10

THREE DIMENSIONS AND MISCELLANY

10.1. INTRODUCTION

In this final chapter, four miscellaneous topics are discussed: three dimensions, line segment obstacles, point obstacles, and mirrors.

10.2. THREE DIMENSIONS

Very little is known about art gallery theorems in three dimensions. In this section we present three negative results that collectively show that there is a vast difference between the problem in two and three dimensions, and one positive result concerning convex polyhedra.

10.2.1. Untetrahedralizable Polyhedra

The reason that progress in three dimensions has been difficult is that the main tool used throughout this book for two-dimensional problems—triangulation—does not generalize. Lennes proved in 1911 the surprising theorem that there exist polyhedra (even of genus zero, i.e., without holes) whose interior cannot be partitioned into tetrahedra whose vertices are selected from the polyhedra vertices (Lennes 1911). Schönhardt later gave a simpler example (Schönhardt 1928), which we present here, based on Bagemihl's exposition (Bagemihl 1948).

Let a, b, and c be the vertices (labeled counterclockwise) of an equilateral triangle of unit edge length in the xy-plane. Let a', b', and c' be the vertices of abc when translated up to the plane $z = 1$, as shown in Fig. 10.1a. Define an intermediate polyhedron P' as the hull of the two triangles, including the diagonal edges ab', bc', and ca', as well as the vertical edges aa', bb', and cc', and the edges in the two triangles abc and $a'b'c'$. Now twist the top triangle $a'b'c'$ 30° counterclockwise in the plane $z = 1$, rotating and stretching the attached edges accordingly. The result is shown in Fig. 10.1b; a view from $z = \infty$ is shown in Fig. 10.1c. Call the resulting polyhedron P.

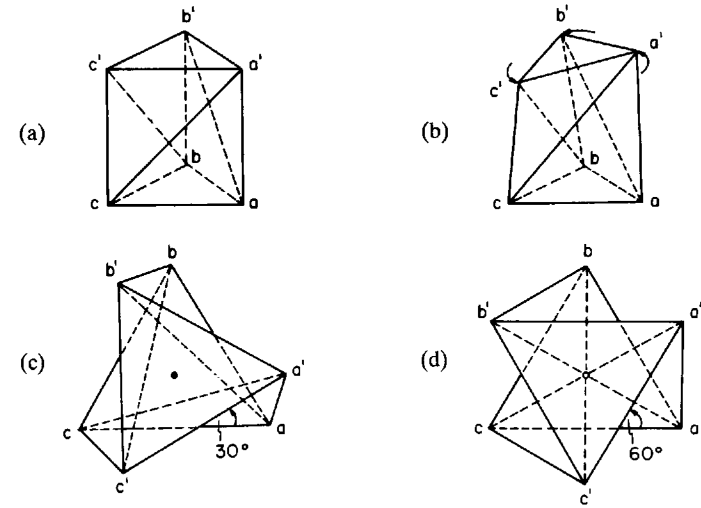

Fig. 10.1. Schönhardt's untetrahedralizable polyhedron, constructed by twisting the top of a triangular prism (a) by 30°, producing (b), shown in top view (c); a twist of 60° would cause face intersections (d).

First note that P is indeed a valid polyhedron: it would take a twist of 60° (shown in an overhead view in Fig. 10.1d) to "pinch off" the interior. Now we show that any tetrahedron whose vertices are selected from those of P includes points exterior to P. This is established with the help of two claims:

(1) Every open segment whose endpoints are vertices of P but which is not an edge of P, is exterior to P.
(2) Every triangle whose sides are edges of P is a face of P.

P has 6 vertices and 12 edges. Since $\binom{6}{2} = 15$, only three segments need be checked to verify claim (1): ac', ba', and cb'. All three are clearly seen to be exterior from Fig. 10.1c. Claim (2) can be checked at a single vertex, say a, as all have the same local connections. And indeed, it is the case that for every pair of edges of P incident to a, either a third edge of P forms a face of P, or there is no third edge of P forming a triangle.

Now, by claim (1), every edge of an interior tetrahedron T must be an edge of P. By claim (2), this means that every face of T is a face of P. But since P is a valid polyhedron, this implies that $T = P$, a contradiction to the fact that P has 6 vertices.

Schöhardt proved that this is the smallest example of an untetrahedralizable polyhedron. Bagemihl extended this example to construct a polyhedron of n vertices with the same properties for every $n \geq 6$. As far as I am aware there is no characterization of which polyhedra are tetrahedralizable. It seems likely that there is a nice art gallery theorem for tetrahedralizable polyhedra; this remains an area for future exploration.

10.2.2. $\Omega(n^{3/2})$ Guards Necessary

It seems almost obvious that guards posted at every vertex of a polyhedron cover the entire interior. But this would only be obvious if every polyhedron were tetrahedralizable. For then every tetrahedron would have a guard in a corner (in fact in all four corners), and the tetrahedra would cover the interior. In the absence of tetrahedralization, however, the "obviousness" of complete coverage is less clear. In fact, we describe in this section a polyhedron constructed by Seidel that has these two properties:

(1) Guards placed at every vertex do *not* cover the interior.
(2) $\Omega(n^{3/2})$ guards are necessary, where n is the number of vertices.

The polyhedron that realizes these properties is orthogonal and of genus zero. It may be constructed as follows.

Start with a cube of side length L. On the front face mark squares of side length 1 in a regular $k \times k$ array, with $1 + \varepsilon$ separation between each row and column, where $\varepsilon \ll 1$, as illustrated in Fig. 10.2. Thus L should be chosen to be larger than $(2 + \varepsilon)k$. Attach a $1 \times 1 \times (L - \varepsilon)$ rectangular box behind each square inside the cube, and remove the square on the front face. The result is a deep dent at each square that does not quite reach the back face of the cube. Apply the same procedure for the right face, and for the top face, staggering the $k \times k$ arrays so that none of the box dents intersect. The resulting polyhedron has $n = 8(3k^2 + 1)$ vertices.

Figure 10.3 shows a top-view cross section of the interior. Point x in the figure is confined inside a $(1 + \varepsilon) \times (1 + \varepsilon) \times (1 + \varepsilon)$ cube bound by six box dents, two from each of three directions; x is at the center of this cube. This cube space is not closed, but has $\frac{1}{2}$ ε-cracks along all 12 edges. Nevertheless, it should be clear that x is not visible from any vertex if ε is chosen to be much smaller than 1.

This establishes the first claimed property. The second claim follows by noting that there are $(k - 1)^3$ equivalent points x, no two of which are visible from the same point. Thus at least $g = (k - 1)^3$ guards are necessary, and

$$g = \approx (n/24)^{3/2} \approx n^{3/2}/118 = \Omega(n^{3/2}).$$

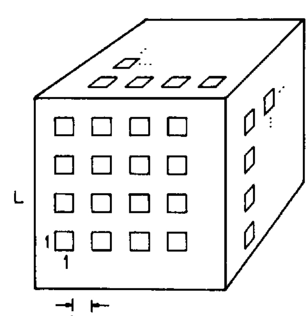

Fig. 10.2. Exterior view of Seidel's polyhedron showing array of dents.

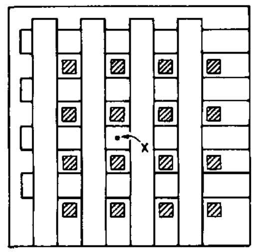

Fig. 10.3. Cross section of Seidel's polyhedron: point x is not visible to any vertex.

Note that the fact that a guard at each vertex does not suffice for coverage implies that Seidel's example is not tetrahedralizable. Finally, the example may be "turned inside-out" to establish the same bound for exterior visibility.

10.2.3. Convex Partitions

In the absence of tetrahedralization, it is natural to attempt to approach three-dimensional art gallery problems through convex partitions, which proved useful in two dimensions (Section 1.4). Our final negative result is that there are polyhedra that require $\Omega(n^2)$ convex pieces in any convex partition of a polyhedron of n vertices. This result was established by Chazelle (1984), who also provided an algorithm that finds a partition into at most $\frac{1}{2}r^2 + \frac{1}{2}r + 1$ convex pieces, where r is the number of reflex edges of the polyhedron, in $O(nr^3)$ time. Chazelle's example may be constructed as follows.

Start with a cube aligned with orthogonal xyz coordinate axes. Cut k thin notches into the bottom face, parallel to the xz-plane. Similarly cut k notches into the top face, parallel to the yz-plane. The result is shown in Fig. 10.4 for $k = 2$. The two sets of notches do not quite meet. The top edges of the notches in the bottom face lie on the hyperbolic paraboloid $z = xy$, and the bottom edges of the notches in the top face lie on $z = xy + \varepsilon$, the same surface shifted up by ε, where $\varepsilon \ll 1$. A hyperbolic paraboloid can be generated by two sets of orthogonal lines (Thomas 1962),

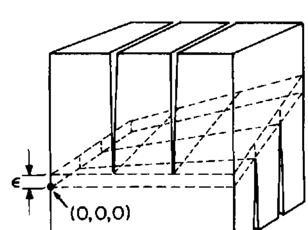

Fig. 10.4. Any convex partition of Chazelle's polyhedron requires a quadratic number of pieces.

so the edges can be chosen to lie on these surfaces. Chazelle proved that the intersection of the warped shape between the two hyperbolic paraboloids with any convex subset of the polyhedron can only have such a small volume that $\Omega(n^2)$ pieces are necessary to make up the volume of the shape. His proof is long and difficult and will not be presented here. His conclusion is that at least $n^2/66 = \Omega(n^2)$ convex pieces are necessary in any convex partition of the polyhedron just described.

10.2.4. Satellite Sentries

The only non-trivial art gallery theorem known for three dimensions is for the very special case of exterior visibility for guards confined to the surface of a convex polyhedron. The equivalent problem in two dimensions is trivial: $\lceil n/2 \rceil$ boundary guards are *always* necessary and sufficient to guard the exterior of a convex polygon. But in three dimensions the situation is not as straightforward. First, there are several quantities that might serve as the basis for a theorem: V, E, and F, the number of vertices, edges, and faces of the polyhedron. It seems that F is the most natural measure, and we will use it in this section.

The theorem is obtained by using matchings in the graph of the dual of the polyhedron. We will need the following theorem of Nishizeki on the size of maximum matchings in planar graphs.

LEMMA 10.1 [Nishizeki 1977]. If G is a connected planar graph of n nodes, with minimum vertex degree $\delta \geq 3$, and with connectivity $\kappa \geq 2$, then for all $n \geq 14$, the number of edges in a maximum matching of G is greater than or equal to $\lceil (n+4)/3 \rceil$, and for $n < 14$, the number of edges is $\lfloor n/2 \rfloor$.

Nishizeki obtained many similar results for different values of δ and κ, all of which are best possible (Nishizeki and Baybars 1977; Nishizeki 1977). We will have occasion to use this powerful theorem in the next section as well.

We may now prove the art gallery theorem.

THEOREM 10.1 [Grünbaum and O'Rourke 1983]. $\lfloor (2F-4)/3 \rfloor$ vertex guards are sometimes necessary and always sufficient to see the exterior of a convex polyhedron of F faces, for $F \geq 10$.

Proof.

Necessity. Let Q be any *simple* polyhedron of f faces, that is, having all vertices of degree 3. From Euler's formula $v - e + f = 2$, and $2e = 3v$, it follows that $v = 2f - 4$. From Q construct a polyhedron P by "truncating" all vertices of Q, that is, replace each vertex of Q by a small triangle so that none of the new triangles share common points. This procedure is illustrated in Fig. 10.5 when Q is a cube. P has $F = f + v = 3f - 4$ faces. Each of the new triangular faces requires its own guard, so the total number required is at least $v = 2f - 4$. But $\lfloor (2F-4)/3 \rfloor = \lfloor (6f-12)/3 \rfloor = 2f - 4$. This establishes necessity when $F \equiv 2 \bmod 3$, since $3f - 4 \equiv 2 \bmod 3$. The

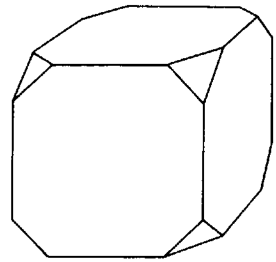

Fig. 10.5. The result of truncating a cube at every vertex.

other two cases (mod 3) can be shown as follows. If one of Q's vertices is not cut off, then P has $F = 3f - 5$ faces, and needs $2f - 5 = \lfloor[2(3f - 5) - 4]/3\rfloor = \lfloor(2F - 4)/3\rfloor$ guards. If two of Q's vertices are not cut off, then P has $F = 3f - 6$ faces, and needs $2f - 6 = \lfloor[2(3f - 6) - 4]/3\rfloor = \lfloor(2F - 4)/3\rfloor$ guards. Thus for all values of F, polyhedra exist that require $\lfloor(2F - 4)/3\rfloor$ guards.

Sufficiency. Let G be the dual graph of the surface of the polyhedron P; G has F nodes. G is planar and its minimum vertex degree is three because each face of P must have at least three edges. A *polyhedral graph* is the graph determined by the vertices and edges of a convex polyhedron. G has connectivity of at least three since polyhedral graphs are 3-connected by Balinski's theorem (Grünbaum 1975), and G is polyhedral because it is the dual of a polyhedral graph. Therefore, Lemma 10.1 applies and shows that, for $F \geq 14$, there is a matching M in G of at least $m = \lceil(F + 4)/3\rceil$ edges. Now place a guard on one of the endpoints of the edge of P corresponding to each edge in the matching. This covers $2m$ faces. Assign a separate guard to each of the $F - 2m$ faces of P. The result is complete coverage with $m + F - 2m = F - \lceil(F + 4)/3\rceil$ guards. This quantity is identical to $\lfloor(2F - 4)/3\rfloor$. For $F < 14$, there is a matching of $m = \lfloor F/2\rfloor$ edges, which by the same argument leads to coverage with $\lceil F/2\rceil$ guards. For $F \geq 10$, $\lceil F/2\rceil \leq \lfloor(2F - 4)/3\rfloor$. This establishes the theorem, then, for all $F \geq 10$. □

The necessity holds for all $F \geq 5$, and although I suspect sufficiency also holds in the range $5 \leq F \leq 9$, I have not verified this yet.

10.3. LINE SEGMENT OBSTACLES

Throughout this book we have concentrated on polygons, but "art gallery-like" questions may be posed for other types of obstacles. In this section we prove an art gallery theorem for n non-intersecting line segments. Visibility is defined as follows: a guard at point x sees point y if the line segment xy does not *cross* the interior of any line segment obstacle; xy may be collinear with a segment, or touch one of its endpoints. Sufficiency follows easily using the same technique just employed for convex polyhedra. Necessity is less obvious, but fortunately a counterexample to a hypothesis on the prison yard problem considered in Chapter 6 may be modified to yield the critical example.

THEOREM 10.2 [O'Rourke 1985]. $\lfloor 2n/3 \rfloor$ point guards are sometimes necessary and always sufficient to cover the plane in the presence of n line segment obstacles, where the guards may be positioned anywhere in the plane, under the following assumptions:

(1) No two segments are parallel (and therefore none are collinear).
(2) No three lines determined by segments intersect in a common point.
(3) $n \geq 5$.

Proof.
Sufficiency. Partition the plane into $n + 1$ regions in a manner similar to that used in Sections 1.4 and 6.5.2 (see Lemma 6.5): extend each segment in both directions until it hits either another segment or a previous segment extension. The induced convex partition is dependent on the order in which the extensions are made, but it always has $n+1$ regions by the non-collinearity assumption (1). Form a graph G from this partition as was done in Section 6.5.2, as follows. Associate a node of G with each convex region of the partition, and connect two nodes by an arc of G if their regions share a common boundary point. An example is shown in Fig. 10.6.

It is easy to see that assumption (2) ensures that G is a planar graph, and indeed a triangulation, since every face of G (except the exterior face) can be associated with the intersection of two segment lines, and a neighborhood of this intersection point touches three mutually adjacent regions, corresponding to a triangle in G. Without the non-degeneracy assumption, either G would not necessarily be a triangulation, or it would not necessarily

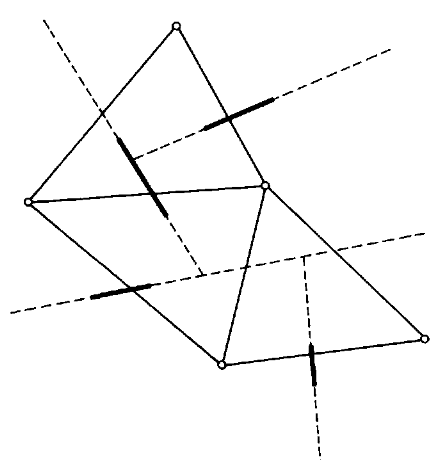

Fig. 10.6. A convex partition of the plane induced by a set of line segments (shown bold) and its dual graph.

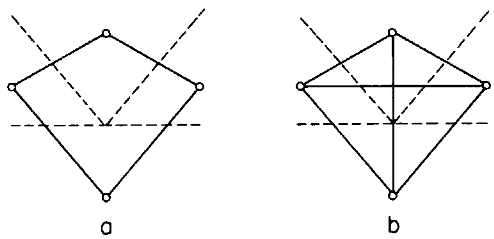

Fig. 10.7. If three segments (dashed) meet at a point, either dual graph is not a triangulation (a) or it is not necessarily planar (b).

be planar, depending on whether adjacency in G required a finite length of common boundary or just a common point, respectively (see Figs. 10.7a and 10.7b). Although these degeneracies are actually "in our favor," the proof is more straightforward if they are assumed not to occur.

We would like to apply Lemma 10.1 to G, which requires a minimum vertex degree δ of 3. However, G may have $\delta = 2$ as illustrated in Fig. 10.6. Since G is a triangulation graph, any nodes of degree 2 must be on the exterior face. Augment G to G' by adding a pseudo-node p adjacent to every node of G on the exterior face. Since G must have at least three nodes on its exterior face, p has degree three or more, and since p is connected to every degree 2 node of G, G' has $\delta \geq 3$.

To show that G' is 2-connected, assume to the contrary that removal of one node disconnects G'. Let x be such an articulation point of G'. Then the convex region R associated with x must divide the plane into two parts that share no boundary points. But this is only achievable if R has parallel edges running to infinity in both directions, which is not possible by the non-parallel assumption (1).

Now apply Lemma 10.1 to the $(n+2)$-node graph G', for $n \geq 12$, to obtain a matching M of $m = \lceil (n+6)/3 \rceil = \lceil n/3 \rceil + 2$ edges. Each edge of M not incident on p may be associated with a boundary point shared between two convex regions. Placing a guard at such a point clearly covers the two incident regions since they are convex. At most one edge of M may be incident to p. If there is such an edge, a guard may be used to cover the region associated with the other endpoint. Thus m guards associated with the matching edges cover at least $2m - 1$ regions. Covering the remaining $(n+1) - (2m-1)$ regions each with their own guard results in total coverage with

$$m + (n+1) - (2m-1) = n - m + 2 = n - \lceil n/3 \rceil = \lfloor 2n/3 \rfloor$$

guards. For $n < 12$, Lemma 10.1 guarantees a matching of size $\lceil (n+2)/2 \rceil = \lceil n/2 \rceil + 1$ edges, which by the same argument yields coverage with $\lfloor n/2 \rfloor + 1$ guards. Since $\lfloor n/2 \rfloor + 1 \leq \lfloor 2n/3 \rfloor$ for $n \geq 5$, sufficiency is established.

Necessity. Although experimentation with small values of n would lead

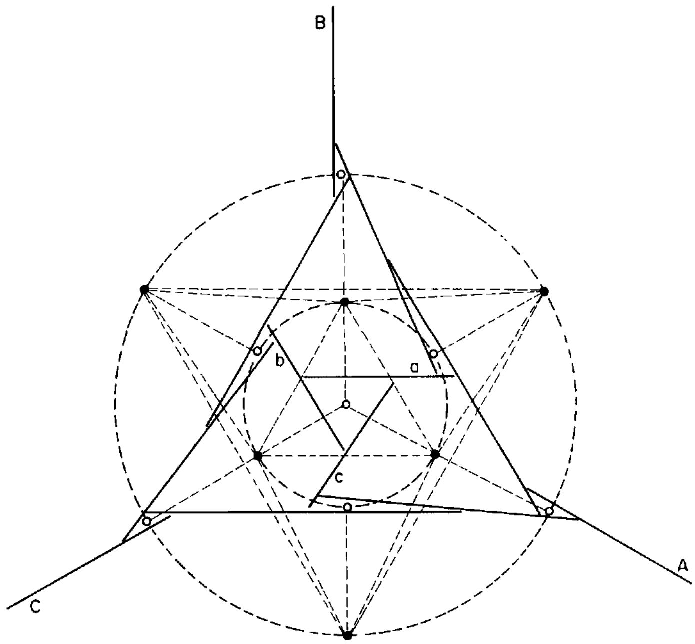

Fig. 10.8. A pattern of 12 line segments that require 7 point guards.

one to expect that at most $\lceil n/2 \rceil$ guards are necessary, the dependence of the sufficiency proof on matching suggests examining graphs with no perfect matching. And indeed, Fig. 6.19, which we used as a counterexample to an approach to the prison yard problem, can be used to establish necessity. Consider the 12 segments and induced convex partition shown in Fig. 10.8. The 13 node dual graph has the property that removal of 6 nodes (solid in the figure) disconnects the graph into seven odd components. Moreover, coverage of three nodes with one guard leaves a graph of 10 nodes that has no perfect matching, because removal of 4 nodes disconnects the remainder into 6 odd components (Section 6.5.2). It is clear that each of the seven triangular regions corresponding to the disconnected nodes (open in the figure) requires their own guard. Since $7 > \lceil 12/2 \rceil$, this example shows that $\lceil n/2 \rceil$ are not sufficient.

In order to show $\lfloor 2n/3 \rfloor$ necessity, we nest the pattern inside of itself as follows. Note that the pattern of segments in Fig. 10.8 has just three edges, A, B, and C, that extend to infinity. Thus the central triangular region formed by edges a, b, and c can be replaced by a copy of the pattern, with A, B, and C replacing the roles of a, b, and c, respectively. If this nesting is repeated k times, $n' = 9k + 3$ segments will be used. Each nesting adds six triangular region that each requires a guard. Since the innermost central triangular region also needs its own guard, $g' = 6k + 1$ guards are necessary.

A final modification yields the critical example. Add three more segments

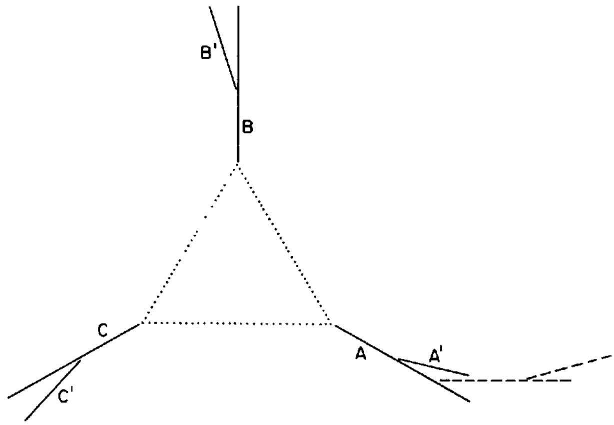

Fig. 10.9. Additional segments added to the pattern of Fig. 10.8, which is nested within the dotted triangle.

A', B', and C' that angle off of A, B, and C to infinity, as shown in Fig. 10.9. The cone bound by A and A' requires its own guard, and similarly for the B and C cones. Thus the figure has $n = n' + 3 = 9k + 6$ segments and requires $g = g' + 3 = 6k + 4$ guards. Since $\lfloor 2n/3 \rfloor = \lfloor (18k + 12)/3 \rfloor = 6k + 4$, the formula has been established when $n \equiv 0 (\text{mod } 3)$. This example also establishes the $n \equiv 1(\text{mod } 3)$ case, since incrementing n by 1 does not increase the value of $\lfloor 2n/3 \rfloor$. The $n \equiv 2(\text{mod } 3)$ case can be settled by adding two more segments, shown dashed in Fig. 10.9, forcing the need for another guard. Here $n = 9k + 8$ and $\lfloor 2n/3 \rfloor = 6k + 5$. Thus for every $n \geq 15$, there exists an arrangement that requires $\lfloor 2n/3 \rfloor$ guards. Removing edge A' establishes the same formula for $n = 14$. □

It remains to be explored whether the theorem also holds for the degenerate cases or small values of n ruled out by the theorem's assumptions. Using Lemma 10.1 for $n < 14$ easily establishes that $\lfloor n/2 \rfloor + 1$ guards are sufficient for $n < 14$, which, for $n \geq 5$, is no greater than $\lfloor 2n/3 \rfloor$, but the necessity of $\lfloor 2n/3 \rfloor$ guards for each $n < 14$ has not been established.

If the guards are restricted to vertices the situation changes dramatically.

THEOREM 10.3 [Boenke and Shermer 1986]. n vertex guards are sometimes necessary and always sufficient to cover the plane in the presence of n line segment obstacles.

Proof. Necessity is established by an arrangement of segments around a circle, as illustrated in Fig. 10.10. Each of the indicated triangular regions is visible only to the two segment endpoints at the base of the triangle. Note the similarity between this example and that used to establish necessity for the prison yard problem (Fig. 6.1).

For sufficiency, partition the plane into $n + 1$ convex regions as in Theorem 10.2. Each region has at least one segment endpoint on its boundary, and each endpoint borders on two regions. Place a guard at any

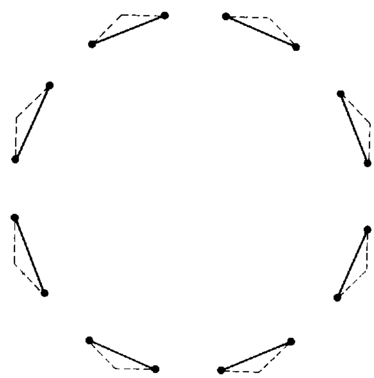

Fig. 10.10. An arrangement of 8 line segments that require 8 vertex guards.

endpoint. This covers two regions. Cover the remaining $n-1$ regions with a guard at an endpoint on their boundaries. □

10.4. POINT OBSTACLES

It may seem that there can be no interesting art gallery questions if the line segment obstacles considered in the previous section are reduced to points, but this is only because we have assumed throughout most of this book that there are no collinear degeneracies. Permitting collinearities and defining visibility to be blocked by points yields two interesting combinatorial-geometric problems, both at least partially unsolved since they were posed in the 1950s and 1960s.

Let P be a set of n points in the plane, not all on a line. Such a point set will be called *non-collinear.* Note that any number $k < n$ of points in P may be collinear. Define points x and y to be visible to one another if the open line segment xy contains no points of P. Let $p^* \in P$ be a point that sees at least as many points of P as any other, and let $M(P)$ be this maximum number. Note that $M(P) = n - 1$ if no three points of P are collinear, with p^* any point of P. Finally define $m(n)$ to be the minimum of $M(P)$ over all point sets of size n. Without the non-collinearity stipulation, $m(n)$ would be 2 for all $n > 2$, since $M(P)$ would be 2 for all sets of collinear points, with p^* any non-extreme point. But if not all points are on a line, it seems a very difficult problem to find $m(n)$. Dirac posed the problem in 1951 (Dirac 1951) and conjectured that $m(n) = \lfloor n/2 \rfloor$.[1]

1. His original problem was somewhat different: he sought the minimum over all configurations of the maximum number of lines determined by two points that pass through a third. This is not exactly the same problem, because if the two determining points of a line are on opposite sides of the third, the third sees both, but if they are on the same side, the third only sees the closest.

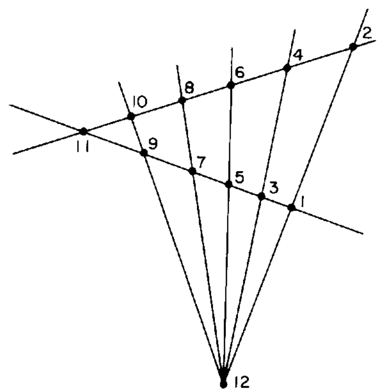

Fig. 10.11. A configuration in which no point can see more than $8 = 12/2 + 2$ other points.

Figure 10.11 shows a configuration that achieves $M(P) = \frac{1}{2}n + 2$ for even n, so $m(n) \le \lfloor n/2 \rfloor + 2$. Dirac offered this simple proof that $m(n) > \sqrt{n}$.

Let p^* be a point that sees a maximum number k of other points. Let L be a line determined by p^* and one of these k points. We claim that L cannot contain more than k points. For suppose it did contain $k' > k$ points. Then because not all points of P are collinear, there is a point $p \in P$ not on L. For each point p_i on L, either p sees p_i, or p sees a point p_i' such that p, p_i', and p_i are collinear in that order. Clearly if p_i and p_j are two distinct points on L, then p_i' and p_j' (if they exist) are distinct also. Thus p sees $k' > k$ points, contradicting the assumption that k is the maximum.

Now count the number of points P in the following way. Each of the k lines through p^* and the k points it sees contains at most $k-1$ points distinct from p^*. Thus $n \le k(k-1)+1$. Therefore, $k > \sqrt{n}$.

Very recently Szemerédi and Trotter proved that $m(n) > cn$ (Moser 1985), but the precise value of c is yet to be determined.

A second art gallery question for point obstacles was posed by Moser in 1966 (Moser 1985). Let P be a set of n non-collinear points. How many guards located at points of P are needed to see the unguarded points of P? Again the problem is trivial if no three points are collinear: one guard suffices. And again the other extreme, all points on one line, is uninteresting: $\lceil (n+1)/2 \rceil$ are necessary. Moser conjectured that $O(\log n)$ guards suffice for points arranged in an $n \times n$ rectangular lattice. More precisely, let $G(P)$ be the minimum number of points of P that collectively see the other points, and let $g(n)$ be the maximum of $G(P)$ over all sets of n non-collinear points P. We may extend Moser's conjecture to the statement that $g(n) = O(\log n)$.

It seems that progress has only been made in the special case of lattice points. Let L_n be an $n \times n$ square array of integer lattice points. Then, for example, $G(L_5) = 2$, as shown in Fig. 10.12. Abbott (1974) proved that

$$\frac{\ln n}{2 \ln \ln n} < G(L_n) < 4 \ln n.$$

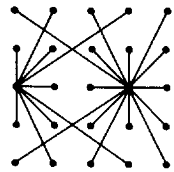

Fig. 10.12. A 5×5 lattice in which two points can see all the other points.

His proofs are number-theoretic; the natural logs in the lower bound come from the prime number theorem. His lower bound establishes that $g(n) > (\ln n^2)/(2 \ln \ln n^2)$; that is, this many guards are sometimes necessary, but the sufficiency of $O(\log n)$ guards has only been established for L_n, and even here Abbott's proof is non-constructive, and does not yield an explicit placement of guards.

10.5. MIRRORS

Having opened this book with a problem posed by Klee, it seems appropriate to close with another Klee problem.[2] Let P be a polygon, and imagine that all of its edges are perfect mirrors. Is there always at least one interior point from which P is completely illuminable by a point light bulb? Is P always illuminable from *each* of its points? Assume that the light bulb sends out rays in all directions, and that the standard "angle of reflection = angle of incidence" law of reflection holds. Further assume that a light ray is absorbed if it hits a vertex. Surprisingly, these problems are unsolved for polygons. However, Klee showed the answers to be "no" if curved (differentiable) arcs are permitted. Figure 10.13 shows a region that is not illuminable from the point x, which is the center of both the upper and lower circular arcs. This shows that not every region is illuminable from each of its points. However, the region is easily seen to be illuminable from, for example, point y. Figure 10.14 shows a region that is not illuminable from any of its points. In the figure, a and b, and a' and b', are foci of ellipses forming the upper and lower arcs, respectively. An ellipse with foci

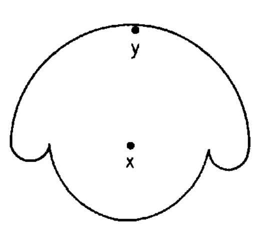

Fig. 10.13. A region not illuminable from x, but illuminable from y.

2. The original poser of the problem is unknown; Klee popularized the problem in two articles (Klee 1969, 1979).

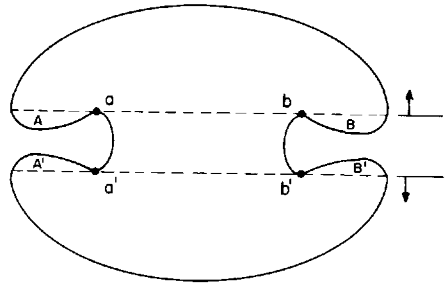

Fig. 10.14. A region not illuminable from any one point.

a and *b* has the following properties:

(1) A ray through *a* immediately reflects through *b*, and vice versa.
(2) A ray that intersects the open segment (a, b) immediately reflects and intersects (a, b) again.
(3) A ray that crosses the major axis but does not intersect the closed segment $[a, b]$ immediately reflects to cross the axis without hitting $[a, b]$ again.

Thus any light source above the $a'b'$ major axis will not illuminate regions A' or B', and similarly for below the *ab* axis, by property (2). And a light source in *A* will bounce into *B* and back again by property (3), never illuminating A' or B'.

Although the problem remains unsolved for polygonal regions, some progress has been made in understanding the behavior of single light rays in a *rational* polygon, one whose angles are all rational multiples of π. (Orthogonal polygons are a very special case of rational polygons.) A single light ray is more usually called a "billiard ball" in the now rather substantial literature on the subject. One of the more accessible results is the following.

THEOREM 10.4 [Boldrighini *et al.* 1978; Kerckhoff *et al.* 1985]. Let x be a point in a rational polygon P, and θ a direction. Then, except for a countable number of "exceptional" directions θ, the path of a billiard ball issuing from x in the direction θ is spatially dense in P, that is, passes arbitrarily close to every point of P.

One implication of this result is that every rational polygon is illuminable from each of its points in the sense that no finite area region will be left unilluminated; whether an isolated point could remain in the dark is unclear.

For irrational polygons, almost nothing is known. It is not even known if every triangle admits a dense billiard path.

10.6. TABLE OF THEOREMS

We conclude with a table of the major art gallery theorems discussed in this book.

Table 10.1. Art gallery theorems

Visibility	Polygon Shape	Holes	Guard Type	Lower Bound (necessary)	Upper Bound (sufficient)	Section Discussed
interior	arbitrary	0	vertex	$\lfloor n/3 \rfloor$		1.2.1
				r		1.4.1
		1		$\lfloor (n+1)/3 \rfloor$		5.2
		h		$\lfloor (n+h)/3 \rfloor$	$\lfloor (n+2h)/3 \rfloor$	5.1
		0	diag epts	$\lfloor n/4 \rfloor$		3.2.1
			edge	$\lfloor (n+1)/4 \rfloor$	$\lfloor n/3 \rfloor$	3.2.2
			edge epts	$\lfloor 2n/7 \rfloor$	$\lfloor n/3 \rfloor$	
	star		vertex	$\lfloor n/3 \rfloor$		4.2
				$\lfloor r/2 \rfloor + 1$		
			line	1		
			diagonal	2		
			edge	$\lfloor n/5 \rfloor$	$\lfloor n/3 \rfloor$	
				$\lfloor r/2 \rfloor + 1$		
	orthogonal	0	vertex	$\lfloor n/4 \rfloor = \lfloor r/2 \rfloor + 1$		2.2.2
		1		$\lfloor n/4 \rfloor$		5.3
		2	point	$\lfloor n/4 \rfloor$		
		h	vertex	$\lfloor n/4 \rfloor$	$\lfloor (n+2h)/4 \rfloor$	
		0	diagonal	$\lfloor (3n+4)/16 \rfloor$		3.3
exterior	arbitrary		vertex	$\lceil n/2 \rceil$		6.2.1
			point	$\lceil n/3 \rceil$		6.2.3
	orthogonal		vertex	$\lfloor n/4 \rfloor + 1$		6.2.2
interior+ exterior	arbitrary		vertex	$\lceil n/2 \rceil$	$\lfloor 2n/3 \rfloor$	6.3.1
					$\lceil n/2 \rceil + r$	
	orthogonal			$\lceil n/4 \rceil + 1$	$\lfloor 7n/16 \rfloor + 5$	6.3.2
	segments		point	$\lfloor 2n/3 \rfloor$		10.3
			vertex	n		