4

MISCELLANEOUS SHAPES

4.1. INTRODUCTION

Five generic shapes of polygons have been usefully distinguished in the literature: convex, orthogonal, star, spiral, and monotone.[1] Convex polygons obviously do not lead to interesting art gallery theorems, since the answer to most questions is 1, and orthogonal polygons were discussed in Chapter 2. In this chapter we cover the three "miscellaneous" shapes: star, spiral, and monotone. Other shapes could be considered—for example, orthogonal spiral—but highly specialized shapes do not lead to interesting theorems. The three shapes considered here have arisen in "practice," and each can be recognized in linear time.

We examine vertex guards (as a function of both n, the number of vertices, and r, the number of reflex vertices), edge, diagonal, and line guards. The results obtained are summarized in Table 4.1. Most of the results are easy to obtain (with one exception), and are often established by a single example.

Table 4.1

Guard Type	Star	Spiral	Monotone
vertex	$\lfloor n/3 \rfloor$	$\lfloor n/3 \rfloor$	$\lfloor n/3 \rfloor$
	$\lfloor r/2 \rfloor + 1$	$\lfloor r/2 \rfloor + 1$	$\lfloor r/2 \rfloor + 1$
edge	$\geq \lfloor n/5 \rfloor$		
diagonal	2	$\lfloor (n+2)/5 \rfloor$	
line	1		$\lfloor (n+2)/5 \rfloor$

1. Orthogonal is often called "rectilinear," and star is usually called "star-shaped" in the literature.

4.2. STAR POLYGONS

A *star polygon* P is a polygon that may be covered by a single guard: there is a point $x \in P$ such that every point of P is visible from x. The set of all points of P that can see every point of P is called the *kernel* of P. Thus a star polygon is one with a non-null kernel. It is easy to see that the kernel is the intersection of all the interior half-planes determined by the edges of P (*interior half-planes* are towards the left in a counterclockwise traversal of the boundary). Thus the kernel is convex. This characterization leads to an $O(n \log n)$ algorithm for constructing the kernel by using Shamos's half-plane intersection algorithm (Shamos 1978). Lee and Preparata showed, however, that the kernel can be constructed in $O(n)$ time (Lee and Preparata 1979). Thus the question of whether a given polygon is a star can be answered in linear time by checking if the kernel is empty. This linear-time recognition capability increases the usefulness of the star class.

Although every star polygon can be covered by one point guard, more interesting questions arise if restrictions are placed on the guard. If the guards are restricted to vertices only, then $\lfloor n/3 \rfloor$ are sometimes necessary, as can be seen by warping the "comb" example (Fig. 1.2) to a star "sun burst" shape shown in Fig. 4.1. Note that, aside from the spike apex, only the two vertices at the base of each spike can see the spike completely. That $\lfloor n/3 \rfloor$ is sufficient of course follows from Chvátal's theorem (Theorem 1.1).

In terms of the number of reflex vertices, a slight modification of Fig. 4.1, shown in Fig. 4.2, establishes the necessity of $\lfloor r/2 \rfloor + 1$ vertex guards. Sufficiency is established as follows. Let x be a point in the kernel. Connect x to every reflex vertex by a line segment, as illustrated in Fig. 4.3. Let y be a reflex vertex. Note that xy resolves the reflex vertex at y, leaving convex angles on either side. Thus the r "spokes" from x partition the polygon into r pieces, at least $r-1$ of which are convex. There may be at most one piece non-convex at x. Suppose all pieces are convex. Then placing a guard at every second reflex vertex covers the polygon with $\lceil r/2 \rceil$ guards. If there is one non-convex piece, then cover it with two guards, one at each (former) reflex vertex on its boundary, and again place a guard at every second reflex vertex in the remainder. The result is coverage by $2 + \lceil (r-3)/2 \rceil = \lceil (r+1)/2 \rceil$ guards. Thus $\lceil (r+1)/2 \rceil = \lfloor r/2 \rfloor + 1$ guards suffice in either case.

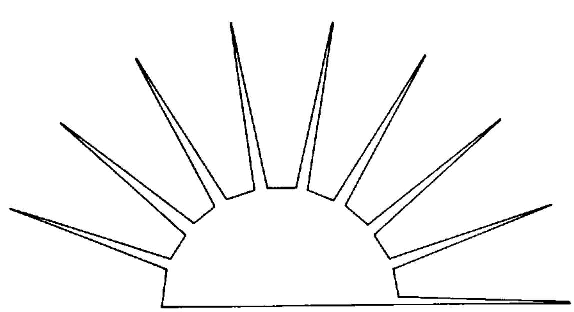

Fig. 4.1. A star polygon that requires $\lfloor n/3 \rfloor$ vertex guards.

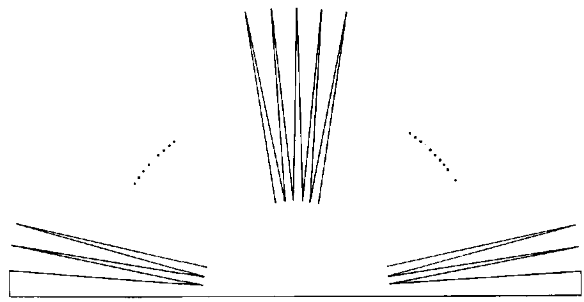

Fig. 4.2. A star polygon that requires $\lfloor r/2 \rfloor + 1$ vertex guards.

Since r may be as large as $n-3$ (see Fig. 1.26), this result may be worse than Chvátal's $\lfloor n/3 \rfloor$, but it is better whenever $r < 2\lfloor n/3 \rfloor - 2$.

For edge guards, the only result known is that at least $\lfloor n/5 \rfloor$ edge guards are necessary. This is established by another "sun burst" example due to Toussaint, shown in Fig. 4.4. In this figure, the endpoints of each edge on the lower semicircle are diametrically opposed to the vertices separating the spikes. Thus, if the spikes are long enough, the apex of each spike is visible from only one edge on the lower semicircle. For example, apex A in Fig. 4.4 is only visible to e. Of course, A is also visible from the edges adjacent to its two base vertices a and b. But the conclusion remains that each spike requires its own edge guard. The figure has $n = 5s$ vertices if there are s spikes, and therefore establishes that $s = \lfloor n/5 \rfloor$ edge guards are necessary. Whether this many edge guards is always sufficient remains an open problem.

Mobile guards are more powerful in star polygons. If the patrol is unrestricted (a *line* guard in the notation of the previous chapter), then clearly one guard suffices: just choose a line that intersects the kernel. If the patrol is restricted to vertex-to-vertex diagonals or edges (*diagonal* guards), then it may be that no diagonal intersects the kernel, as in Fig. 4.5. But if the kernel does not intersect any diagonal, then it must lie inside one triangle T of any triangulation. Then placing guards along any two sides of T will cover the entire polygon, since any line through the kernel must intersect the boundary of T in two locations, and at least one of these must lie on a side covered by a guard. That two diagonal guards are sometimes necessary is established by either of the polygons shown in Fig. 4.6, due to

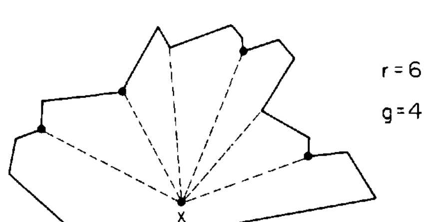

Fig. 4.3. A partition of a star made by connecting a kernel point x to every reflex vertex.

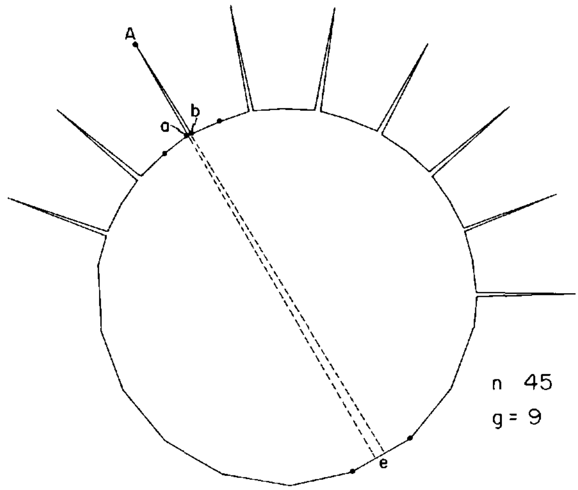

Fig. 4.4. A star polygon requiring $\lfloor n/5 \rfloor$ edge guards.

Shermer and Suri. In both figures, not all vertices are visible from any single diagonal. For example, in Fig. 4.6a, diagonal (4, 9) cannot see 2, and in Fig. 4.6b, diagonal (13, 14), cannot see 1 or 11.

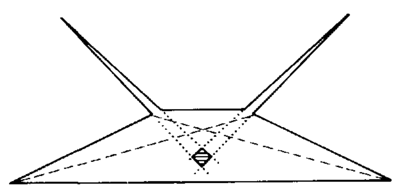

Fig. 4.5. No diagonal intersects the kernel (shaded).

The following theorem summarizes the results of this section.

THEOREM 4.1 [Toussaint 1982]. For coverage of a star polygon of n vertices and r reflex vertices, $\lfloor n/3 \rfloor$ and $\lfloor r/2 \rfloor + 1$ vertex guards are necessary and sufficient, $\lfloor n/5 \rfloor$ edge guards are necessary, and 2 diagonal guards are necessary and sufficient.

4.3. SPIRAL POLYGONS

A *reflex chain* of a polygon is a sequence of consecutive reflex vertices. A *spiral polygon* is a polygon with at most one reflex chain. Feng and Pavlidis studied decomposition of polygons into spiral pieces for its application to character recognition (Feng and Pavlidis 1975; Pavlidis and Feng 1977). Spiral polygons are easily recognized in linear time with a single boundary traversal.

Spiral polygons may require $\lfloor n/3 \rfloor$ vertex or point guards, although here the example is not a simple distortion of the comb shape. The generic example consists of $2k$ equally spaced vertices on the circumference of a circle, and k more vertices on a slightly larger concentric circle. See Fig. 4.7 for an instance with $k = 6$ and $n = 3k = 18$ vertices. There are $k + 2$ convex vertices and $r = 2k - 2$ reflex vertices. Let the vertices on the inner circle occur at multiples of α degrees; then the vertices on the outer circle occur at multiples of 2α. The outer radius is chosen close enough to the inner radius so that each convex vertex on the outer circle (not near either junction between the convex and reflex chains) can see just three vertices on the inner circle, and each reflex vertex on the inner circle can see just two vertices on the outer circle. Placing guards at each vertex on the outer circle, or every other vertex on the inner circle, both result in complete coverage with $k = \lfloor n/3 \rfloor$ guards. It is easily seen that no advantage is gained

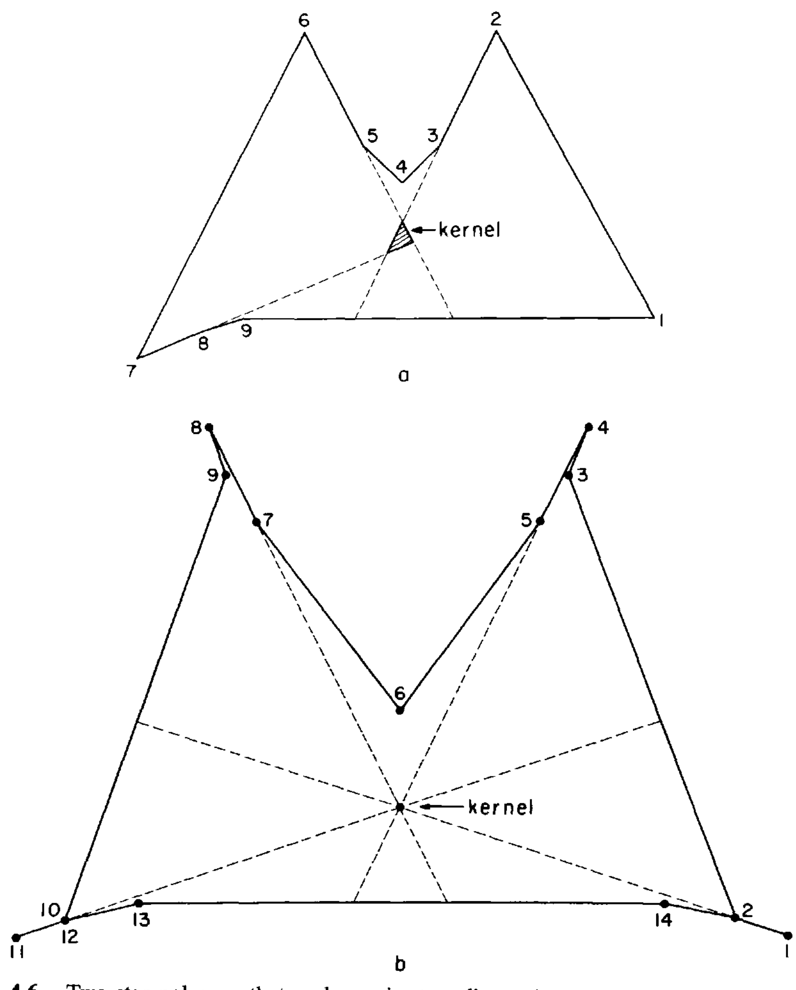

Fig. 4.6. Two star polygons that each require two diagonal guards, due to Shermer (a) and Suri (b).

by mixing guards on the outer and inner circle, or by placing guards on points other than vertices. This establishes that $\lfloor n/3 \rfloor$ point guards are necessary, and Chvátal's theorem gives sufficiency.

In terms of the number of reflex vertices r, the same example establishes $\lfloor r/2 \rfloor + 1$ necessity: $r = 2k - 2$, so $\lfloor r/2 \rfloor + 1 = k$. Sufficiency is established with Chazelle's "naive" convex partitioning, obtained by bisecting every reflex vertex, as shown in Fig. 4.8. The resulting $r + 1$ convex regions can be covered by guards placed at every other reflex vertex, and one additional guard for the last convex region if r is even—that is, $\lfloor r/2 \rfloor + 1$ guards always suffice.

Although the edge guard problem for spiral polygons has not been investigated, the diagonal guard problem has been solved: $\lfloor (n+2)/5 \rfloor$ are necessary and sufficient. It is somewhat easier to understand the example that establishes necessity after seeing the sufficiency proof, so we will proceed with sufficiency first. It depends on two observations: a spiral polygon may be triangulated so that the dual to the triangulation is a path, and a five triangle polygon may always be covered by one diagonal guard. The first observation is proved in the next lemma; the second observation is Lemma 3.4.

LEMMA 4.1. There exists a triangulation of any spiral polygon whose dual is a path—that is, a tree with just two leaves.

Proof. Let a spiral polygon have c convex and r reflex vertices. The proof is by induction on c and r. The induction hypothesis is that every spiral polygon with $c' < c$ and $r' \leq r$, or $c' \leq c$ and $r' < r$, has a path triangulation dual such that the convex vertices a and b that are adjacent to the ends of the reflex chain are the apexes of the leaf triangles.

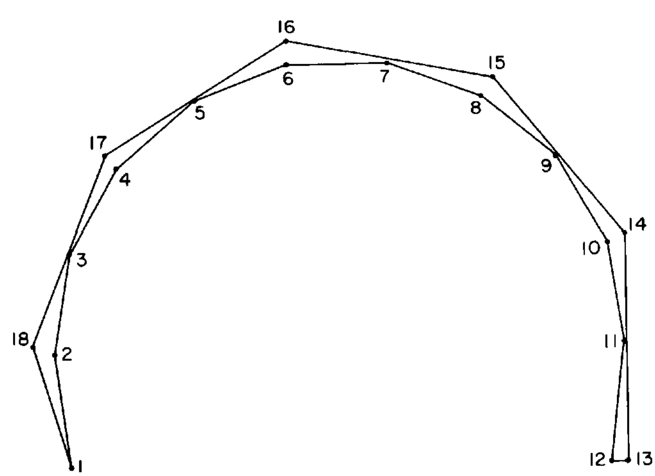

Fig. 4.7. A spiral polygon that requires $\lfloor n/3 \rfloor$ point guards.

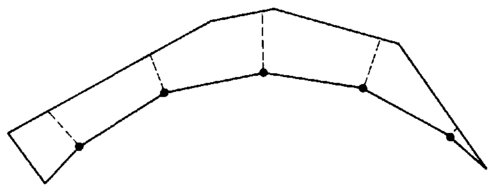

Fig. 4.8. A partition of a spiral polygon into $r + 1$ convex pieces.

If $r = 0$, then P is a convex polygon, every triangulation is possible, and the additional clause of the hypothesis hold vacuously. If $r = 1$, or if $c = 3$ (which is its minimum value), then the basis for the induction is established by the triangulations shown in Figs. 4.9a and 4.9b, respectively.

Assume now that the induction hypothesis holds. Let x be the reflex and y the convex vertex adjacent to a, the convex vertex defined in the hypothesis. Then it is clear that x must be able to see y. Cut off the triangle xay, forming a new spiral polygon P' with c' convex and r' reflex vertices. There are two cases to consider, depending on whether the new angle at x is reflex or convex.

Case 1 (x is reflex in P' (Fig. 4.10a).) Then $c' = c - 1$ and $r' = r$; y becomes a' in P'.

Case 2 (x is convex in P' (Fig. 4.10b).). Then $c' = c$ and $r' = r - 1$; x becomes a' in P'.

In either case the induction hypothesis applies, yielding a path with xy an edge of a leaf triangle. Attaching a node for xay to this yields a path for P satisfying the induction hypothesis. □

We can now prove the claimed sufficiency theorem easily. Let a spiral polygon have n vertices, and chose a triangulation whose dual is a path of $t = n - 2$ nodes as guaranteed by the above lemma. By Lemma 3.4, a septagon may always be covered by one diagonal guard. Thus each five triangles in the path may be covered by one diagonal guard. This yields a total coverage by $\lceil t/5 \rceil = \lceil (n-2)/5 \rceil = \lfloor (n+2)/5 \rfloor$ guards.

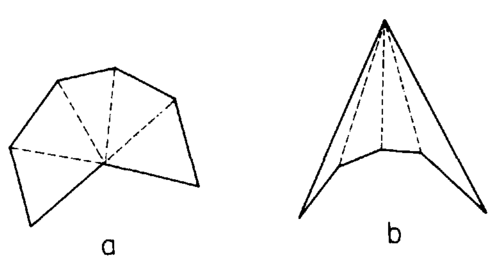

Fig. 4.9. Path triangulations of spiral polygons with one reflex vertex (a) and three convex vertices (b).

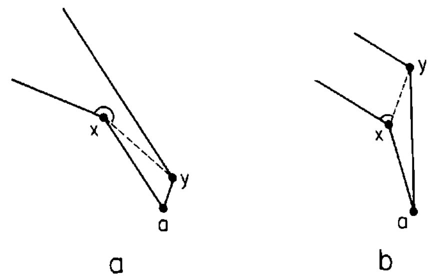

Fig. 4.10. Cutting off an ear from a spiral.

Necessity of this many guards is established by constructing a spiral polygon that only has one triangulation, whose dual is a path. This may be accomplished with $n = 10k + 3$ vertices, with $5k + 2$ convex and $5k + 1$ reflex, arranged as illustrated in Fig. 4.11 for $k = 1$. It should be clear that the only triangulation of this polygon is the one shown. Coverage of five triangles by one guard is then the best possible, and since there are $t = n - 2 = 10k + 1$ triangles, $\lceil t/5 \rceil = \lfloor (n+2)/5 \rfloor$ guards are necessary.

These results on spiral polygons are summarized in the following theorem (Aggarwal 1984).

THEOREM 4.2 [Aggarwal 1984]. For a cover of a spiral polygon of n vertices, r of which are reflex, $\lfloor n/3 \rfloor$ and $\lfloor r/2 \rfloor + 1$ point guards are necessary and sufficient, and $\lfloor (n+2)/5 \rfloor$ diagonal guards are necessary and sufficient.

4.4. MONOTONE POLYGONS

The results known for monotone polygons match those for spiral polygons exactly, suggesting that they are in some sense equally restrictive classes. Necessity for point and mobile guards follows by noting that the critical spiral polygons (Figs. 4.7 and 4.11) are also monotone if the reflex angles are chosen close to 180°. Also, of course, the comb example is monotone. The sufficiency proofs, however, are different. That $\lfloor r/2 \rfloor + 1$ vertex guards

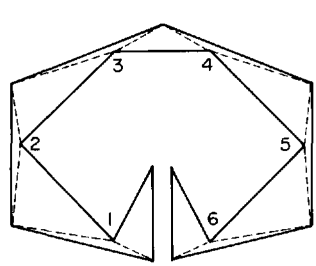

Fig. 4.11. A spiral polygon requiring $\lfloor (n+2)/5 \rfloor$ diagonal guards.

suffice is easily established by drawing a horizontal line through every reflex vertex (assuming the axis of monotonicity is vertical). This partitions the polygon into $r+1$ convex pieces, each with a reflex vertex on its boundary. Placing guards at every other reflex vertex in a vertical sort establishes that $\lceil (r+1)/2 \rceil = \lfloor r/2 \rfloor + 1$ guards suffice.

The sufficiency proof for $\lfloor (n+2)/5 \rfloor$ mobile guards is the only one amongst these specialized results that is difficult. The interested reader is referred to Aggarwal's thesis (Aggarwal 1984), which contains a 15-page proof. The guards used in his proof are not always diagonal guards, so the result is only established for line guards. In summary we have this theorem.

THEOREM 4.3 [Aggarwal 1984]. For a cover of a monotone polygon of n vertices, r of which are reflex, $\lfloor n/3 \rfloor$ and $\lfloor r/2 \rfloor + 1$ point guards are necessary and sufficient, and $\lfloor (n+2)/5 \rfloor$ line guards are necessary and sufficient.

Finally we mention again that Preparata and Supowit found a linear-time algorithm for computing the set of directions with respect to which a polygon is monotone (Preparata and Supowit 1981).