On the Development of the Intersection of a Plane with a Polytope

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Abstract

Define a "slice" curve as the intersection of a plane with the surface of a polytope, i.e., a convex polyhedron in three dimensions. We prove that a slice curve develops on a plane without self-intersection. The key tool used is a generalization of Cauchy's arm lemma to permit nonconvex "openings" of a planar convex chain.

1 Introduction

Although the intersection of a plane Π with a polytope P is a convex polygon Q within that plane, on the surface of P, this "slice curve" can be nonconvex, alternatively turning left and right. The development of a curve on a plane is determined by its turning behavior on the surface. Thus slice curves develop (in general) to nonconvex, open chains on a plane. The main result of this paper is that slice curves always develop to simple curves, i.e., they do not self-intersect.

Our main tool is a generalization of an important lemma Cauchy used to prove the rigidity of polytopes. Cauchy's arm lemma says that if $n-2$ consecutive angles of a convex polygon are opened but not beyond π, keeping all but one edge length fixed and permitting that "missing" edge e to vary in length, then e lengthens (or retains its original length). We employ a generalization of this lemma to permit opening of the angles beyond π, as far reflex as they were originally convex. The conclusion remains the same: e cannot shorten. We will see that this conclusion follows from a theorem of Axel Schur [Sch21].

The first part of this paper (Section 2) concentrates on this generalization of Cauchy's lemma. The issue of self-intersection is addressed in Section 3, and the curve development result is proved in Section 4.

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2 Cauchy's Arm Lemma Extended

Let $A = (a_0, a_1, \ldots, a_n)$ be an n-link polygonal chain in the plane with n fixed edge lengths $\ell_i = |a_i a_{i+1}|$, $i = 0, \ldots, n-1$. We call the vertices a_i the *joints* of the chain, a_0 (which will always be placed at the origin) the *shoulder*, and a_n the *hand*. Define the *turn angle* α_i at joint a_i, $i = 1, \ldots, n-1$ to be the angle in $[-\pi, \pi]$ that turns the vector $a_i - a_{i-1}$ to $a_{i+1} - a_i$, positive for left (counterclockwise) and negative for right (clockwise) turns.

Define an open polygonal chain A to be *convex* if its joints determine a (nondegenerate) convex polygon, i.e., all joints are distinct points (in particular, $a_n \neq a_0$), all joints lie on the convex hull of A and they do not all lie on a line. Note there is no chain link between a_n and a_0. The turn angles for a convex chain all lie in $[0, \pi)$; but note this is not a sufficient condition for a chain to be convex, for it is also necessary that the angles at a_0 and a_n be convex.

We can view the configuration of a polygonal chain A to be determined by two vectors: the fixed edge lengths $L = (\ell_0, \ldots, \ell_{n-1})$ and the variable turn angles $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$, with the convention that a_0 is placed at the origin and a_n horizontally left of a_0. Let $\mathcal{C}_L(\alpha) = A$ be the configuration so determined. We use α to represent the angles of the initial configuration, and β and γ to represent angles in a reconfiguration.

Let $D(r) = \{p \ : \ |pa_0| < r\}$ be the open disk of radius r centered on the shoulder joint a_0. Define $a = |a_n a_0|$, the length of the "missing" link, the original hand-to-shoulder distance. Finally, we will call $D(a)$ the *forbidden (shoulder) disk*. We may state Cauchy's arm lemma in the following form:

Theorem 0 *If $A = \mathcal{C}_L(\alpha)$ is a convex chain with fixed edge lengths L, and turn angles α, then in any reconfiguration to $B = \mathcal{C}_L(\beta)$ with turn angles $\beta = (\beta_1, \ldots, \beta_{n-1})$ satisfying*

$$\beta_i \in [0, \alpha_i] \tag{1}$$

we must have $|b_n b_0| \geq |a_n a_0|$, i.e., the hand cannot enter the forbidden disk $D(a)$.

Cauchy's lemma is sometimes known as Steinitz's lemma, because Steinitz noticed and corrected an error in the proof a century after Cauchy [Cro97, p. 235]. Many proofs of Cauchy's lemma are now known, e.g., [SZ67, Sin97] and [AZ98, p. 64].

Our main tool is a generalization of Cauchy's lemma that replaces the 0 in Eq. (1) by $-\alpha_i$, and is otherwise identical:

Theorem 1 *If $A = \mathcal{C}_L(\alpha)$ is a convex chain with fixed edge lengths L, and turn angles α, then in any reconfiguration to $B = \mathcal{C}_L(\beta)$ with turn angles $\beta = (\beta_1, \ldots, \beta_{n-1})$ satisfying*

$$\beta_i \in [-\alpha_i, \alpha_i] \tag{2}$$

we must have $|b_n b_0| \geq |a_n a_0|$, i.e., the hand cannot enter the forbidden disk $D(a)$.

The intuition is illustrated in Fig. 1; further examples are provided in Fig. 9. Although the chain may become nonconvex, Eq. (2) ensures that the movement constitutes a form of straightening. Note that Theorem 1 makes no claim about steadily increasing hand-shoulder separation during some continuous movement to B; indeed a continuous opening could first increase and later decrease the separation. Rather the claim is that a final configuration satisfying Eq. (2) cannot place the hand in the forbidden disk.

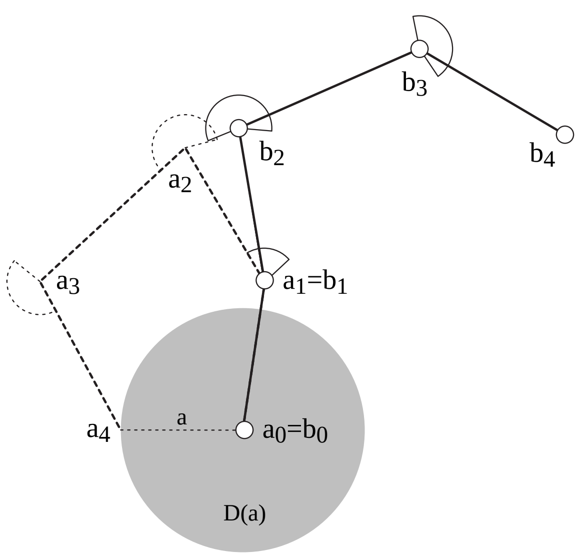

Figure 1: A reconfiguration of a 4-link convex chain A to chain B, satisfying Eq. (2), leaves b_n outside $D(a)$. The valid angle ranges are marked by circular arcs.

As pointed out by Connelly in [Con82, p. 30], Schur generalized Cauchy's theorem to the smooth case [Sch21]. Rather than requiring angles to open, Schur requires the curvature to decrease. Some mentions of Schur's theorem in the literature, e.g., in [Gug63, p.31], phrase it as the smooth, planar equivalent of Cauchy's lemma, which does not capture the nonconvexity permitted in the statement of Theorem 1. But others, notably the exposition by Chern [Che89, p. 119], state it as a generalization also to space curves, and employ the absolute value of curvature, implicitly permitting nonconvexity. Chern also states it (without proof) in the piecewise-smooth case, which directly encompasses polygonal chains. Although it remains unclear whether these authors intended to capture the precise statement of Theorem 1, we will show that Chern's version of Schur's proof can be used to establish the theorem. Consequently, Theorem 1 can be seen as a consequence of Schur's theorem, if not a direct corollary to it.

As with Cauchy's arm lemma, one may expect many different proofs of such a fundamental result. We offer three proofs of Theorem 1 in this paper. The first and second are induction proofs, following the same general outline. The third follows Chern's proof of Schur's theorem, specializing the smooth, space-curve argument to the nonsmooth, planar situation. The reader uninterested in these proofs may skip to Section 3.

2.1 First Proof of Theorem 1

Although we impose no restriction on self-intersection of the chain, we will show in Theorem 2 that the chain remains simple. Note that, because we fix a_0 to the origin, and the first turn angle is at joint a_1, in any reconfiguration the first edge of the chain is fixed.

Our first proof of Theorem 1 requires a few preparatory lemmas. We start with the simple observation that negating the turn angles reflects the chain.

Lemma 1 *If a chain $A = \mathcal{C}_L(\alpha)$ is reconfigured to $B = \mathcal{C}_L(\beta)$ with $\beta_i = -\alpha_i$, $i = 1, \ldots, n-1$, then B is a reflection of A through the line M containing a_0a_1, and $|b_nb_0| = |a_na_0|$.*

Proof: Reflecting A through line M does indeed negate each turn angle: $\beta_1 = -\alpha_1$ is immediate, and all others have their sense reversed by the reflection. Because β determines the configuration uniquely, this reflection is indeed the configuration determined by that β vector. Because M passes through $a_0 = b_0$, b_n remains at the same distance from b_0 as a_n is from a_0. □

Call a reconfiguration $B = \mathcal{C}_L(\beta)$ of a convex chain $A = \mathcal{C}_L(\alpha)$ which satisfies the constraints of Eq. (2) a *valid reconfiguration*, and call the vector of angles β *valid angles.* Define the *reachable region* $R_L(\alpha)$ for a convex chain $A = \mathcal{C}_L(\alpha)$ to be the set of all hand positions b_n for any valid reconfiguration $B = \mathcal{C}_L(\beta)$. One can view Theorem 1 as the claim that $R_L(\alpha) \cap D(a) = \emptyset$. It is well known [HJW84][O'R98, p. 326] that the reachable region for a chain with no angle constraints is a shoulder-centered closed annulus, but angle-constrained reachable regions seem unstudied.

For the first proof we need two technical lemmas.

Lemma 2 *The configuration of a chain $A = \mathcal{C}_L(\alpha)$ is a continuous function of its turn angles α.*
Proof: The coordinates of each joint a_i can be written as a trigonometric polynomial (rotation and translation of each link), with terms multiplying sin() and cos() applied to angles, and constants depending on the lengths L. Since all the constituents of these polynomials are continuous functions of the angles, each joint, and so all joints, are also. □

Lemma 3 *$R_L(\alpha)$ is a closed set.*
Proof: The $(2n-2)$-dimensional configuration space S of all chains $B = (b_0, b_1, \ldots, b_n)$ with valid $\beta = (\beta_1, \ldots, b_{n-1})$ is the image of the trigonometric polynomials mentioned in the previous proof as the angles vary over the compact domain

$$[-\alpha_1, \alpha_1] \times \cdots \times [-\alpha_{n-1}, a_{n-1}]$$

Because the image of a continuous function on a compact domain is compact, and because the function is continuous by Lemma 2, S is compact. In Euclidean space, a compact set is closed and bounded; so S is closed. $R_L(\alpha)$ is just the 2-dimensional b_n-slice through S, and so it is closed as well. □

We use this lemma to help identify, among potential counterexamples, the "worst" violators. Define a configuration $B = \mathcal{C}_L(\beta)$ to be *locally minimal* if there is a neighborhood N of β such that, for all $\beta' \in N$, the determined hand position b'_n is no closer to the shoulder: $|b'_n a_0| \geq |b_n a_0|$. Thus the hand's distance to the shoulder is locally minimal.

Lemma 4 *Let $B = \mathcal{C}_L(\beta)$ be a reconfiguration of convex chain $A = \mathcal{C}_L(\alpha)$ with $b_n \in D(a)$. Then either $b_n = a_0$, or there is some locally minimal configuration $B' = \mathcal{C}_L(\beta')$ with $b'_n \in D(a)$.*
Proof: Suppose $b_n \neq a_0$. Inflate a circle $C(r)$ about a_0, starting with radius $r = 0$, until some point of $R_L(\alpha)$ is first encountered. Because $b_n \in D(a)$, this event will occur before $r = a$. Because $R_L(\alpha)$ is closed by Lemma 3, there is some definite, smallest radius r_0, $0 < r_0 < a$, when the circle first hits the reachability region. A configuration corresponding to any point in $C(r_0) \cap R_L(\alpha)$ satisfies the lemma. □

The above lemma will provide a "hook" to reduce n in the induction step. We separate out the base of the induction in the next lemma.

Lemma 5 *Theorem 1 holds for $n = 2$.*

Proof: A 2-link chain's configuration is determined by single angle at a_1. The reachable region is a single circular arc exterior to $D(a)$, centered on a_1, of radius ℓ_1. See Fig. 2. □

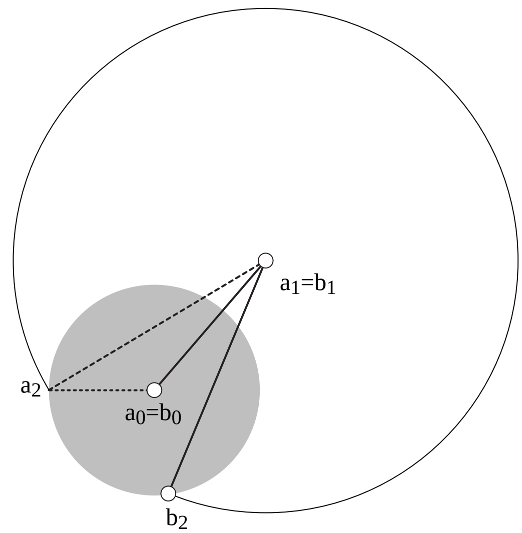

Figure 2: $R_L(\alpha)$ for a 2-link chain is a circle arc centered on $a_1 = b_1$.

We now prove Theorem 1 by induction.
Proof: Lemma 5 establishes the theorem for $n = 2$. Assume then that the theorem holds for all chains of $n-1$ or fewer links. We seek to establish it for an n-link chain $A = \mathcal{C}_L(\alpha)$, $n > 2$. Assume, for the purposes of contradiction, that A may be reconfigured so that the hand falls inside the forbidden disk $D(a)$. We seek a contradiction on a shorter chain. By Lemma 4, one of two cases holds: the hand reaches a_0, or there is a locally minimal configuration.

1. Suppose $B = \mathcal{C}_L(\beta)$ is such that $b_n = a_0$, as illustrated in Fig. 3(c). There are two possibilities. Either $\ell_{n-1} = |a_{n-1}a_n| < |a_{n-1}a_0| = a'$, when a_{n-1} is left of the bisector of a_0a_n (Fig. 3(a)), or a_{n-1} is right of the bisector (Fig. 3(b)). In the latter case, because a_1 cannot be left of the bisector, if we relabel the chain in reverse, then we again have $\ell_{n-1} < a'$. (Note that if both a_{n-1} and a_1 are on the bisector, then $a_{n-1} = a_1$ and the chain has only two links.)

   Now consider the chains A' and B' obtained by removing the last links $a_{n-1}a_n$ and $b_{n-1}b_n$. First, A' is a convex chain of $n-1$ links, so the induction hypothesis applies and says that A' cannot be validly reconfigured to place b_{n-1} closer to a_0 than $a' = |a_{n-1}a_0|$. B' places b_{n-1} at distance ℓ_{n-1} from a_0, which we just observed is less than a'. It remains to argue that B' is a valid reconfiguration of A', i.e., that it satisfies Eq. (2). However, this is satisfied for $i = 1, \ldots, n-2$ because these angles are not changed by the shortening, and after shortening there is no constraint on β_{n-1}. Thus B' is a valid reconfiguration of A' but places the hand in the forbidden disk, a contradiction.

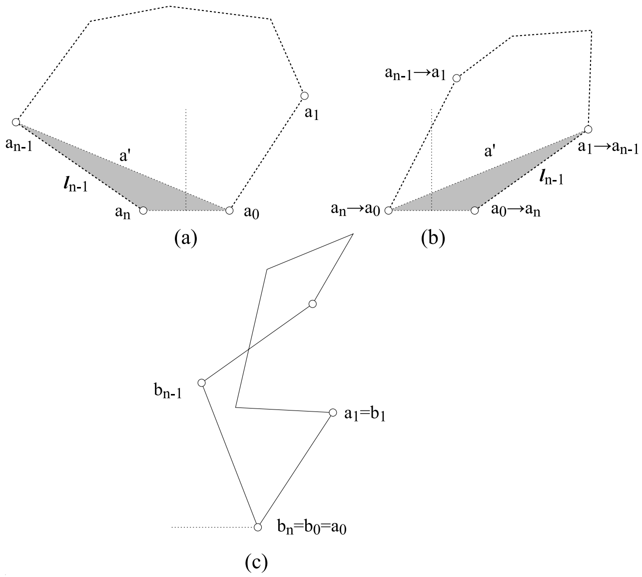

Figure 3: Case 1: $b_n = a_0$. (Drawing (c) is not accurate.)

2. We may henceforth assume, by Lemma 4, that there is a locally minimal configuration $B = \mathcal{C}_L(\beta)$ that places $b_n \in D(a)$. Again we seek to shorten the chain and obtain a contradiction.

   First we establish that at least one[1] β_i is at the limit of its valid turn range: $\beta_i = \pm\alpha_i$. Suppose to the contrary that all β_i, $i = 1, \ldots, n-1$, are strictly interior to their allowable turn ranges: $\beta_i \in (-\alpha_i, \alpha_i)$. Let M be the line containing $b_0 b_n$. Consider two cases:

   (a) Some b_i, $i = 1, \ldots, n-1$, does not lie on M. Then because β_i is not extreme, the subchain $(b_{i+1}, \ldots, b_n)$ may be rotated about b_i in both directions. Because b_i is off M, one direction or the other must bring b_n closer to b_0, contradicting the fact that b_n is locally minimal.

   (b) All b_i lie on M. Then there must be some b_i which is extreme on M. For this b_i, $\beta_i = \pm\pi$. But $\alpha_i \in [0, \pi)$: the nondegeneracy assumption bounds α_i away from π, and so bounds β_i away from $\pm\pi$.

   Henceforth let b_i be a joint whose angle β_i is extreme. If $\beta_i = -\alpha_i$, then reflect B about $b_0 b_n$ so that $\beta_i = \alpha_i$ is convex. By Lemma 1, this does not change the distance from b_n to the shoulder, so we still have $b_n \in D(a)$.

   We are now prepared to shorten the chains. Let A' and B' be the chains resulting from removing a_i and b_i from A and B respectively:

$$A' = (a_0, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \quad (3)$$
$$B' = (b_0, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n) \quad (4)$$

   A crucial point to notice is that $|b_{i-1}b_{i+1}| = |a_{i-1}a_{i+1}|$ because $\beta_i = \alpha_i$; this was the reason for focusing on an extreme β_i. Therefore B' is a reconfiguration of A'. Of course both A' and B' contain $n-1$ links, so the induction hypothesis applies. Moreover, because $i \leq n-1$, the b_i removed does not affect the position of b_n. So $b_n \in D(a)$ by hypothesis. To derive a contradiction, it only remains to show that B' is a valid reconfiguration of A', i.e., one that satisfies the turn constraints (2).

   Let α'_{i+1} be the turn angle at a_{i+1} in A'. We analyze this turn angle in detail, and argue later that the situation is analogous at a_{i-1}. Let θ be the angle of the triangle $\triangle_i = \triangle(a_i, a_{i+1}, a_{i-1})$ at a_{i+1}; see Fig. 4(a). Because A is a convex chain, cutting off $\triangle_i$ from A increases the turn angle at a_{i+1} in A':

$$\alpha'_{i+1} = \theta + \alpha_{i+1} \quad (5)$$

   Now consider the turn angle β'_{i+1} at b_{i+1} in B'. Although here the turn could be negative, as in Fig. 4(b), it is still the case that the turn is advanced by θ by the removal of $\triangle_i$:

$$\beta'_{i+1} = \theta + \beta_{i+1} \quad (6)$$

   We seek to prove that $\beta'_{i+1} \in [-\alpha'_{i+1}, \alpha'_{i+1}]$. Substituting the expressions from Eqs. (5) and (6) into the desired inequality yields:

$$\begin{array}{ccccc} -\alpha'_{i+1} & \leq & \beta'_{i+1} & \leq & \alpha'_{i+1} \\ -\alpha_{i+1} - \theta & \leq & \beta_{i+1} + \theta & \leq & \alpha_{i+1} + \theta \\ -\alpha_{i+1} - 2\theta & \leq & \beta_{i+1} & \leq & \alpha_{i+1} \end{array}$$

   And this holds because $\theta > 0$ and $\beta_{i+1} \in [-\alpha_{i+1}, \alpha_{i+1}]$ (because B is a valid reconfiguration of A). The intuition here is that the nesting of the turn angle ranges at a_{i+1} in A and A' (evident in Fig. 4(a)) carries over, rigidly attached to $\triangle_i$, to B, so that satisfying the tighter constraint in B also satisfies the looser constraint in B'.

   Although the situation is superficially different at a_{i-1} because our definition of turn angle depends on the orientation of the chain, it is easily seen that the turn constraint is identical if the orientation is reversed. Another way to view this is that we can

[1] In fact I believe that all must be extreme, but the proof only needs one.

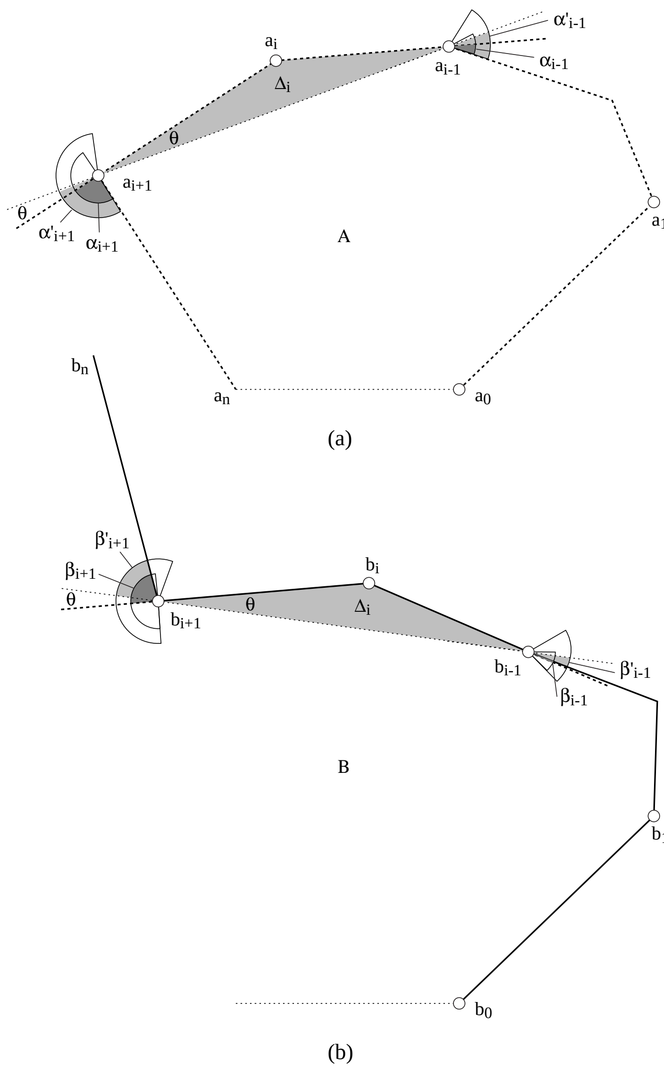

Figure 4: (a) Shortening the chain A by removal of a_i determines new, larger turn angles α'_{i+1} and α'_{i-1} at a_{i+1} and a_{i-1} respectively. (b) Here the turn angles β_{i+1} and β'_{i+1} are negative.

base the turn angles on $\triangle_i$. Thus the equations derived above hold again, except with $i+1$ replaced by $i-1$, and θ replaced by the angle of $\triangle_i$ at a_{i-1}.

We have thus established that B' is a valid reconfiguration of A'. By the induction hypothesis, the hand b_n of B' cannot enter the forbidden disk $D(a)$. But by assumption it is in that disk. This contradiction shows that our assumption that $b_n \in D(a)$ cannot hold, and establishes the theorem.

□

The following corollary extends the distance inequality to every point of the chain.

Corollary 1 *Let $A = \mathcal{C}_L(\alpha)$ be a convex chain as in Theorem 1, and let p_1, $p_2 \in A$ be any two distinct points of the chain. Then in any valid reconfiguration B, the points q_1, $q_2 \in B$ corresponding to p_1 and p_2 satisfy $|q_1 q_2| \geq |p_1 p_2|$, i.e., they have not moved closer to one another.*

Proof: Without loss of generality, assume that p_1 occurs before p_2 on A, i.e., is a shorter length along A from a_0. Let A' be the chain A with the portion prior to p_1, and the portion after p_2, removed: $A' = (p_1, \ldots, a_i, \ldots, p_2)$. Let B' be the corresponding clipped version of B: $B' = (q_1, \ldots, q_2)$. A' is a convex chain because A is. B' is a valid reconfiguration of A', for none of the angle ranges satisfied by B have been altered. Applying Theorem 1 with p_1 and q_1 playing the role of the shoulder, and p_2 and q_2 the role of the hand, establishes the claim. □

2.2 Second Proof of Theorem 1

We now sketch a second proof, which avoids reliance on locally minimal configurations. The proof is again inductive, by contradiction from a shortened chain, and relies on the same detailed argument concerning the turn angle ranges. None of those details will be repeated.

Proof: Let $A = \mathcal{C}_L(\alpha)$ be the given convex chain, and $C = \mathcal{C}_L(\gamma)$ a valid reconfiguration that places $c_n \in D(a)$, in contradiction to the theorem. We first construct an "intermediate" configuration $B = \mathcal{C}_L(\beta)$ with $\beta_i = |\gamma_i|$ for all $i = 1, \ldots, n-1$, i.e., B is a convex chain formed by flipping all turns in C to be positive. Note that, because γ is a valid angle vector for A, $\gamma_i \in [-\alpha_i, \alpha_i]$, and so $\beta_i \in [0, \alpha_i]$. As this is exactly the Cauchy arm opening condition, Eq. (1), we may apply Theorem 0 to conclude that $b = |b_n b_0| \geq |a_n a_0| = a$.

Now consider chain B. It may be a convex chain, but it is possible that it is not, as in Fig. 5. In this

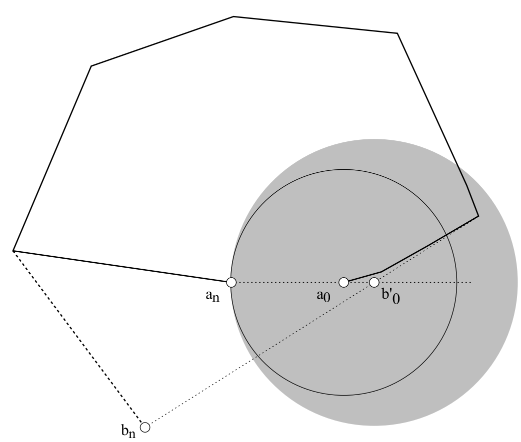

Figure 5: B might not be a convex chain, but another chain starting at b'_0 (lying on the line through $a_0 a_n$) is.

latter case we replace B with the chain starting at b'_0, which by construction is convex, and whose forbidden disk is a superset of the forbidden disk for A. To keep this sketch short, we do not further analyze the modifications necessary in this case.

Now we focus attention on chains B and C. Because $\gamma_i = \pm\beta_i$, $\gamma_i \in [-\beta_i, \beta_i]$. Therefore, C is a valid reconfiguration of B. But here is the point: every angle γ_i of C is extreme with respect to B, and so there is no need to invoke local minimality.

Choose an i and remove b_i from B and c_i from C, obtaining shorter chains B' and C'. Applying the argument from the previous section verbatim, we conclude that C' is a valid reconfiguration of B'. But because B' has $n-1$ links, the induction hypothesis applies and shows that c_n cannot enter the forbidden disk $D(b)$, with $b = |b_n b_0|$. Because $b \geq a$, c_n cannot be in $D(a)$ either. This contradicts the assumption and establishes the theorem. □

2.3 Third Proof of Theorem 1

We follow Chern's proof [Che89] of Schur's Theorem, at times quoting Chern nearly word-for-word. Although in some ways the specialization of his proof to our planar, nonsmooth instance destroys some of its elegance, the exercise does establish that Theorem 1 follows in spirit if not in fact from Schur's Theorem.

We adjust notation slightly to more closely track Chern's proof. In particular, we will use * to label quantities of the reconfigured chain. The given chain is A, and the reconfigured chain $A^* = B$ is some valid reconfiguration.

Proof: Chern/Schur's proof is carried out largely in the domain of the "tangent indicatrixes" of A and A^*. For any oriented curve C, draw unit-length vectors parallel to the tangent vectors of C, with the vectors based at an origin O. Their end-points describe a curve Γ on the unit sphere centered on O called the *tangent indicatrix* of C. If C is a curve in d dimensions, the tangent indicatrix lies on a $(d-1)$-dimensional sphere. For planar curves, the indicatrix lies on a unit circle. For polygonal curves, we turn the tangent continuously at a vertex through the turn angle there, thus resulting in a connected indicatrix. The indicatrix may be "layered" if it turns back over itself. An example (Γ^*) is shown in Fig. 6. Because A is a convex chain, Γ is a simple arc. But A^* is nonconvex and Γ^* doubles back over itself.

We will employ two nonnegative measures on an indicatrix. Let p_1 and p_2 be any two points of an indicatrix Γ, with p_1 prior to p_2 according to the curve's orientation. The *arc length* $l(p_1, p_2)$ is the length of the curve Γ from p_1 to p_2. Note that if the curve doubles back over itself, this length measure includes this doubling—there is no subtraction. Second, the *spherical distance* $\sigma(p_1, p_2)$ is the length of the shortest path on the sphere from p_1 to p_2. For plane curves, when the sphere is a circle, the spherical distance is the shortest circle arc connecting p_1 and p_2. σ is always in $[0, \pi]$. These definitions imply

$$\sigma(p_1, p_2) \leq l(p_1, p_2) \quad (7)$$

$$\sigma(p_1^*, p_2^*) \leq l(p_1^*, p_2^*) \quad (8)$$

where p_1^* and p_2^* are the points of Γ^* corresponding to p_1 and p_2 of Γ. For example, in Fig. 6, the arc length between 1 and 5 in Γ is approximately 241°, but the spherical distance between those points is 119°.

It should be clear that the arc length l measures the amount the tangent turns in absolute value. The absolute value prevents cancellation when the indicatrix doubles back over itself. Thus $l(p_1, p_2) = \sum |\alpha_i| = \sum \alpha_i$, where the sum is over all the vertices between, at which the tangent turn is concentrated. Note here the absolute value is unnecessary because A is convex and so $\alpha_i \geq 0$. Because $A^* = B$ is a valid reconfiguration of A, Eq. (2) holds: $|\beta_i| \leq a_i$. So we have

$$\sum |\beta_i| \leq \sum a_i \quad (9)$$

$$l(p_1^*, p_2^*) \leq l(p_1, p_2) \quad (10)$$

For example, in Fig. 6, the total arc length of Γ^* is about 68°, considerably less than Γ's length of 241°.

We must address here an issue that does not arise in Chern's proof, which assumes smoothness of the original curve. Without a smoothly turning tangent, there may be no unique correspondence between $p \in \Gamma$ and $p^* \in \Gamma^*$. For example, let $q = a_i$ be a vertex of A. If we select $p \in \Gamma$ to be the tangent determined by $a_{i-1}a_i$ and $p^* \in \Gamma^*$ to be the tangent determined by $a_i a_{i+1}$, then it could be this choice renders Eq. (10) false. For example, suppose $A = A^*$, p_1 is interior to $a_0 a_1$, and p_2 is the p just described. Then the β-sum in Eq. (9) includes $\beta_i = \alpha_i$ but the α-sum excludes α_i, and so the inequality is false. We repair this problem in the one instance of the proof that it matters by imagining an intermediate, zero-length link of the chain A parallel to the relevant tangent, and which is present as well in A^*. If the turn angle at that vertex in A is α, the turn is partitioned to $\alpha = \alpha' + \alpha''$, and correspondingly in A^* we have $\beta = \beta' + \beta''$ with $|\beta'| \leq \alpha'$ and $|\beta''| \leq \beta''$. The effect is just as if we had a short link in A aligned perfectly with the tangent p.

Let $Q \in A$ be a point through which a tangent is parallel to the missing link $a_0 a_n$. Let p_0 be the image of Q on Γ. Then, for any other point $p \in \Gamma$, the tangent can turn at most π between p_0 and p:

$$l(p_0, p) \leq \pi \quad (11)$$

This means that p_0 and p lie within the same semicircle, and because the convexity of A ensures that Γ does

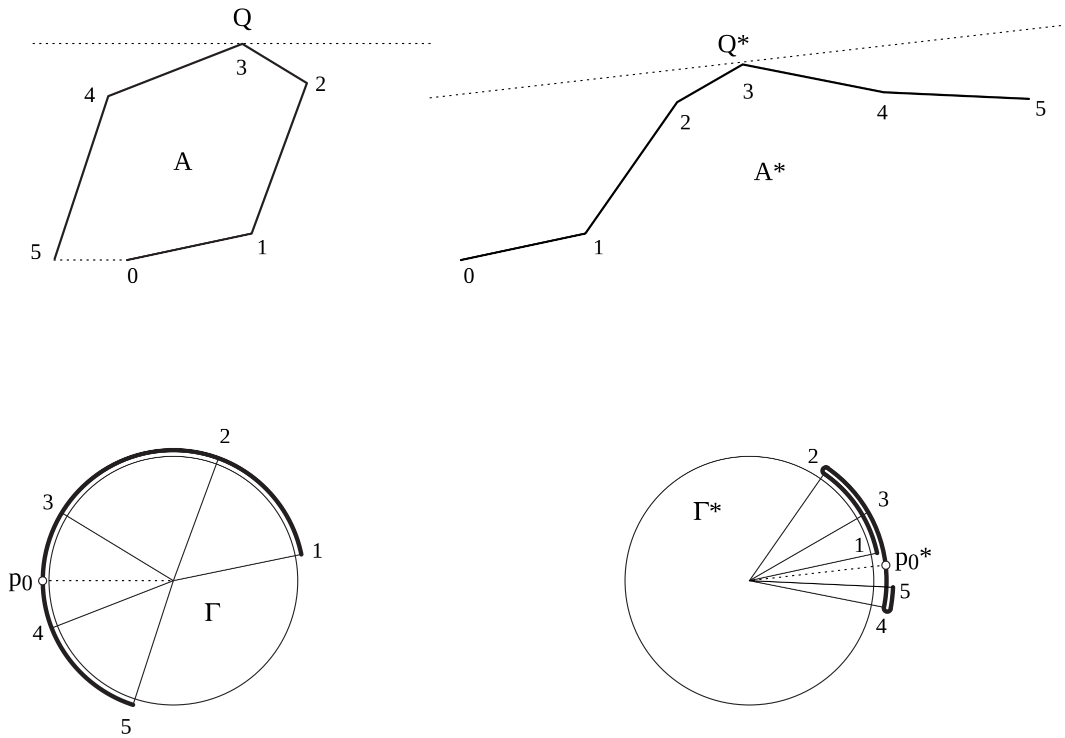

Figure 6: A is a convex chain, and A^* a valid reconfiguration. Γ and Γ^* are their respective tangent indicatrixes.

not double back over itself, in this circumstance the spherical distance and the arc length coincide:

$$\sigma(p_0, p) = l(p_0, p) \tag{12}$$

Now we need the corresponding point $p_0^* \in \Gamma^*$, and we follow the plan mentioned earlier to obtain a valid corresponding point. See Fig. 6; here $\alpha_3 = 30° + 22°$, and so we choose a tangent at Q^* that turns no more than 30°, leaving a further turn of no more than 22° to the next link.

The issue of carefully choosing a point p^* of Γ^* corresponding to $p \in G$ does not arise, as we may select p to be directly on one of the link tangents. From Eqs. (8) and (10), using p_0 and p as the two arbitrary points p_1 and p_2, we have

$$\begin{aligned} \sigma(p_0^*, p^*) &\leq l(p_0^*, p^*) && (13)\\ l(p_0^*, p^*) &\leq l(p_0, p) && (14)\\ l(p_0, p) &= \sigma(p_0, p) && (15) \end{aligned}$$

that is,

$$\sigma(p_0^*, p^*) \leq \sigma(p_0, p) \leq \pi \tag{16}$$

The distance $\sigma(p_0, p)$ may be interpreted as the angle between the tangent specified by p and the line determined by p_0. Thus $\cos(\sigma(p_0, p))$ is the projection of a link of the chain whose tangent is p onto that line. Thus the distance between the endpoints of the chain $a = |a_0 a_n|$ may be computed as

$$a = \int_0^L \cos(\sigma(p_0, p)) ds \tag{17}$$

where p varies with parameter s over the entire length L of chain A. For a polygonal chain, this reduces to a sum

$$a = \sum_{i=0}^{n-1} \ell_i \cos(\theta_i) \tag{18}$$

where θ_i is the angle of link i from the line through Q, i.e., $\sigma(p_0, p)$. This expression can be viewed as computing the (horizontal) x-coordinates of a_0 and a_n working in both directions from Q.

Now we look at the corresponding expression for Γ^*:

$$\int_0^L \cos(\sigma(p_0^*, p^*)) ds \tag{19}$$

This is again a projection of the curve A^*, and therefore of its missing edge, but onto the line determined by p_0^*. This line has no particular geometric significance; in particular, it is not necessarily parallel to the chord between the endpoints of A^*. However, because a projection is never longer than the original, it provides a lower bound on that chord length $a^* = |b_0 b_n|$:

$$a^* \geq \sum_{i=0}^{n-1} \ell_i \cos(\theta_i^*) \tag{20}$$

where θ_i^* is the angle of link i of A^* from the line determined by p_0^*. Finally, we observe from Eq. (16) that $\theta_i^* \leq \theta_i \leq \pi$, and because the cosine function is monotone decreasing over $[0, \pi]$, we have

$$\cos(\theta_i^*) \geq \cos(\theta_i) \tag{21}$$

With Eqs. (18) and (20) this finally implies that $a^* \geq a$, i.e., the distance between the endpoints has increased (or stayed the same). □

As Connelly remarks [Con82], this proof can be viewed as a variant of Zaremba's "shadow" proof of Cauchy's arm lemma [SZ67].

3 Noncrossing of Straightened Curve

Define a polygonal chain to be *simple* if nonadjacent segments are disjoint, and adjacent segments intersect only at their single, shared endpoint. By our nondegeneracy requirement, convex chains are simple. In particular, any opening of a convex chain via Cauchy's arm lemma (Theorem 0) remains simple because it remains convex. We now establish a parallel result for the generalized straightening of Theorem 1. We generalize slightly to permit the convex chain to start with the hand at the shoulder.

Theorem 2 *If $A = (a_0, \ldots, a_n) = \mathcal{C}_L(\alpha)$ is a closed convex chain with n fixed edge lengths L and turn angles α, closed in the sense that $a_n = a_0$, then any valid reconfiguration to $B = \mathcal{C}_L(\beta)$ is a simple polygonal chain.*

Proof: Suppose to the contrary that B is nonsimple. Let q_2 be the first point of B, measured by distance along the chain from the shoulder b_0, that coincides with an earlier point $q_1 \in B$. Thus q_1 and q_2 represent the same point of the plane, but different points along B. See Fig. 7. Because B is nonsimple, these "first touching points" exist,[2] and we do not have both $q_1 = b_0$ and $q_2 = b_n$ (because that would make B a simple, closed chain). Let p_1 and p_2 be the points of A corresponding to q_1 and q_2. Corollary 1 guarantees that $|q_1 q_2| \geq |p_1 p_2|$. But $|q_1 q_2| = 0$, and because the q's do not coincide with the original hand and shoulder, $|p_1 p_2| > 0$. This contradiction establishes the claim. □

One could alternatively prove this theorem by induction on the length of the chain, showing that in a continuous motion to B, the first violation of simplicity is either impossible by the induction hypothesis, or directly contradicts Theorem 1.

[2]The proof works for any self-intersection point. We only choose the first for definiteness.

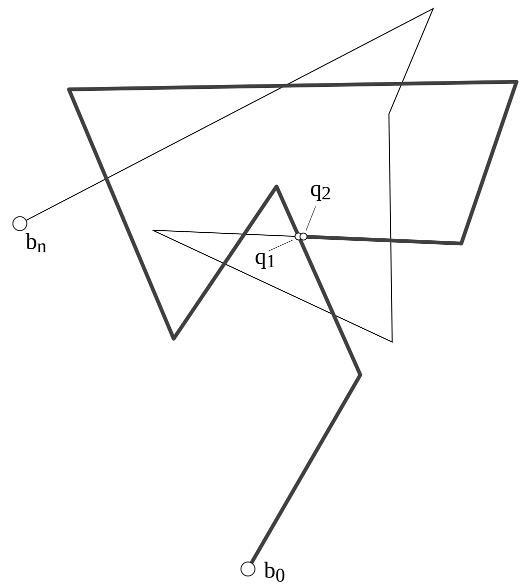

Figure 7: Violation of Theorem 1. $q_1 = q_2$ is the first point of self-contact; the initial portion of B, up to q_2, is highlighted.

Corollary 2 *A valid reconfiguration of an open convex chain remains simple.*

Proof: Theorem 2 guarantees that even the final missing edge between a_n and a_0 is not crossed, so the corollary is obtained by simply ignoring that last edge. □

4 Application to Curve Development

A curve Γ on the surface of a convex body may be "developed" on a plane by rolling the convex body on the plane without slippage so that the curve is always the point of contact. Here we will only consider polygonal curves on the surface of convex polyhedra (polytopes). An earlier result is that a closed convex polygonal curve on a polytope, i.e., one whose turns are all leftward on the surface, develops to a simple path [OS89]. Here we prove that particular (nonconvex) curves also develop without self-intersection: *slice* curves, those that are the intersection of a polytope with a plane; see Fig. 8.

Orient Γ to be counterclockwise from above. Let $c_0, c_1, \ldots, c_n$ be the *corners* of Γ, the points at which Γ crosses a polytope edge with a dihedral angle different from π, or meets a polytope vertex. Define the *right surface angle* $\theta(p)$ at a point $p \in \Gamma$ to be the total incident face angle at p to the right of the directed curve Γ at p. Only at a corner c_i of Γ is the right surface angle θ_i different from π. Note that θ_i could be greater or less than π, i.e., the slice curve could turn right or left on the surface.

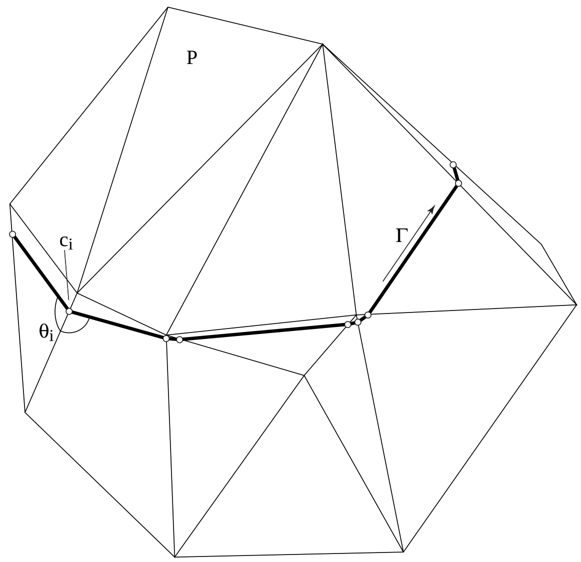

Figure 8: Γ is the intersection of a plane (not shown) with polytope P.

Define the *right development* of Γ to be a planar drawing of the polygonal chain Γ as the chain $B = (b_0, b_1, \ldots, b_n)$ with the same link lengths, $|b_i b_{i+1}| = |c_i c_{i+1}|$ for $i = 0, \ldots, n-1$, and with exterior angle θ_i to the right of b_i the same as the surface angle to the right of Γ at c_i on P, for all $i = 1, \ldots, n-1$. Define *left development* similarly. Note that if Γ avoids all polytope vertices, then there is no difference between the left and right development of Γ, for the sum of the right and right surface angles at any point is always 2π. Define the *development* of Γ to be the right development of Γ.

Theorem 3 *Let* $\Gamma = P \cap \Pi$ *be a closed curve on the surface of a polytope* P *that is the intersection of* P *with a plane* Π. *Then* Γ *develops on a plane to a simple (noncrossing) polygonal curve.*

Proof: We first dispense with the degenerate intersections, where there is zero volume of P to one side of Π. Then Π must intersect P in either a face, an edge, or a vertex. In all cases, Γ develops as is, and there is nothing to prove. Henceforth we assume that the slice is nondegenerate.

Let Q be the convex polygon in plane Π bound by Γ. Let $\phi_i \in (0, \pi)$ be the internal convex angle of Q at c_i. Our aim is to prove that these internal angles are related to the right surface angles θ_i as follows:

$$\phi_i \leq \theta_i \leq 2\pi - \phi_i \tag{22}$$

First note that, by our nondegeneracy assumption, the intersection of P with the halfspace below (and including) Π is a polytope; call it P_0. P_0 has Q as a top face, and the corners of Γ as vertices. The total angle incident to vertex c_i of P_0 is $\phi_i + \theta_i$, because P_0 includes the entire right surface angle at c_i. Because P_0 is a polytope, this sum must be at most 2π, and from $\phi_i + \theta_i \leq 2\pi$ the right hand inequality of Eq. (22) follows.

Repeating the argument to the other side, let P_1 be the intersection of P with the halfspace above Π. Because the left surface angle at c_i on P is no more than $2\pi - \theta_i$ (with equality if c_i is not a vertex of P), then the total angle incident to vertex c_i of P_1 is no more than $\phi_i + 2\pi - \theta_i$. Because P_1 is a polytope, this sum must be at most 2π, and from $\phi_i + 2\pi - \theta_i \leq 2\pi$ the left hand inequality of Eq. (22) follows.

Let $A = (a_0, a_1, \ldots, a_n)$ be a polygonal chain representing convex polygon Q, with a_i corresponding to c_i. The turn angle α_i at a_i is $\alpha_i = \pi - \phi_i$, i.e.,

$$\phi_i = \pi - \alpha_i \tag{23}$$

Let $B = (b_0, b_1, \ldots, b_n)$ be the development of Γ in the plane, again with b_i corresponding to c_i. The turn angle β_i at b_i is determined by the right surface angle: $\beta_i = \theta_i - \pi$, i.e.,

$$\theta_i = \pi + \beta_i \tag{24}$$

Substituting Eqs. (23) and (24) into Eq. (22) yields

$$\begin{array}{ccccc} \pi - \alpha_i & \leq & \pi + \beta_i & \leq & 2\pi - (\pi - \alpha_i) \\ -\alpha_i & \leq & \beta_i & \leq & \alpha_i \end{array}$$

i.e., $\beta_i \in [-\alpha_i, \alpha_i]$. Thus we see that B is a valid reconfiguration of A, and Theorem 2 applies and establishes the claim that it is simple. □

The examples in Fig. 9 can all be viewed as developments of slice curves.

Because Schur's Theorem generalizes to smooth curves, Theorem 3 should generalize to slice curves for any convex body B.

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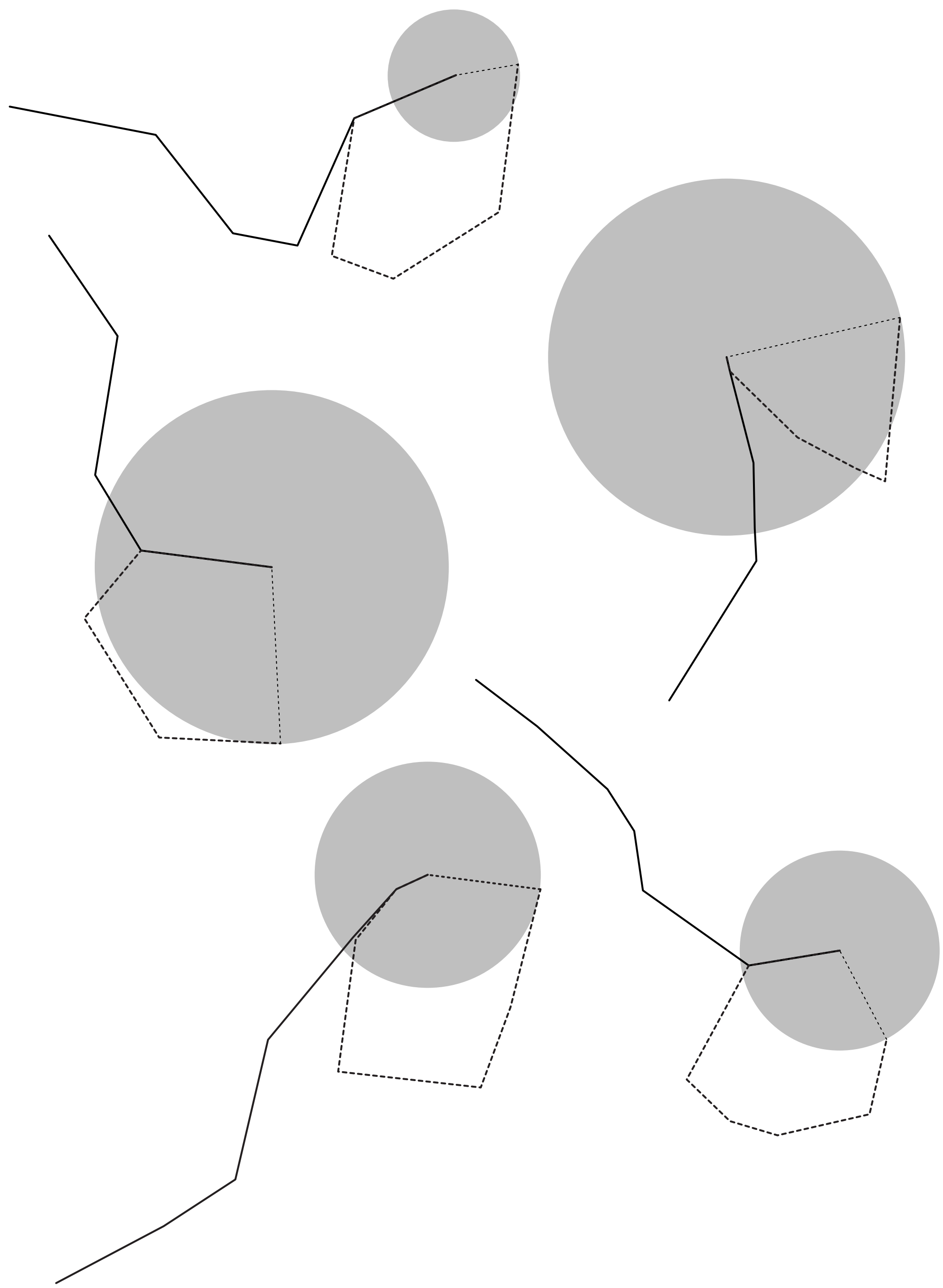

Figure 9: Examples of valid reconfigurings of convex chain A (dashed) to B (solid). In each case, the forbidden shoulder disk is shown.