CSC 240 Computer Graphics
Video 19: Ray Tracing Part 2

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Some slides & content courtesy Sara Mathieson
Ray Tracing Pseudocode

Loop over all pixels
  Create ray from eye through pixel center
Loop over all world objects
  Calculate intersection with ray
  Keep if closest
  Color pixel accordingly
Ray Representation

We represent light rays via a parametric vector equation

$$\vec{R}(t) = \vec{R}_0 + t\vec{R}_d$$

Example: \(\vec{R}(t) = \begin{bmatrix} 0 \\ 2 \\ 10 \end{bmatrix} + t \begin{bmatrix} 4/13 \\ 3/13 \\ -12/13 \end{bmatrix}\)

This is really three equations:

\[
\begin{align*}
x(t) &= 0 + \frac{4t}{13} \\
y(t) &= 2 + \frac{3t}{13} \\
z(t) &= 10 - \frac{12t}{13}
\end{align*}
\]

Questions:

- What is the starting point of this ray? \((0,2,10)\)
- Where is this ray at \(t = 13\)? \((4,5,-2)\)
Ray-Sphere Intersection

Easiest object to intersect with is a sphere
… we see lots of ray-traced sphere examples

How do we tell if a ray intersects with a sphere?

- Sphere surface equation:

\[(x - s_x)^2 + (y - s_y)^2 + (z - s_z)^2 = r^2\]

- Vector version:

\[\vec{p} - \vec{s} = \vec{p} - \vec{s} \cdot (\vec{p} - \vec{s}) = r^2\]

\[||\vec{p} - \vec{s}||^2 = r^2\]

\[\vec{p} = (x, y, z) \text{ is a point on the surface}\]
\[\vec{s} = (s_x, s_y, s_z) \text{ is the sphere's center}\]

Recall: if \(\vec{v} = (x, y, z)\), then
\[\vec{v} \cdot \vec{v} = \|v\|^2 = x^2 + y^2 + z^2\]
Ray-Sphere Intersection

Sphere equation: \[ \| \mathbf{p} - \mathbf{s} \|^2 = r^2 \]

Ray equation: \[ \mathbf{p} = \mathbf{R}_0 + t\mathbf{R}_d \]

Let \[ \mathbf{R}_0' = \mathbf{R}_0 - \mathbf{s} \]

\[ \| \mathbf{R}_0' + t\mathbf{R}_d \|^2 = r^2 \]

\[ (\mathbf{R}_0' + t\mathbf{R}_d) \cdot (\mathbf{R}_0' + t\mathbf{R}_d) = r^2 \]

\[ \mathbf{R}_0' \cdot \mathbf{R}_0' + 2t\mathbf{R}_0' \cdot \mathbf{R}_d + t^2 \mathbf{R}_d \cdot \mathbf{R}_d = r^2 \]

\[ \mathbf{R}_0' \cdot \mathbf{R}_0' + 2t\mathbf{R}_0' \cdot \mathbf{R}_d + t^2 \mathbf{R}_d \cdot \mathbf{R}_d = r^2 \]

- Quadratic equation in \( t \) -- zero, one, or two real solutions!

Plug the second into the first and solve for \( t \)

Then use \( t \) in the ray equation to find \( \mathbf{p} \)

Vector dot product

Distribute
Ray-Sphere Intersection

\[ \vec{R}_0' \cdot \vec{R}_0' + 2t\vec{R}_0' \cdot \vec{R}_d + t^2\vec{R}_d \cdot \vec{R}_d = r^2 \]

Let:

\[ a = \vec{R}_d \cdot \vec{R}_d, \quad b = 2\vec{R}_0' \cdot \vec{R}_d, \quad c = \vec{R}_0' \cdot \vec{R}_0' - r^2 \]

Solve via quadratic equation:

\[ t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

- zero, one, or two solutions
- Ignore solutions with \( t < 0 \) (Why?)
- \( b^2 - 4ac > 0 \): 2 solutions
- \( b^2 - 4ac = 0 \): 1 solution
- \( b^2 - 4ac < 0 \): 0 solutions
Ray-Sphere Intersection

Suppose \( R_0 = (0,0,0) \) and \( R_d = (1,0,0) \) and our sphere has radius 5 with center \( s = (10,3,0) \)

What is (are) the intersection point(s)?

1. Compute \( R_0' = R_0 - s \)
   \[
   R_0' = (0,0,0) - (10,3,0) = (-10, -3, 0)
   \]

2. Compute \( a = R_d \cdot R_d, \quad b = 2R_0' \cdot R_d, \quad c = R_0' \cdot R_0' - r^2 \)
   \[
   a = (1,0,0) \cdot (1,0,0) = 1 + 0 + 0 = 1
   \]
   \[
   b = 2(-10, -3, 0) \cdot (1,0,0) = 2(-10 \cdot 1 - 3 \cdot 0 + 0) = -20
   \]
   \[
   c = (-10, -3, 0) \cdot (-10, -3, 0) - 5^2 = 100 - 9 + 0 - 25 = 84
   \]

3. Compute \( t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)
   \[
   t = \frac{20 \pm \sqrt{(-20)^2 - 4(84)}}{2(1)} = 6 \text{ or } 14
   \]

4. Compute \( \hat{p} = R_0 + tR_d \)
   \[
   \hat{p} = (0,0,0) + 6(1,0,0) = (6,0,0)
   \]

\( \hat{p} = (6,0,0) \)
Questions

1. If you solve the ray-sphere intersection and get two answers for $t$, which one should you use?
   *Use the smallest positive value of $t$.*

2. What does it mean if one of the solutions for $t$ is negative and the other is positive?
   *The starting point of the ray is inside the sphere.*

3. If the ray intersects with multiple spheres, how do you decide which one matters?
   *Use the smallest positive value of $t$.*
Ray-Triangle Intersection

Where does a ray intersect with a triangle?

- Corners of triangle define a plane
  \[ \vec{p} = \vec{A} + u(\vec{B} - \vec{A}) + v(\vec{C} - \vec{A}) \]

- Set equal to ray formula and solve:
  \[ \vec{R}_0 + t\vec{R}_d = \vec{A} + u(\vec{B} - \vec{A}) + v(\vec{C} - \vec{A}) \]

- Yields three equations for three unknowns
  \[
  \begin{align*}
  x_0 + tx_d &= a_x + u(b_x - a_x) + v(c_x - a_x) \\
  y_0 + ty_d &= a_y + u(b_y - a_y) + v(c_y - a_y) \\
  z_0 + tz_d &= a_z + u(b_z - a_z) + v(c_z - a_z)
  \end{align*}
  \]

- Point lies inside triangle iff \( t, u, v \geq 0 \) and \( u + v \leq 1 \)
Rays & Surfaces

What about other, more complex shapes?

- Could work out math for other shapes, e.g., NURBS
- In practice, best to specialize on one primitive
- Everything is converted to triangle meshes
  - Repeat triangle intersections \textit{ad infinitum}
  - Speedup: grouping & bounding
    (No intersection with bounding sphere means no need to test polygons inside)
Recall that a more realistic simulation requires following rays as they reflect and refract through the scene

- How do we find the reflection angle?
- How do we find the refraction angle?
Recursive Ray Tracing: Mirror Reflection

At reflective surfaces, some light bounces off

- Angle of incidence $\hat{E}$ and reflected ray $\hat{R}$ match
- Coplanar with surface normal $\hat{N}$

$$\hat{R} = \hat{E} - 2(\hat{E} \cdot \hat{N})\hat{N}$$
Recursive Ray Tracing: Transmission

At translucent/transparent surfaces, some light passes through

- Amount transmitted is property of material: \( N = \text{index of refraction} \)
- Angle change depends on relative indices of refraction and incoming angle

\[ N_1 \sin \theta_1 = N_2 \sin \theta_2 \quad \text{(Snell’s Law)} \]

\[ \theta_2 = \sin^{-1} \left( \frac{N_1}{N_2} \sin \theta_1 \right) \]

\[ \theta_2 = \sin^{-1} \left( \frac{N_1}{N_2} \sin \cos^{-1} (\hat{n} \cdot \vec{R}_d) \right) \]

Index of refraction

Angle of incidence

\( \theta_1 \)

\( \theta_2 \)

\( N_1 \)

\( N_2 \)

\( \hat{n} \)

\( \vec{R}_d \)

N=1 for air
1.3 water
1.5 glass
2.4 diamond

Dot product of \( \vec{R}_d \) and surface normal
Multiple Ray Tracing

A pixel has area: Multiple rays from the focal point pass through it
- Each sees a slightly different view, may strike different objects
- Pixel averages results of all rays passing through it
- In practice: single center ray, minigrid, or multiple random selections
  Adaptive method: use extra rays where big differences observed
As a stunt, people have written ray tracers that fit on a business card!

Code at left makes image on right!

c++ -03 card.cpp
./a.out > card.ppm
Questions

1. In what situation might you be unable to solve for $t$ when intersecting the ray equation with the plane of a triangle? *When the ray and the plane are parallel.*

2. Why do we also need to solve for $u$ and $v$? (What do they tell us?) *We need $u$ and $v$ to tell us whether the intersection point lies inside the triangle.*

3. A light ray in air strikes a glass surface at an angle, as shown. Does the transmitted ray bend up, down, or stay the same? *It bends down. (This is called refraction.)*

If the index of refraction of the new material was smaller than the original, it would bend up.
Review

After watching this video, you should be able to…

- Compute the intersection of a ray and a sphere
- Compute the intersection of a ray and a triangle
- Use a simple technique to rule out intersections with many objects at once
- Compute the new direction vector of a mirror reflection
- Compute the new direction vector of a transmitted ray with refraction
- Describe how to improve image quality by tracing additional rays

Music: https://www.bensound.com