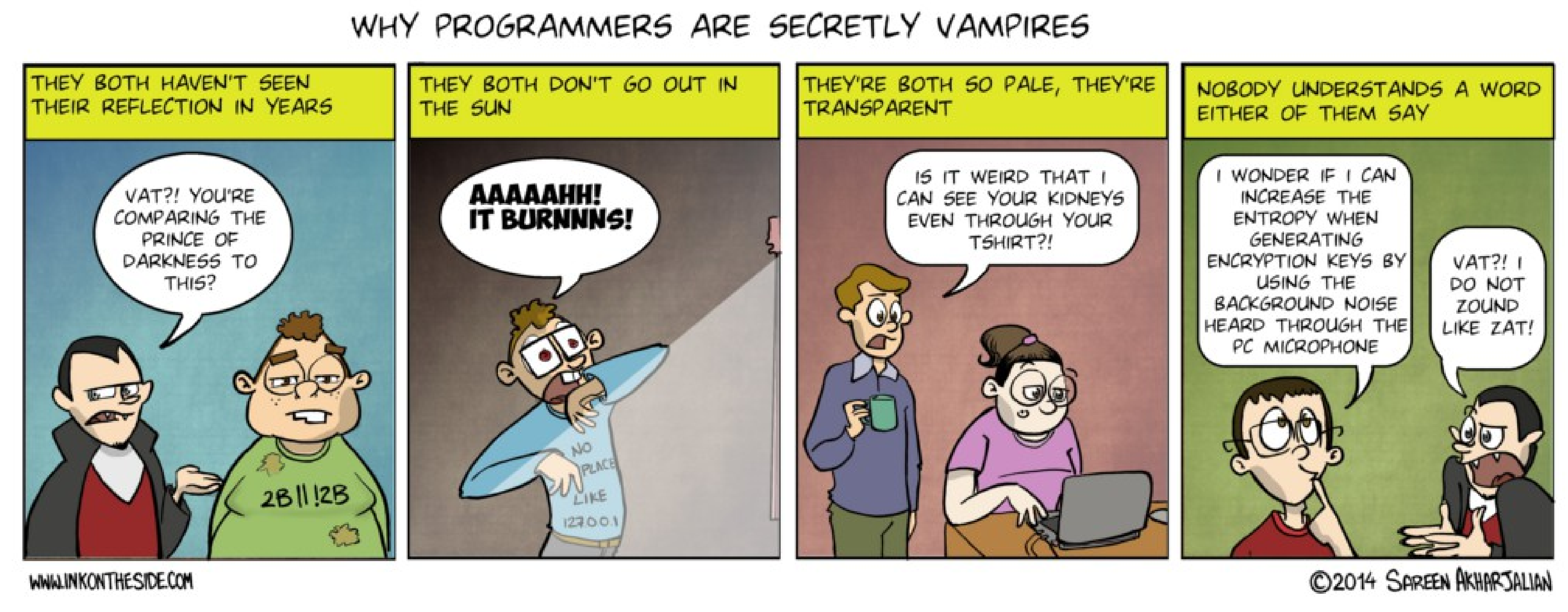
Specific Transformations

Identity Transformation

Identity Transformation

- The **identity** transformation leaves points unchanged
$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
  - Note: $\begin{bmatrix} v_x \; v_y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} v_x \; v_y \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$
- For any other matrix *M*, define its inverse M^{-1} such that
$$M^{-1}M = I = MM^{-1}$$
- What is the inverse of I?

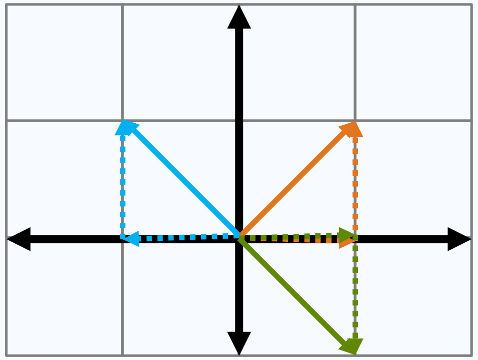
Reflection Transformation



http://inkontheside.com/2014/01/20/why-programmers-are-secretly-vampires/

Reflection Transformation

- Consider the vector $\begin{bmatrix}1\\1\end{bmatrix}$. What is its reflection across the *y* axis?
  - Note $\vec{v} = 1e_1 + 1e_2$
  - Reflection $\vec{v}' = -1e_1 + 1e_2$
  - Only the *x* component is negated
- How about across the x axis?



$$\vec{v}' = -1e_1 + 1e_2$$

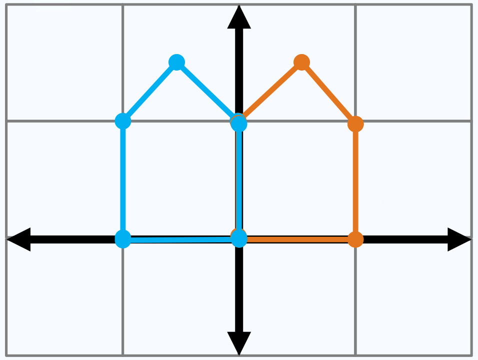
- A matrix can express the treatment of each component:

$$F_{horiz} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}, F_{vert} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

- What is the inverse of a reflection? $F_{horiz} \cdot F_{horiz} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} \cdot \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

Reflection Transformation

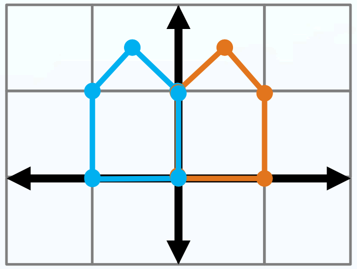
- Can apply transformation to a set of points
- $P = \begin{bmatrix} 0 & 1 & 1 & 0.5 & 0 \\ 0 & 0 & 1 & 1.5 & 1 \end{bmatrix}$



$$P' = FP = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} P = \begin{bmatrix} 0 & -1 & -1 & -0.5 & 0 \\ 0 & 0 & 1 & 1.5 & 1 \end{bmatrix}$$

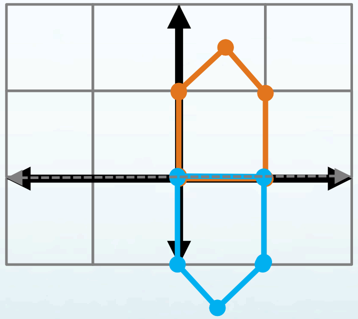
Reflection Transformation

- What matrix gives a reflection across the y axis?



$$F_{horiz} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

- What matrix gives a reflection across the x axis?



$$F_{vert} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

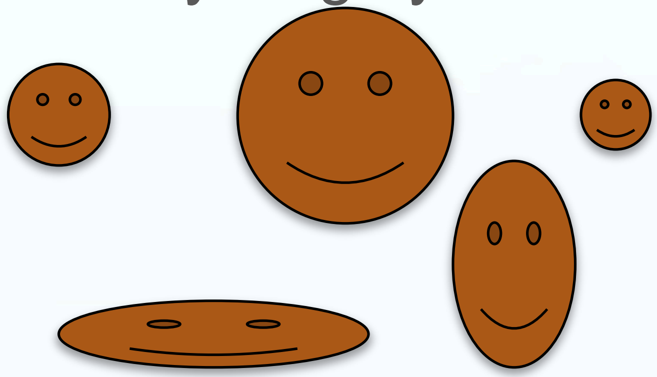
- What matrix gives a reflection across an arbitrary line?

Let's revisit this question later…

Scaling Transformation

Scaling Transformation

- Scaling grows or shrinks everything by a scalar multiple



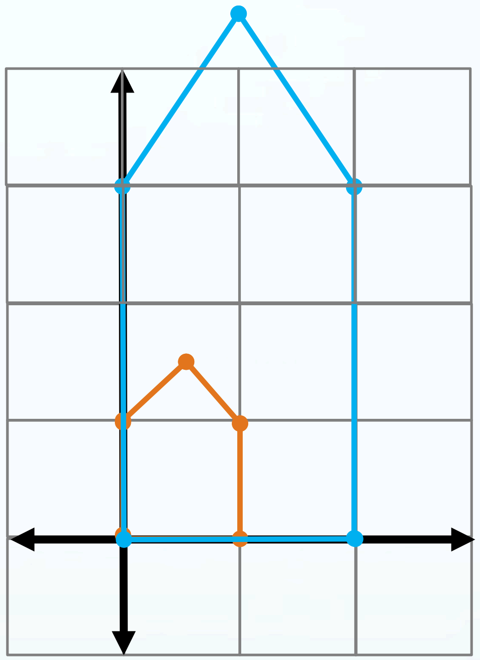
- Can differ by axis

- Matrix: $S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$

- Inverse: $S^{-1} = \begin{bmatrix} 1/s_x & 0 \\ 0 & 1/s_y \end{bmatrix}$

Scaling Transformation

- Applied to set of points:
  $$P = \begin{bmatrix} 0 & 1 & 1 & 0.5 & 0 \\ 0 & 0 & 1 & 1.5 & 1 \end{bmatrix}$$
- Suppose $s_x = 2$ and $s_y = 3$
  $$S = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

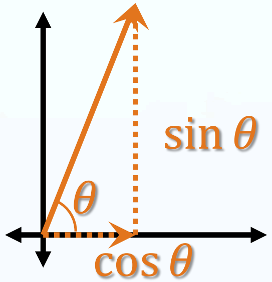


$$P' = SP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} P = \begin{bmatrix} 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 3 & 4.5 & 3 \end{bmatrix}$$

Rotation Transformation

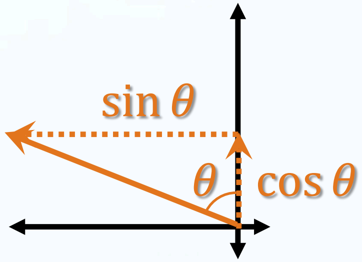
Rotation Transformation

- Rotation changes each basis vector in a certain way



$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$



- Final result is combination of both

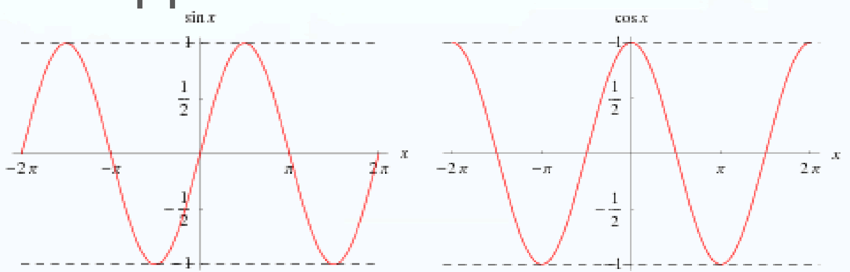
$$\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = v_x e_1 + v_y e_2 \to v_x \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + v_y \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} v_x \cos\theta - v_y \sin\theta \\ v_x \sin\theta + v_y \cos\theta \end{bmatrix}$$

- We can express this more succinctly as a matrix

$$\vec{v}' = R\vec{v} \text{ where } \mathrm{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotation Transformation

- What is the inverse of a rotation?
  - ➢ Rotation in the opposite direction



$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

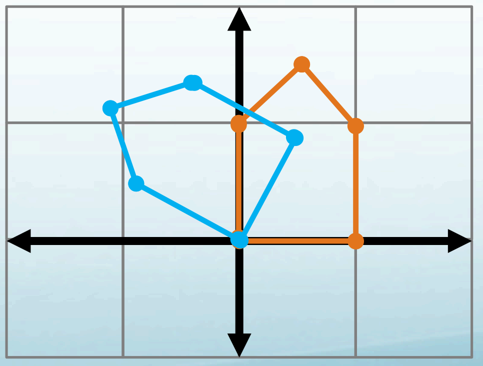
- Example: rotate by 60°

$$P = \begin{bmatrix} 0 & 1 & 1 & 0.5 & 0 \\ 0 & 0 & 1 & 1.5 & 1 \end{bmatrix}$$

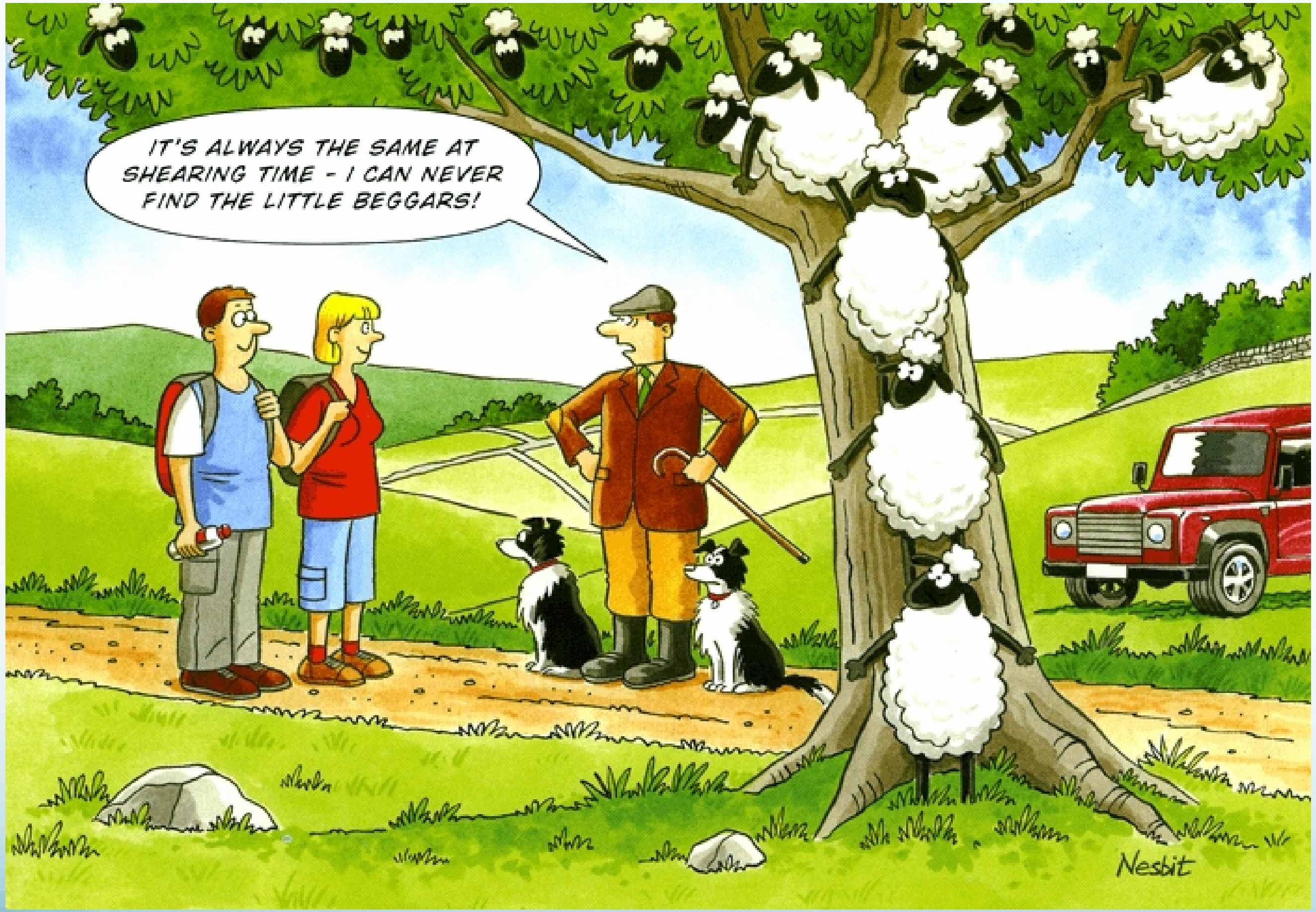
$$R = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} 0.50 & -0.87 \\ 0.87 & 0.50 \end{bmatrix}$$

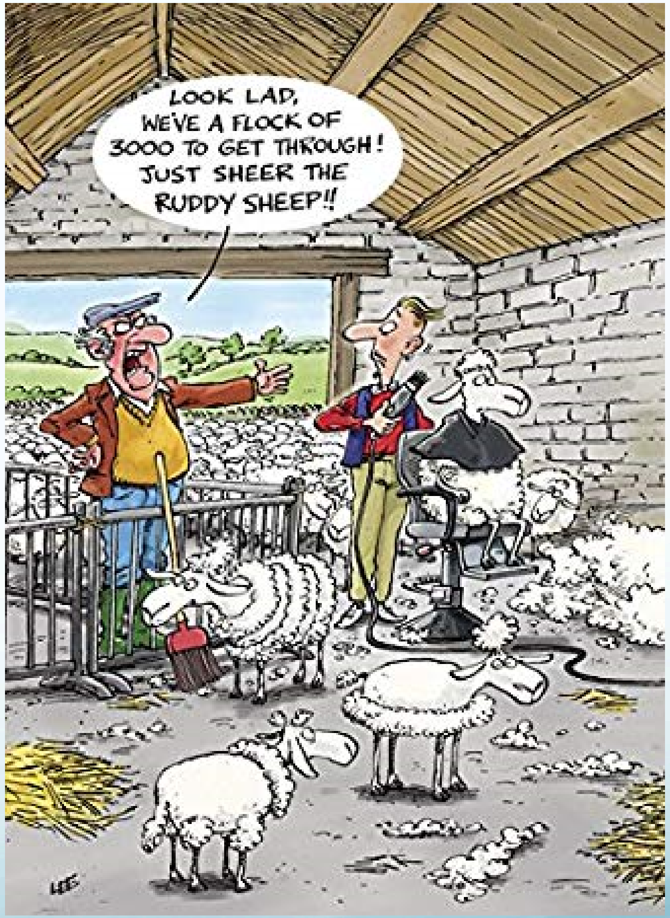
$$P' = RP = \begin{bmatrix} 0.50 & -0.87 \\ 0.87 & 0.50 \end{bmatrix} P$$

$$= \begin{bmatrix} 0 & 0.50 & -0.37 & -1.05 & -0.87 \\ 0 & 0.87 & 1.37 & 1.18 & 0.50 \end{bmatrix}$$



Shear Transformation





http://abcdefghigklmnopqrstuvwxyz.net/page/930/
https://www.amazon.co.uk/Sheep-Shearing-Humour-Greetings-Birthday/dp/B073G1DKLW

Shear Transformation

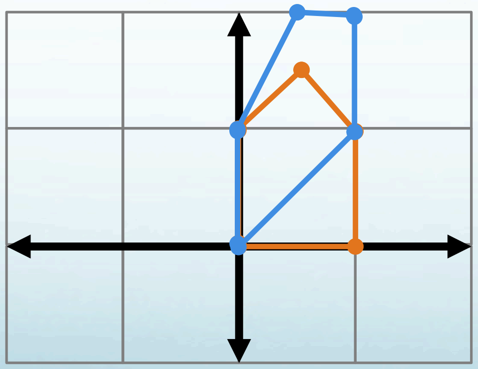
- Shear is a slant effect, where one dimension is augmented by a linear function of the other

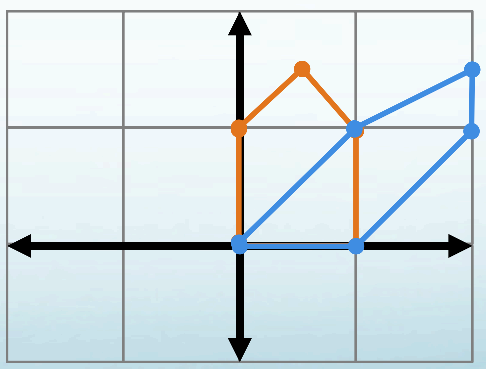
X-Shear: $H_x = \begin{bmatrix} 1 & 0 \\ k_x & 1 \end{bmatrix}$

Y-Shear: $H_y = \begin{bmatrix} 1 & k_y \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0.5 & 0 \\ 0 & 0 & 1 & 1.5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1.5 & 0 \\ 0 & 1 & 2 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0.5 & 0 \\ 0 & 0 & 1 & 1.5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1.5 & 1 \end{bmatrix}$$





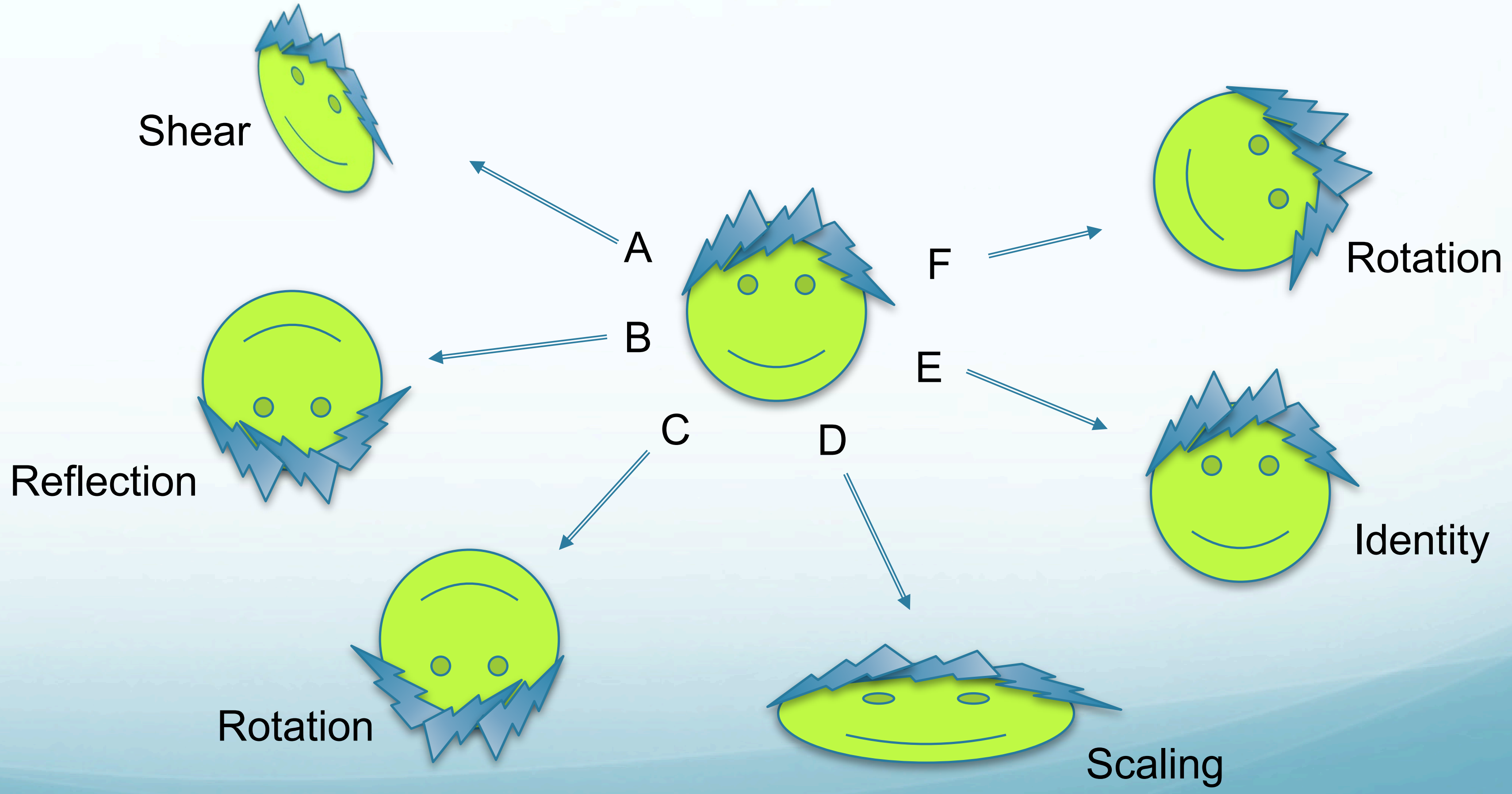
*Note: X-Shear plus rotation & scaling can simulate Y-Shear, and vice versa.

Questions

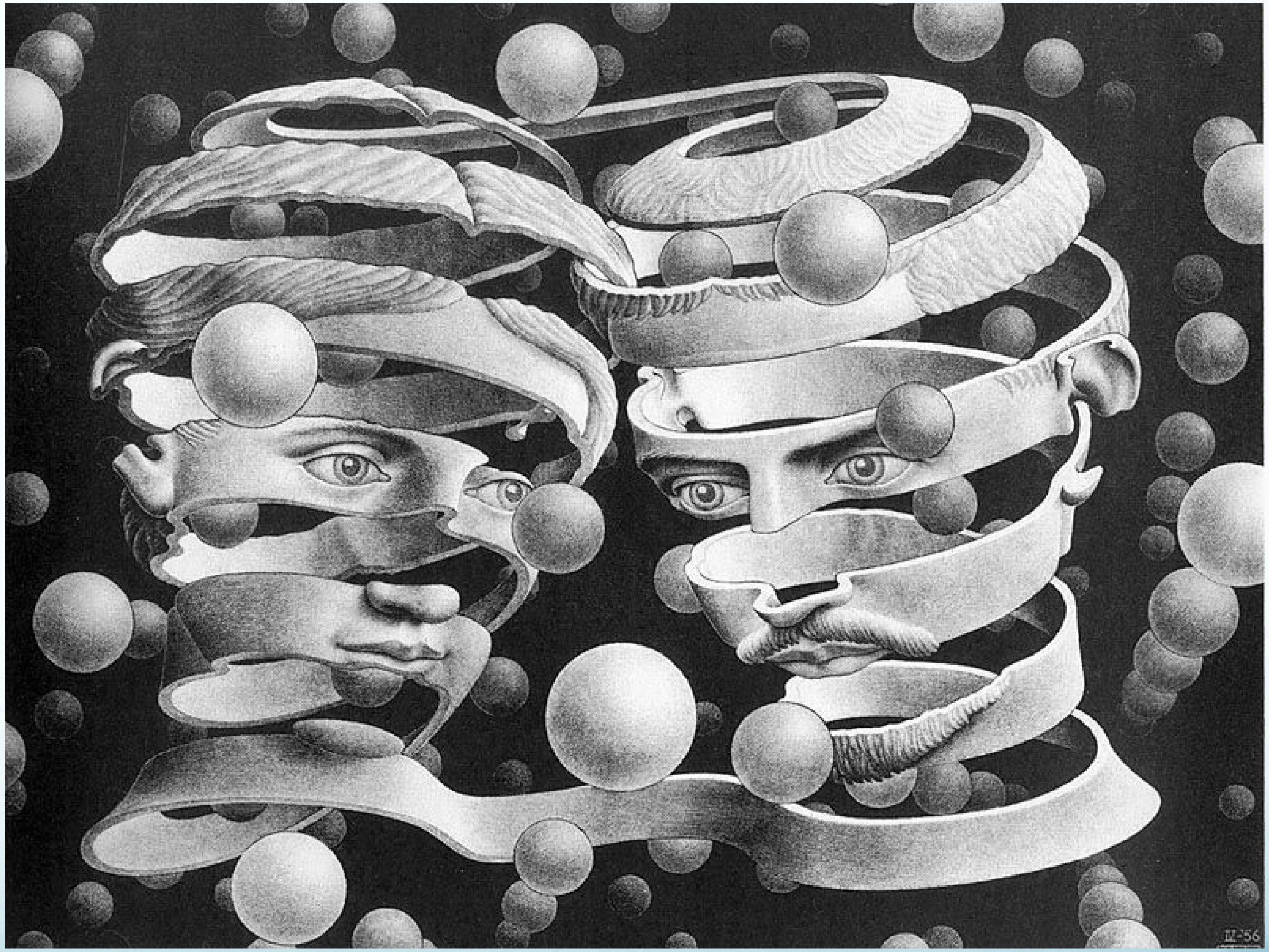
PAUSE NOW & ANSWER

Classify each of the transformations shown as one of the types below

- Identity
- Reflection
- Scaling
- Rotation
- Shear



Translation Transformation

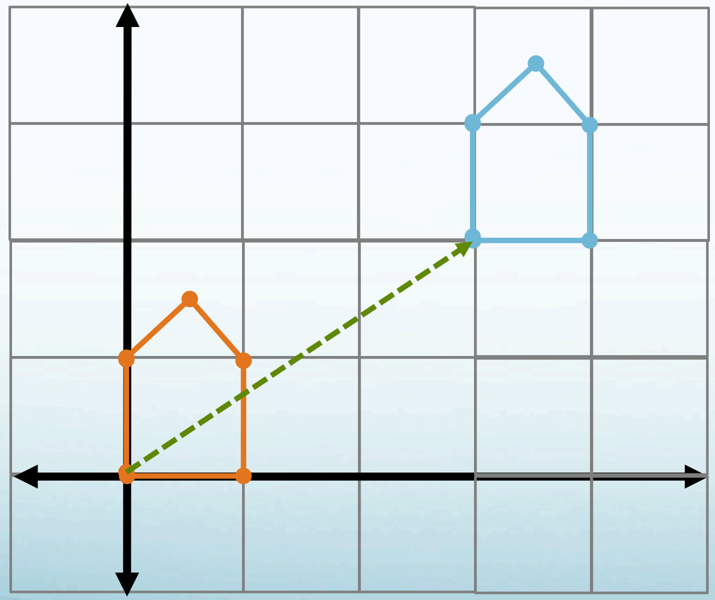


Translation Transformation

- Most natural formulation for translation is addition: $\vec{u} = T\vec{v} + \vec{b}$

$$\begin{bmatrix} 0 & 1 & 1 & 0.5 & 0 \\ 0 & 0 & 1 & 1.5 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 4 & 3.5 & 3 \\ 2 & 2 & 3 & 3.5 & 3 \end{bmatrix}$$

Translate all points by (3,2)



Need a different size translation matrix for each set of points

All other transformations are use multiplication

Can we reformulate translation as multiplication?

Translation Transformation

- Consider how matrix multiplication works

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} av_x + bv_y \\ cv_x + dv_y \end{bmatrix}$$

There's no way to add a constant

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} av_x + bv_y + t_x \\ cv_x + dv_y + t_y \\ 1 \end{bmatrix}$$

…unless we alter the structure

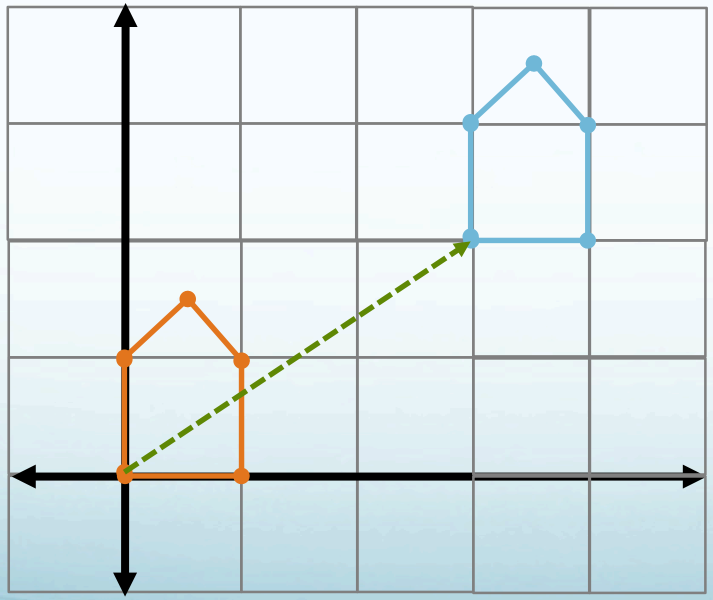
- A standard trick in computer graphics:
  - Express 2D points using **three** coordinates
  - The last coordinate is always 1

"Homogeneous Coordinates"

Translation Transformation

- New formulation for translation

$$\begin{bmatrix}1 & 0 & 3\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 1 & 0.5 & 0\\0 & 0 & 1 & 1.5 & 1\\1 & 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}3 & 4 & 4 & 3.5 & 3\\2 & 2 & 3 & 3.5 & 3\\1 & 1 & 1 & 1 & 1\end{bmatrix}$$



General form:

$$T=\begin{bmatrix}1 & 0 & t_x\\0 & 1 & t_y\\0 & 0 & 1\end{bmatrix}$$

Inverse:

$$T^{-1}=\begin{bmatrix}1 & 0 & -t_x\\0 & 1 & -t_y\\0 & 0 & 1\end{bmatrix}$$

Homogeneous Coordinates

What about all the other 2x2 transformation matrices?

- Modify them into 3x3 homogeneous matrices also

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \rightarrow \begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: if $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_x{}' \\ v_y{}' \end{bmatrix}$

then $\begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_x \\ 1 \end{bmatrix} = \begin{bmatrix} v_x{}' \\ v_x{}' \\ 1 \end{bmatrix}$

If you want to plot the points, use just the first two coordinates

2D Transformation Summary

Scaling

$$S = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translation

$$T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

X Shear

$$H = \begin{bmatrix} 1 & 0 & 0 \\ k_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Y Reflection

$$F = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Questions

PAUSE NOW & ANSWER

Classify each of the transformation matrices shown as one of the types below

- Identity
- Reflection
- Scaling
- Rotation
- Shear
- Translation
- Other/None

A *Translation* $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

B *None* $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

C *Identity* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

D *Shear* $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

E *Scaling* $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

F *Reflection* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

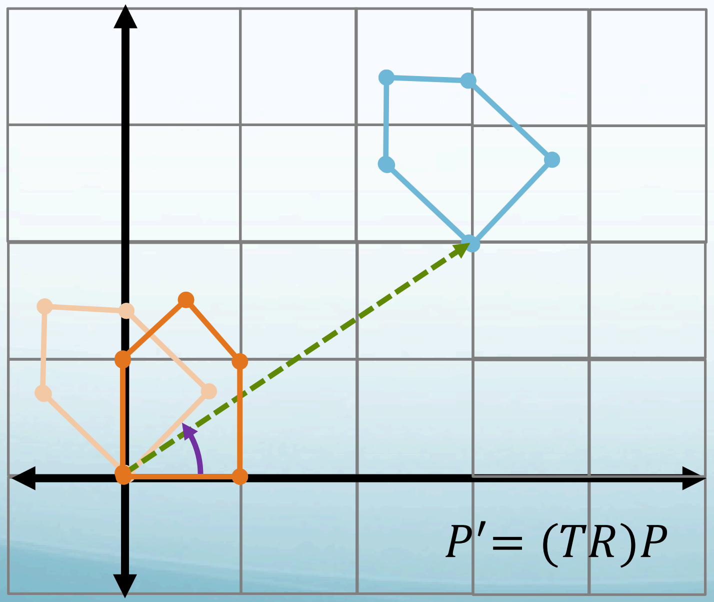
G *Rotation* $\begin{bmatrix} 0.6 & -0.8 & 0 \\ 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

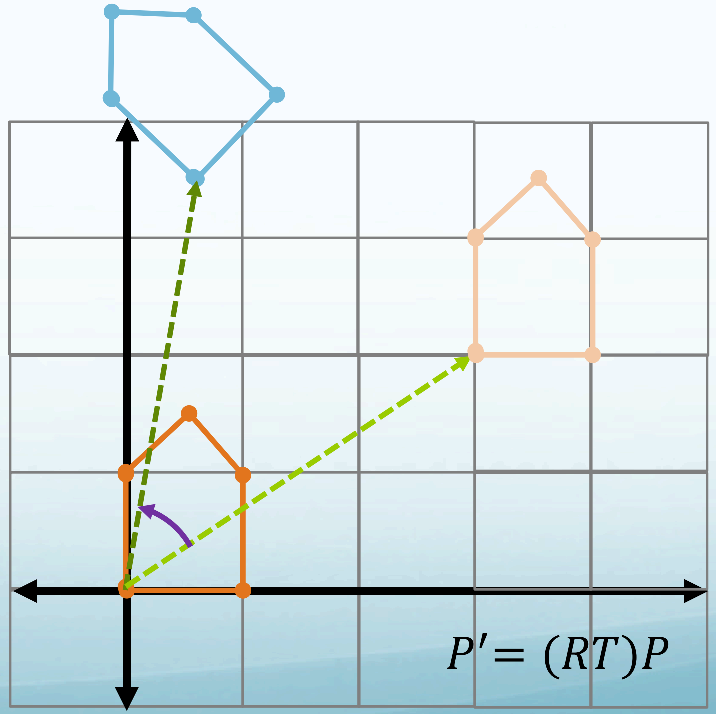
H *None* $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

I *None* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Composition

- Combining transformations is called **composition**
  - Apply multiple matrices in sequence $P' = T \cdot R \cdot P$
  - Using associativity, can combine all into single transformation matrix $P' = (TR)P$
    $M = TR$
  - Order can matter a lot!





Review

After watching this video, you should be able to…

- Define a transformation and describe why they are useful
- Describe 6 major transformation types qualitatively
- Express 6 major transformation types numerically as a matrix
- Convert a regular 2D vector into homogeneous coordinates
- Convert a 2x2 transformation matrix into a 3x3 homogenous equivalent
- Compose transformations by multiplying their matrices