Vertex π-Lights for Monotone Mountains

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Abstract

It is established that $\lceil t/2 \rceil = \lceil n/2 \rceil - 1$ vertex π-lights suffice to cover a monotone mountain polygon of $t = n-2$ triangles. A monotone mountain is a monotone polygon one of whose chains is a single segment, and a vertex π-light is a floodlight of aperture π whose apex is a vertex.

Keywords. art gallery theorems, floodlights, monotone polygons.

1 Introduction

It was established in [ECOUX95] that for any $\alpha < \pi$, there is a polygon that cannot be illuminated with an α-floodlight at each vertex. An *α-floodlight* (or *α-light*) is a light with aperture no more than α. A *vertex α-light* is one whose apex is placed at a vertex, aiming a cone of light of up to α into the polygon. Each vertex may be assigned at most one light. The result of [ECOUX95] is then that n vertex α-lights do not always suffice when $\alpha < \pi$. Let a polygon P have t triangles in any triangulation, $t = n-2$; we will phrase bounds in terms of t. For $\alpha = \pi$, an easy argument shows that t vertex π-lights always suffice: place a light at an ear tip, cut off the ear, and recurse. This raises the question of finding a better upper bound. Urritia phrased the problem this way [Urr97]: is there a $c < 1$ such that cn vertex π-lights always suffice? The largest lower bound is $c = \frac{3}{5}$ via an example of F. Santos.

In this paper we pursue this question, but only in special cases. In particular, we show that $c = \frac{1}{2}$ for spirals and, more interestingly, for monotone mountains. A *monotone mountain* is a monotone polygon one of whose chains is a single segment. More precisely, a *monotone chain* is a polygonal chain whose intersection with any vertical line is at most one point. A monotone mountain consists of one monotone chain, whose extreme (left and right) vertices are connected by a single segment. Note this *base edge* need not be horizontal.[1] Fig. 5 shows a monotone mountain with base edge xy.

Although this is a severely restricted class of polygons, it deserves attention for three reasons: the examples establishing the results of [ECOUX95] (and [OX94]) are "nearly" monotone mountains; the problem is already not completely trivial for monotone mountains; and there is some reason to hope similar techniques will apply to the unrestricted problem.

We start with a result on spiral polygons, where the problem is trivial.

2 Spiral Polygons

A *spiral polygon* consists of two joined polygonal chains: a chain of reflex vertices, and a chain of convex vertices.

Theorem 1 *A spiral polygon S of $t = n-2$ triangles may be covered by $\lceil t/2 \rceil = \lceil n/2 \rceil - 1$ vertex π-lights; some spirals require this many.*

Proof: If S has no reflex vertices, S is convex and can be covered with one vertex π-light at any vertex. So assume S has at least one reflex vertex.

Let x, y, and z be three consecutive vertices of S, with x reflex, y convex, and z convex. Such a triple always exists, because any polygon has at least three convex vertices. The segment xz must be an internal diagonal of the polygon. Therefore at least two triangles are incident to z in any triangulation of S. Placing a light at z, as shown in Fig. 1, therefore covers at least two triangles; because z is convex, the light covers the entire angle at z. Removing the covered

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[1] This definition differs in this respect from that introduced in [OX94], which demanded a horizontal base edge.

triangles leaves a smaller spiral polygon. Repeating this process covers S with at most $\lceil t/2 \rceil$ lights.

Generalizing the polygon shown in Fig. 1 establishes that the bound is tight. □

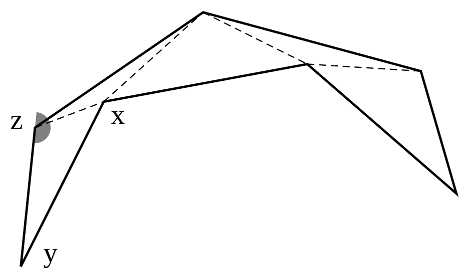

Figure 1: Placing a π-light at z covers at least two triangles. The light is shown as a full π-light, although only the angle interior to the polygon is relevant.

Notice that the procedure implied by this proof places lights only on convex vertices. One reason spiral polygons are so easy is that lights never need be placed on reflex vertices, and so the potentially difficult decision of how to orient a π-light at a reflex vertex need not be confronted.

3 Non-Locality

Monotone mountains are more difficult than spirals for two reasons: reflex vertices cannot be avoided, and the decision of how of orient a light at reflex vertex cannot be made locally. Many art gallery theorems can be proved inductively as follows: cut off a small piece, illuminate that piece, and recurse on the remainder [O'R87]. The reason this paradigm works is that decisions can be made locally: what happens in the small piece is independent of the shape of the remainder of the polygon.

This is not the case with the vertex π-light problem, even for monotone mountains. Consider the polygon shown in Fig. 2, and imagine trying to decide whether to shine the light at z left or right, basing the decision only on the portion of the polygon to the left of z. One can see that no $c < 1$ can be achieved without looking at the structure of the right portion: if the "wrong" decision is made at z (as illustrated), then an arbitrarily large fraction of all remaining vertices will need lights. Although the decision is obvious in this case, as it can be based on the number of triangles incident to z, the effect might be more subtle.

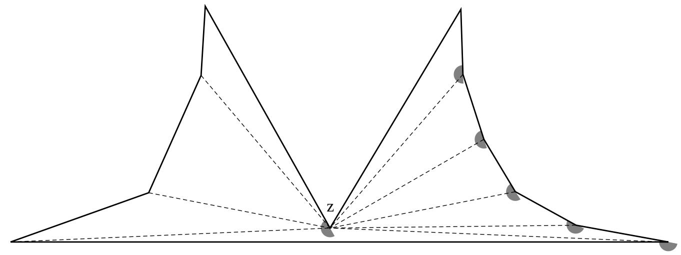

Figure 2: A wrong orienting decision at z can lead to suboptimal coverage.

4 Worst Case

It is clear that if the number of triangles incident to z in Fig. 2 from the left is k and from right is also k, then a lower bound of $c = \frac{1}{2}$ is attained: $t = 2k+1$, and $k+1 = \lceil t/2 \rceil$ lights are necessary, one at z and k on the opposite reflex chain. The same bound is acheived by the shape shown in Fig. 3. In this polygon, the extension of $v_1 v_2$ meets $v_5 v_6$; the extension of $v_2 v_3$ meets $v_4 v_5$; and so on.

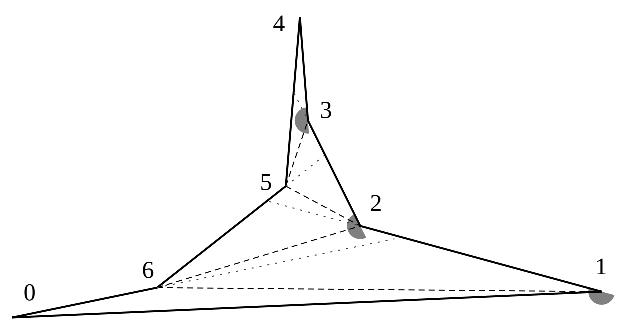

Figure 3: $\lceil t/2 \rceil$ lights are necessary: $t = 5$ and $\lceil 5/2 \rceil = 3$ are needed.

We prove this simple fact for later reference:

Lemma 1 *The generalization of the polygon M in Fig. 3 requires $\lceil t/2 \rceil = \lceil n/2 \rceil - 1$ vertex π-lights.*

Proof: Each vertex on the left chain can only see two vertices on the right chain, and vice versa: v_5 can see v_2 and v_3, because the extensions of $v_1 v_2$ and $v_2 v_3$ straddle v_5; etc. Thus at most (in fact exactly) three triangles are incident to v in a triangulation of M. A π-light at v can only fully cover two of these three triangles, because v is reflex. So each light covers at most two triangles, and $\lceil t/2 \rceil$ are needed overall. □

5 Duality

One way to view the phenomenon illustrated in Fig. 2 is as follows: the polygon naturally partitions into two monotone mountain subpolygons at z. If at light is placed at z and aimed left, then in the right subpolygon, placing a light at z is forbidden (as that would place two lights at one vertex). Moreover, that example shows that a (sub)polygon with one vertex forbidden a light could in fact require one light per triangle.

However, there is an interesting "duality" at play here, in the following sense: if a polygon with one forbidden vertex requires many lights, then placing a light at the forbidden vertex permits it to be covered with few lights. In other words, there is no polygon structure that is both bad with a forbidden vertex and bad without that vertex forbidden.

If M is a monotone mountain with extreme left and right vertices x and y, let $L_{10}(M)$ be the number of vertex π-lights needed to cover M when vertex x is assigned a light and y is forbidden to have a light; and let $L_{01}(M)$ be the number needed when y is assigned a light and x is forbidden. Note that, in these definitions, not only is one vertex forbidden a light, but the other extreme vertex must be assigned a light. The precise statement of duality is captured in the following lemma:

Lemma 2 *For any monotone mountain M of t triangles, $L_{10}(M) + L_{01}(M) \leq t + 1$.*

The generalization of Fig. 4 establishes that the sum is sometimes as large as $t+1$: here $L_{10}(M) = 1$ (v_0 assigned) and $L_{01}(M) = t$ (v_0 forbidden, as illustrated).

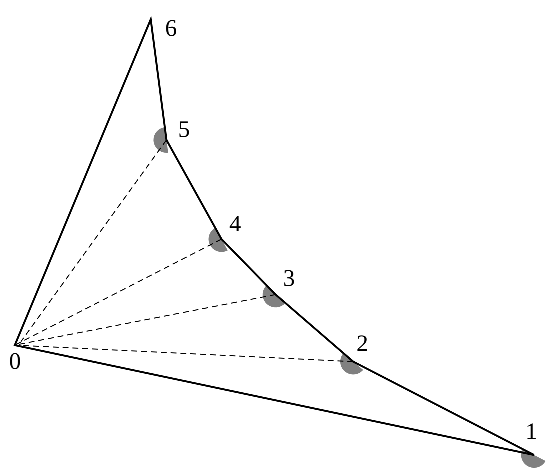

Figure 4: Duality: $L_{10}(M) + L_{01}(M) = t + 1$.

Lemma 2 is the key to the main theorem in the next section. We now prove it via induction.

Proof: Let M be a monotone mountain of t triangles. The induction hypothesis is that $L_{10}(M') + L_{01}(M') \leq t' + 1$ for any monotone mountain M' of $t' < t$ triangles. The base case is a single triangle T, $t = 1$, when $L_{10}(T) = L_{01}(T) = 1$, and so $L_{10}(T) + L_{01}(T) = 2 = t + 1$.

Let the base edge of M be xy, and let z be the vertex first encountered by sweeping the line containing xy vertically; see Fig. 5. It must be the case that both xz and yz are internal diagonals. This provides a natural partition of M into three pieces: $\triangle xyz$, a subpolygon A sharing diagonal xz, and a subpolygon B sharing diagonal yz. Note that it may well be that either A or B is the empty polygon $\emptyset$; if both are empty, $t = 1$ and we fall into the base case of the induction.

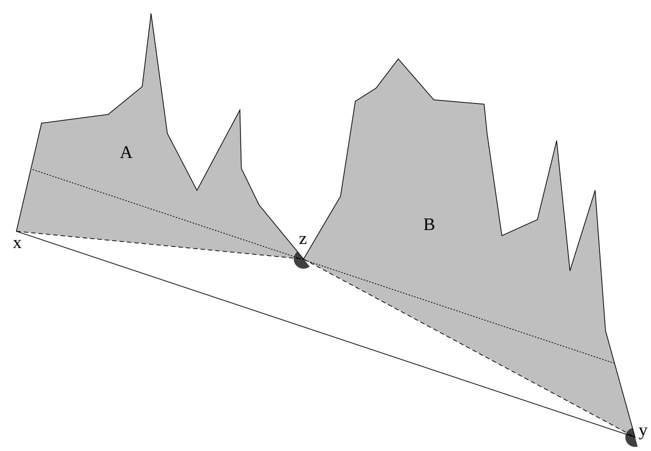

Figure 5: Induction partition of M into A, B, and $\triangle xyz$.

It is clear that A and B are monotone mountains. In particular, the angle at z in A is convex, as is the angle at z in B: for the monotone chain enters z from the left and leaves it from the right (Fig. 6), as do the diagonals xz and zy respectively.

We prove the lemma in two cases.

Case 1: Neither A nor B is empty.

We compute a bound on $L_{10}(M)$, which places a light at x but forbids a light at y. Because the angle at x in M is convex, the light at x covers $\triangle xyz$. This light also serves as a light at x in A. It makes sense in this situation to place a light at z and aim it into B. Doing this gives us an upper bound on $L_{10}(M)$, upper because this sensible light placement and orientation at z might not optimal. This strategy yields

$$L_{10}(M) \leq L_{10}(A) + L_{10}(B)\,. \qquad (1)$$

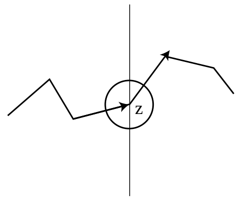

Figure 6: The monotone chain enters each vertex from the left halfplane and leaves in the right halfplane.

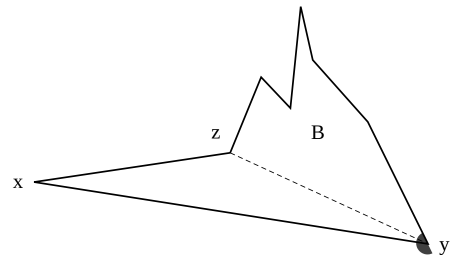

Figure 7: $A = \emptyset$.

Analogous reasoning (again the light at y (illustrated in Fig. 5) covers $\triangle xyz$) yields

$$L_{01}(M) \leq L_{01}(A) + L_{01}(B)\,. \tag{2}$$

Adding Eqs. 1 and 2 yields

$$L_{10}(M)+L_{01}(M) \leq [L_{10}(A)+L_{01}(A)]+[L_{10}(B)+L_{01}(B)]\,.$$

Suppose A contains a triangles and B contains b triangles, so that $t = a + b + 1$. Then applying the induction hypothesis to each yields

$$\begin{aligned} L_{10}(M) + L_{01}(M) &\leq [a+1] + [b+1] \\ L_{10}(M) + L_{01}(M) &\leq t+1\,. \end{aligned}$$

This is the claim to be proved.

It only remains to handle the case where one of A or B is empty.

Case 2: $A = \emptyset$ but B is not empty.

This case is illustrated in Fig. 7; the case with $B = \emptyset$ is symmetric and need not be considered. If a light is placed at x, it serves to cover $\triangle xyz$, and the reasoning is just as before:

$$L_{10}(M) \leq 1 + L_{10}(B)\,.$$

If a light is placed at y, then it covers $\triangle xyz$ (as illustrated in Fig. 7), and there is no need to an additional light to cover the empty A:

$$L_{01}(M) \leq L_{01}(B)\,.$$

Adding yields

$$\begin{aligned} L_{10}(M) + L_{01}(M) &\leq 1 + [L_{10}(B) + L_{01}(B)] \\ L_{10}(M) + L_{01}(M) &\leq 1 + [b+1] \\ L_{10}(M) + L_{01}(M) &\leq t+1\,. \end{aligned}$$

□

6 Main Result

With Lemma 2 in hand, the final step is easy:

Theorem 2 *A monotone mountain polygon M of $t = n-2$ triangles may be covered by $\lceil t/2 \rceil = \lceil n/2 \rceil - 1$ vertex π-lights; some monotone mountains require this many.*

Proof: We know from Lemma 2 that

$$L_{10}(M) + L_{01}(M) \leq t+1\,.$$

Let

$$L(M) = \min\{L_{10}(M), L_{01}(M)\}\,.$$

By the pigeonhole principle,

$$L(M) \leq \lfloor (t+1)/2 \rfloor = \lceil t/2 \rceil\,.$$

Lemma 1 established that this bound can be attained (Fig. 3). □

The proofs of Lemma 2 and Theorem 2 imply a simple algorithm: compute a bound on $L_{10}(M)$ by placing lights at the left corners of A and of B and recursing, and compute a bound on $L_{01}(M)$ similarly. Use the light placement of whichever is smaller. The algorithm is easily seen to be $O(n \log n)$: spend linear time finding z, and recursively process the pieces. This leads to the familiar divide-and-conquer recurrence.

An example is shown in Fig. 8. Here M has $t = 14$ triangles, and $L_{10}(M) + L_{01}(M) \leq 5 + 10 = t + 1$. This example illustrates a number of features of the light placements implied by the bound computation on L_{10} and L_{01}:

1. Every vertex that is not a local maximum is assigned a light in either the L_{10} or L_{01} computation. (Some vertices are assigned a light in both.)

2. All the lights in the L_{10} placement aim to the right; and all those in the L_{01} placement aim to the left.

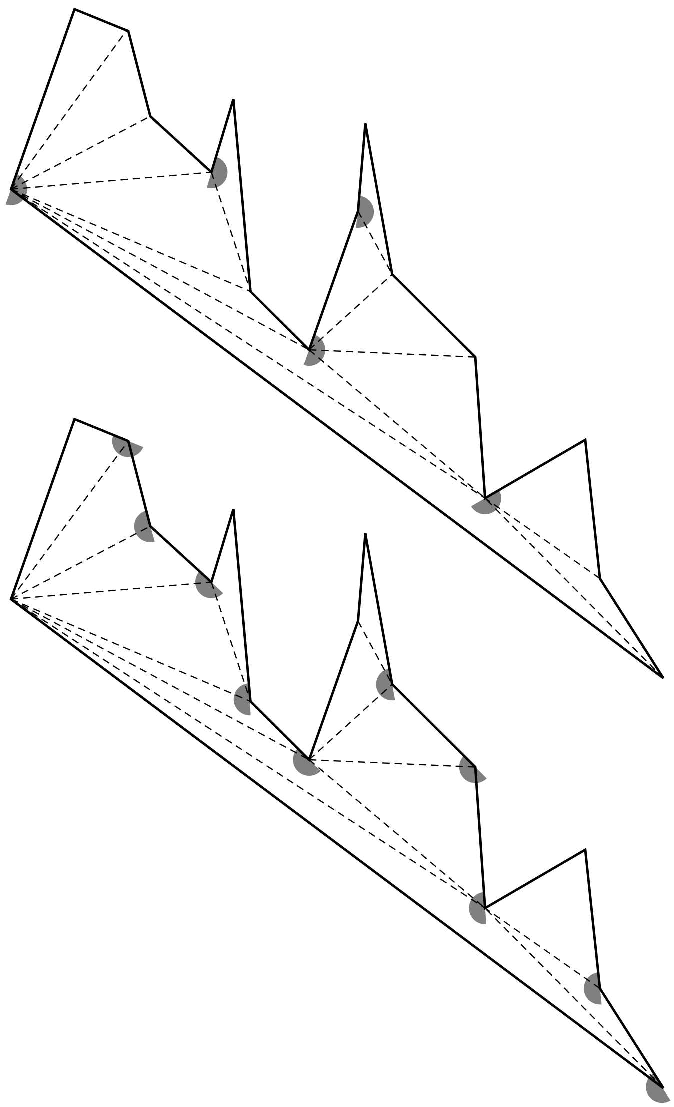

Figure 8: Example: $t = 14$, $L_{10} = 5$ (top), $L_{01} = 10$ (bottom).

3. The sum $L_{10}(M) + L_{01}(M)$ achieved is always exactly $t + 1$, because blindly following the procedure places lights even if they might not be needed (e.g., when M is convex).

4. Lights at reflex vertices are turned either fully counterclockwise (in L_{10}) or clockwise (in L_{10}): intermediate positions are never needed.

7 Discussion

Many of the features present in monotone mountains hold for the problem for general simple polygons as well: for example, non-locality. For other features, it remains unclear: for example, whether every light may be fully turned (observation 4 above). In any case, I believe that a version of the duality described in Lemma 2 holds and will be a key to solving Urrutia's problem. I conjecture $c = \frac{2}{3}$ is achievable.

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