CSC 240 Computer Graphics
Lecture 5: Vectors & Matrices

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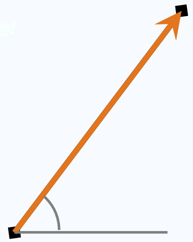
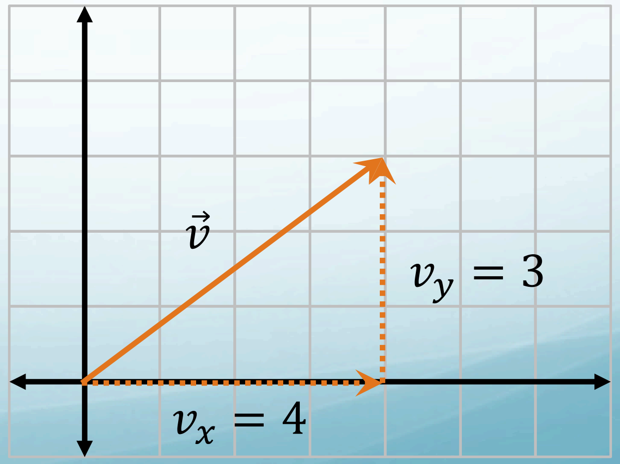
Portions based on slides & content courtesy Sara Mathieson

Vectors

- Used to represent **spatial relationships**
  From…to, offset, position, velocity, etc.
- Characterized by both **magnitude** and **direction**
  - Could use a polar representation (r,θ) in 2D
  - More common is Cartesian representation (x,y) or $\begin{bmatrix} x \\ y \end{bmatrix}$
  - Variables representing vectors: $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ or $\begin{bmatrix} v_x \\ v_y \end{bmatrix}$
- Example: $\vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

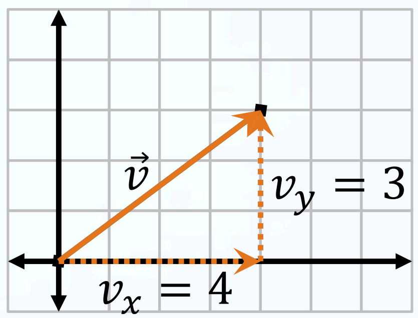
This is a compact way of saying that the vector is equivalent to 4 units in the x axis direction plus 3 units in the y axis direction





Vectors

$$|\vec{v}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$



- Vector magnitude (length): $|\vec{v}| = \sqrt{v_x^2 + v_y^2}$

Euclidean distance

- Vector direction: **unit vector** $\hat{v} = \dfrac{\vec{v}}{|\vec{v}|} = \begin{bmatrix} v_x/|\vec{v}| \\ v_y/|\vec{v}| \end{bmatrix}$

$$\hat{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} / 5 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

(Points in same direction but has length 1)

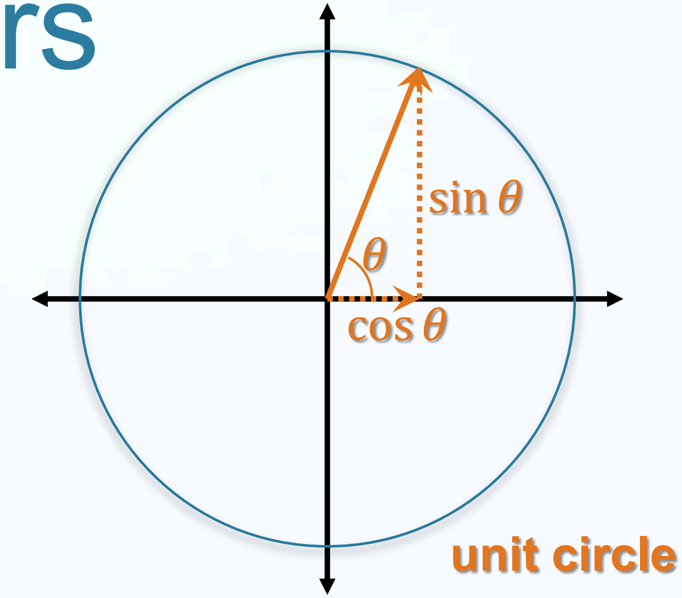
- To verify that unit vector always has length 1, plug unit vector expression into magnitude formula

$$|\hat{v}| = \sqrt{\left(\frac{v_x}{|\vec{v}|}\right)^2 + \left(\frac{v_y}{|\vec{v}|}\right)^2} = \sqrt{\left(\frac{v_x}{\sqrt{v_x^2+v_y^2}}\right)^2 + \left(\frac{v_y}{\sqrt{v_x^2+v_y^2}}\right)^2}$$

$$= \sqrt{\frac{v_x^2}{v_x^2 + v_y^2} + \frac{v_y^2}{v_x^2 + v_y^2}} = \sqrt{\frac{v_x^2 + v_y^2}{v_x^2 + v_y^2}} = \sqrt{1} = 1$$

Unit Vectors

- Unit vectors have length 1
- Components are trigonometric functions of some orientation angle (the direction of the vector!)



$$\hat{u}_\theta = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

- Note that $|\hat{u}_\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$ ✓

Questions

PAUSE NOW & ANSWER

1. What is the magnitude of the vector (5,-12)?

$$\sqrt{5^2 + (-12)^2} = \sqrt{169} = 13$$

2. Find the unit vector that points in the same direction.

$$\left(\frac{5}{13}, \frac{-12}{13}\right)$$

3. What is the unit vector for an angle of 60 degrees ($\frac{\pi}{3}$ radians)?

$$\left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \approx (0.5, 0.866)$$

Matrices

- Represents entities in the form of a grid of numbers
  - Collections of vectors
  - Spatial **transformations**
  - etc.

$$\boldsymbol{M} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

A 3x3 **matrix**

- Use capital-letter variables to represent
- Specify values as grid within square brackets
- Vector can be considered a 1-D matrix (row or column)

$$\vec{\boldsymbol{v}} = \begin{bmatrix} \mathbf{4} \\ \mathbf{3} \end{bmatrix}$$

A 2x1 **column vector**

$$\vec{\boldsymbol{u}} = \begin{bmatrix} \mathbf{5} & \mathbf{-2} \end{bmatrix}$$

A 1x2 **row vector**

(note we specify the number of rows before the number of columns)

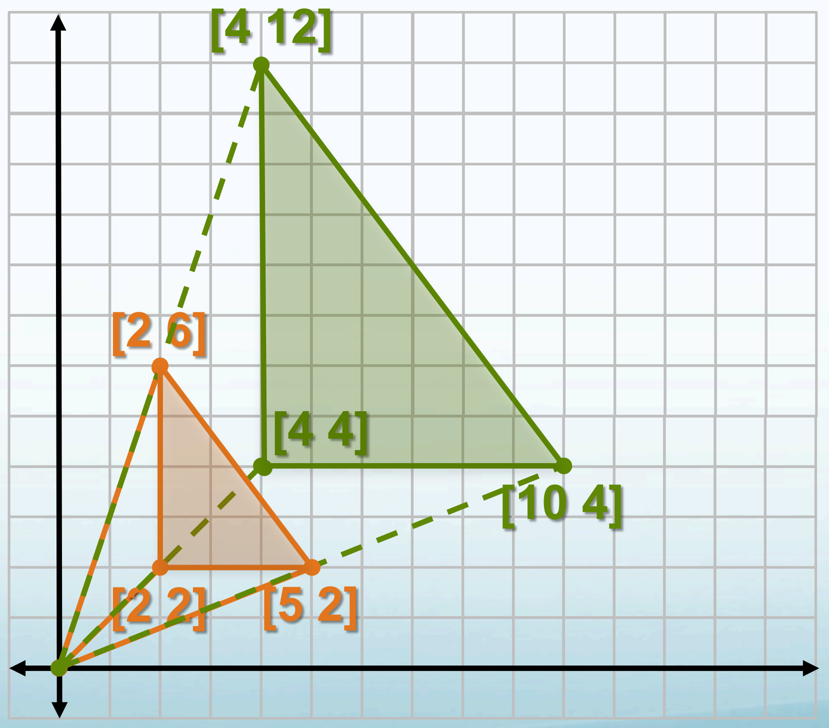
Vector & Matrix Math

Vector Math

- Scalar multiplication: $k\vec{v} = k\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} kv_x \\ kv_y \end{bmatrix}$

Multiply each component by k

What do we get if we multiply each of these vectors by 2?



2*[2 2] = [4 4]
2*[5 2] = [10 4]
2*[2 6] = [4 12]

All are 2x as far from the origin as before!

We've doubled the size of the shape!

Vector Math

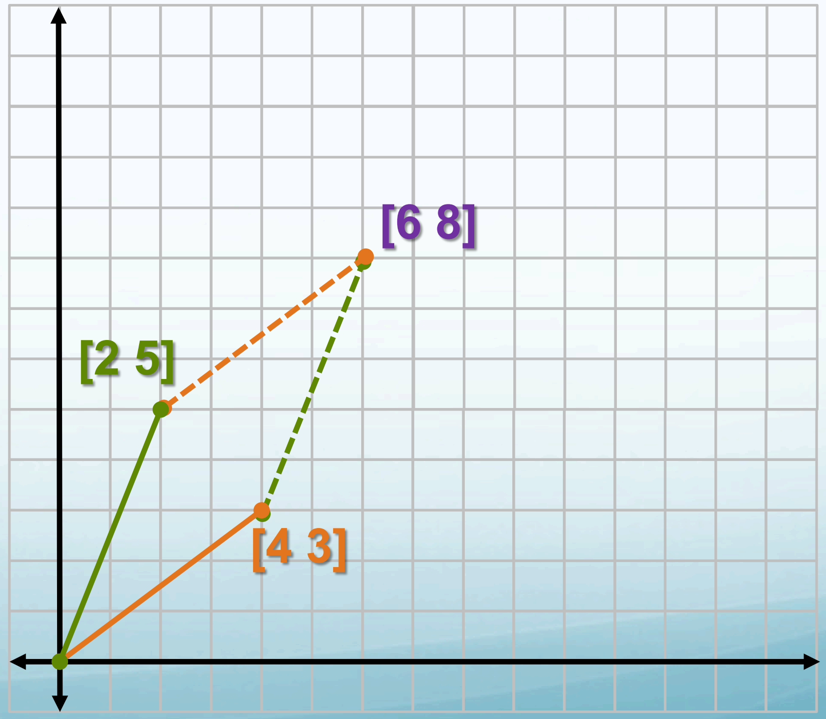
- Vector addition: $\vec{v} + \vec{u} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} + \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} v_x + u_x \\ v_y + u_y \end{bmatrix}$

Add each component separately

Example:

$\begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $= \begin{bmatrix} 4+2 \\ 3+5 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

Place the vectors head to tail – either order works!



Vector Math

Dot product: $\vec{v} \cdot \vec{u} = v_x u_x + v_y u_y$

Multiply the components separately, then add everything up

More on this later, possibly.

Note $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$

(length of a vector is the square root of its self dot product)

Matrix Math

- Scalar multiplication works element-by-element, like with vectors

$$2 \cdot \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 3 \\ 2 \cdot 2 & 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix}$$

- Matrix addition works element-by-element, like with vectors

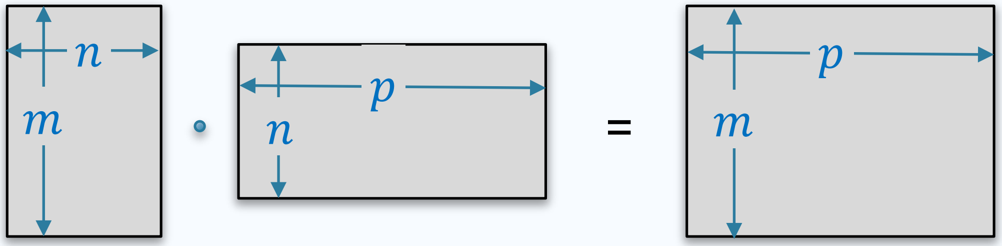
$$\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2+3 & 1-2 \\ 0+1 & -1+2 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix}$$

- Matrix multiplication is completely different!

Matrix Math

- Multiplication is **only** defined for matrices of matching size:

  An $\boldsymbol{m} \times \boldsymbol{n}$ matrix multiplies an $\boldsymbol{n} \times \boldsymbol{p}$ matrix to give an $\boldsymbol{m} \times \boldsymbol{p}$



- Compute element by element:

  Element *(i,j)* is the dot product of row *i* with column *j*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

row 1 column 1
row 1 column 2
row 2 column 1
row 2 column 2

Matrix Math

- Multiplication is defined for matrices of matching size:

  An $m \times n$ matrix multiplies an $n \times p$ matrix to give an $m \times p$

- Compute element by element

  Element (i,j) is the dot product of row i with column j

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + 1 \cdot 1 & 2 \cdot 2 + 1 \cdot 7 \\ 0 \cdot 3 + 4 \cdot 1 & 0 \cdot 2 + 4 \cdot 7 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ 4 & 28 \end{bmatrix}$$

Matrix Math

- Multiplication is defined for matrices of matching size:

  An $m \times n$ matrix multiplies an $n \times p$ matrix to give an $m \times p$

- Compute element by element

  Element *(i,j)* is the dot product of row *i* with column *j*

$$\begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + 1 \cdot 1 & 2 \cdot 2 + 1 \cdot 7 \\ 0 \cdot 3 + 4 \cdot 1 & 0 \cdot 2 + 4 \cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 11 \\ 4 & 28 \end{bmatrix}$$

Matrix Math

- What do we do about this?

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & 5 & 4 \end{bmatrix} \begin{bmatrix} 6 & 3 & 4 \\ 1 & -2 & 4 \end{bmatrix} = ?$$

- Can't multiply matrices whose sizes don't match!

  (**Columns** of first must match **rows** of second)

- When sizes do match, results may surprise.
  - A 10x1 matrix times a 1x10 matrix is a 10x10 matrix
  - A 1x10 matrix times a 10x1 matrix is a 1x1 matrix (just a number, aka a scalar)
  - A 10x1 matrix can't multiply by another 10x1 matrix
  - A 1x10 matrix can't multiply by another 1x10 matrix

Matrix Math

- Matrix multiplication is associative:

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

- It is **NOT** commutative, in general:

$$A \cdot B \neq B \cdot A$$

- For example, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

…but $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Matrix Math

To multiply an $m \times n$ matrix A by an $n \times p$ matrix B:

```
for i from 1 to m

  for j from 1 to p

    compute element (i,j) as the dot product
       of A’s row i and B’s column j

  endfor

endfor
```

This will require another loop:

$C_{ij} = 0$

```
for k from 1 to n
```

$C_{ij} = C_{ij} + A_{ik}*B_{kj}$

```
endfor
```

Questions

PAUSE NOW & ANSWER

1. What is $3 \cdot [2 \quad 1 \quad 3] + (-1) \cdot [2 \quad 1 \quad 3]$? $= [6 \quad 3 \quad 9] + [-2 \quad -1 \quad -3]$

   $[4 \quad 2 \quad 6]$ $= 2 \cdot [2 \quad 1 \quad 3]$

2. What is the size of a 2x9 matrix times a 9x17 matrix times a 17x3 matrix?

   A 2x3 matrix

3. Which of the following matrices can be multiplied?

A. ✓ $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 5 & 4 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ 4 & 1 \\ -2 & 4 \end{bmatrix}$

B. ✗ $[5 \quad 2] \begin{bmatrix} 6 & 3 \\ 4 & 1 \\ -2 & 4 \end{bmatrix}$

C. ✓ $[3 \quad 4] \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

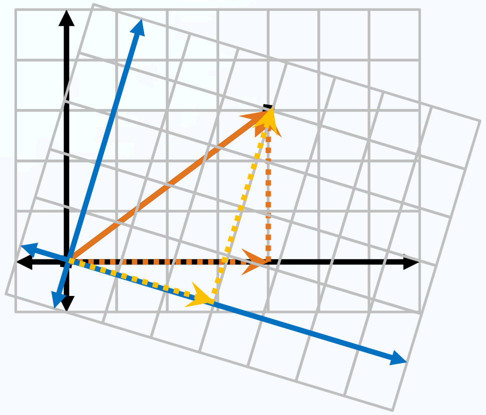
D. ✓ $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} [2 \quad 3]$

E. ✗ $\begin{bmatrix} 3 & 6 \\ 5 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ 4 & 1 \\ -2 & 4 \end{bmatrix}$

Basis Vectors

- $\begin{bmatrix}4\\3\end{bmatrix}$ can be decomposed into $\begin{bmatrix}4\\0\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix}$

  and further into $4 \cdot \begin{bmatrix}1\\0\end{bmatrix} + 3 \cdot \begin{bmatrix}0\\1\end{bmatrix}$
- Any 2-D vector can be composed this way:
  - A scalar times the x-axis unit vector $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$
  - A scalar times the y-axis unit vector $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$
- This is not the only possible decomposition!



New basis unit vectors:
$\begin{bmatrix}0.96\\-0.28\end{bmatrix}$ **and** $\begin{bmatrix}0.28\\0.96\end{bmatrix}$

New decomposition
$3 \cdot \begin{bmatrix}0.96\\-0.28\end{bmatrix} + 4 \cdot \begin{bmatrix}0.28\\0.96\end{bmatrix}$

Transformation matrix:
$\begin{bmatrix}0.96 & -0.28\\0.28 & 0.96\end{bmatrix} \cdot \begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$

These basis vectors are unit vectors for a grid rotated by 37°

If we multiply a vector by the transformation matrix, we get a rotated vector!

Review

After watching this video, you should be able to…

- Determine the magnitude and direction of a given vector
- Generate a unit vector from an ordinary vector, or from a 2D angle
- Recognize a matrix and note its dimensions
- Perform both addition and scalar multiplication on matrices and vectors
- Perform and implement matrix multiplication, where possible
- Envision vectors as a combination of unit basis vectors.

Music: https://www.bensound.com