Refold Rigidity of Convex Polyhedra*

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Abstract

We show that every convex polyhedron may be unfolded to one planar piece, and then refolded to a different convex polyhedron. If the unfolding is restricted to cut only edges of the polyhedron, we identify several polyhedra that are "edge-unfold rigid" in the sense that each of their unfoldings may only fold back to the original. For example, each of the 43,380 edge unfoldings of a dodecahedron may only fold back to the dodecahedron, and we establish that 11 of the 13 Archimedean solids are also edge-unfold rigid. We begin the exploration of which classes of polyhedra are and are not edge-unfold rigid, demonstrating infinite rigid classes through perturbations, and identifying one infinite nonrigid class: tetrahedra.

1 Introduction

It has been known since [LO96] and [DDL+99] that there are convex polyhedra, each of which may be unfolded to a planar polygon and then refolded to different convex polyhedra. For example, the cube may be unfolded to a "Latin cross" polygon, which may be refolded to 22 distinct non-cube convex polyhedra [DO07, Figs. 25.32-6]. But there has been only sporadic progress on understanding which pairs of convex polyhedra[1] have a common unfolding. A notable recent exception is the discovery [SHU11] of a series of unfoldings of a cube that refold in the limit to a regular tetrahedron, partially answering Open Problem 25.6 in [DO07, p. 424].

Here we begin to explore a new question, which we hope will shed light on the unfold-refold spectrum of problems: Which polyhedra $\mathcal{P}$ are *refold-rigid* in

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[1] All polyhedra in this paper are convex, and the modifier will henceforth be dropped.

the sense that any unfolding of $\mathcal{P}$ may only be refolded back to $\mathcal{P}$? The answer we provide here is: NONE—Every polyhedron $\mathcal{P}$ has an unfolding that refolds to an incongruent $\mathcal{P}'$. Thus every $\mathcal{P}$ may be *transformed* to some $\mathcal{P}'$.

This somewhat surprising answer leads to the next natural question: Suppose the unfoldings are restricted to *edge unfoldings*, those that only cut along edges of $\mathcal{P}$ (rather than permitting arbitrary cuts through the interior of faces). Say that a polyhedron $\mathcal{P}$ whose every edge unfolding only refolds back to $\mathcal{P}$ is *edge-unfold rigid*, and otherwise is an *edge-unfold transformer*. It was known that four of the five Platonic solids are edge-unfold transformers (e.g., [DDL+10] and [O'R10]). Here we prove that the dodecahedron is edge-unfold rigid: all of its edge unfoldings only fold back to the dodecahedron. The proof also demonstrates edge-unfold rigidity for 11 of the Archimedean solids; we exhibit new refoldings of the truncated tetrahedron and the cuboctahedron. We also establish the same rigidity for infinite classes of slightly perturbed versions of these polyhedra. In contrast to this, we show that every tetrahedron is an edge-unfold transformer: at least one among a tetrahedron's 16 edge unfoldings refolds to a different polyhedron.

This work raises many new questions, summarized in Section 6.

2 Notation and Definitions

We will use $\mathcal{P}$ for a polyhedron in $\mathbb{R}^3$ and P for a planar polygon. An *unfolding* of a polyhedron $\mathcal{P}$ is development of its surface after cutting to a single (possibly overlapping) polygon P in the plane. The surface of $\mathcal{P}$ must be cut open by a spanning tree to achieve this. An *edge-unfolding* only includes edges of $\mathcal{P}$ in its spanning cut tree. Note that we do not insist that unfoldings avoid overlap.

A *folding* of a polygon P is an identification of its boundary points that satisfies the three conditions of Alexandrov's theorem: (1) The identifications (or "gluings") close up the perimeter of P without gaps or overlaps; (2) The resulting surface is homeomorphic to a sphere; and (3) Identifications result in $\leq 2\pi$ surface angle glued at every point. Under these three conditions, Alexandrov's theorem guarantees that the folding produces a convex polyhedron, unique once the gluing is specified. See [Ale05] or [DO07]. Note that there is no restriction that whole edges of P must be identified to whole edges, even when P is produced by an edge unfolding. We call a gluing that satisfies the above conditions an *Alexandrov gluing*.

A polyhedron $\mathcal{P}$ is *refold-rigid* if every unfolding of $\mathcal{P}$ may only refold back to $\mathcal{P}$. Otherwise, $\mathcal{P}$ is a *transformer*. A polyhedron is *edge-unfold rigid* if every edge unfolding of $\mathcal{P}$ may only refold back to $\mathcal{P}$, and otherwise it is an *edge-unfold transformer*.

3 Polyhedra Are Transformers

The proof that no polyhedron $\mathcal{P}$ is refold rigid breaks naturally into two cases. We first state a lemma that provides the case partition. Let $\kappa(v)$ be the curvature at vertex $v \in \mathcal{P}$, i.e., the "angle gap" at v: 2π minus the total incident face angle $\alpha(v)$ at v. By the Gauss-Bonnet theorem, the sum of all vertex curvatures on $\mathcal{P}$ is 4π.

Lemma 1 *For every polyhedron $\mathcal{P}$, either there is a pair of vertices with $\kappa(a) + \kappa(b) > 2\pi$, or there are two vertices each with at most π curvature: $\kappa(a) \leq \pi$ and $\kappa(b) \leq \pi$.*

Proof: Suppose there is no pair with curvature sum more than 2π. So we have $\kappa(v_1) + \kappa(v_2) \leq 2\pi$ and $\kappa(v_3) + \kappa(v_4) \leq 2\pi$ for four distinct vertices. Suppose neither of these pairs have both vertices with at most π curvature. If $\kappa(v_2) > \pi$, then $\kappa(v_1) \leq \pi$; and similarly, if $\kappa(v_4) > \pi$, then $\kappa(v_3) \leq \pi$. Thus we have identified two vertices, v_1 and v_3, both with at most π curvature. □

We can extend this lemma to accommodate 3-vertex doubly covered triangles as polyhedra, because then every vertex has curvature greater than π.

Lemma 2 *Any polyhedron $\mathcal{P}$ with a pair of vertices with curvature sum more than 2π is not refold-rigid: There is an unfolding that may be refolded to a different polyhedron $\mathcal{P}'$.*

Proof: Let $\kappa(a)+\kappa(b) > 2\pi$, and so the incident face angles satisfy $\alpha(a)+\alpha(b) < 2\pi$. Let γ be a shortest path on $\mathcal{P}$ connecting a to b. Cut open $\mathcal{P}$ with a cut tree T that includes γ as an edge. How T is completed beyond the endpoints of $\gamma = ab$ doesn't matter. (Recall our definition of unfolding does not demand non-overlap.)

Let γ_1 and γ_2 be the two sides of the cut γ, and let m_1 and m_2 be the midpoints of γ_1 and γ_2. Reglue the unfolding by folding γ_1 at m_1 and gluing the two halves of γ_1 together, and likewise fold γ_2 at m_2. All the remaining boundary of the unfolding outside of γ is reglued back exactly as it was cut by T. See Fig. 1.

The midpoint folds at m_1 and m_2 have angle π (because γ is a geodesic). The gluing draws the endpoints a and b together, forming a point with total angle $\alpha(a) + \alpha(b) < 2\pi$. Thus this gluing is an Alexandrov gluing, producing some polyhedron $\mathcal{P}'$. Generically $\mathcal{P}'$ has one more vertex than $\mathcal{P}$: it gains two vertices at m_1 and m_2, and a and b are merged to one. $\mathcal{P}'$ could only have the same number of vertices as $\mathcal{P}$ if $\alpha(a)+\alpha(b) = 2\pi$, which is excluded in this case. □

Lemma 3 *Any polyhedron $\mathcal{P}$ with a pair of vertices each with curvature at most π is not refold-rigid: There is an unfolding that may be refolded to a different polyhedron $\mathcal{P}'$.*

Proof: Let a and b be a pair of vertices with $\kappa(a) \leq \pi$ and $\kappa(b) \leq \pi$, and so $\alpha(a) \geq \pi$ and $\alpha(b) \geq \pi$. Let $\gamma = ab$ be a shortest path from a to b on

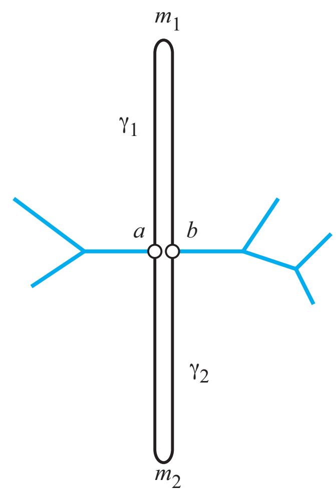

Figure 1: Gluing when $\alpha(a) + \alpha(b) < 2\pi$.

$\mathcal{P}$. Because the curvature at each endpoint is at most π, there is at least π surface angle incident to a and to b. This permits identification of a rectangular neighborhood R on $\mathcal{P}$ with midline ab, whose interior is vertex-free.

Now we select a cut tree T that includes ab and otherwise does not intersect R. This is always possible because there is at least π surface angle incident to both a and b. So we could continue the path beyond ab to avoid cutting into R. Let T unfold $\mathcal{P}$ to polygon P. We will modify T to a new cut tree T'.

Replace ab in T by three edges ab', $b'a'$, $a'b$, forming a zigzag 'Z'-shape, $Z = ab'a'b$, with $Z \subset R$. We will illustrate with an unfolding of a cube, shown in Fig. 2, with ab the edge cut between the front (F) and top (T) faces of the cube.

We select an angle ε determining the Z according to two criteria. First, ε is smaller than either $\kappa(a)$ and $\kappa(b)$. Second, ε is small enough so that the following construction sits inside R. Let $R' \subset R$ be a rectangle whose diagonal is ab; refer to Fig. 3. Trisect the left and right sides of R', and place a' and b' two-thirds away from a and b respectively. The angle of the Z at a' and at b' is ε. $\triangle ab'a'$ and $\triangle ba'b'$ are congruent isosceles triangles; so $|ab'| = |b'a'| = |a'b|$.

The turn points a' and b' of Z have curvature zero on $\mathcal{P}$ (because $Z \subset R$ and R is vertex-free). Let P' be the polygon obtained by unfolding $\mathcal{P}$ by cutting T', and label the pair of images of each Z corner $a_1, a'_1, \ldots, b_2$, as illustrated in Fig. 2. Now we refold it differently, to obtain a different polyhedron $\mathcal{P}'$. "Zip" P' closed at the reflex vertices a'_2 and b'_1. Zipping at a'_2 glues $a'_2b'_2$ to a'_2b_2, so that now $b_2 = b'_2$; zipping at b'_1 glues $b'_1a'_1$ to b'_1a_1, so that now $a_1 = a'_1$. (See the insert of Fig. 2.) Finally, the two "halves" of the new $a'_1b'_1 = a_2b_2$ are glued together, and the remainder of T' is reglued just as it was in T.

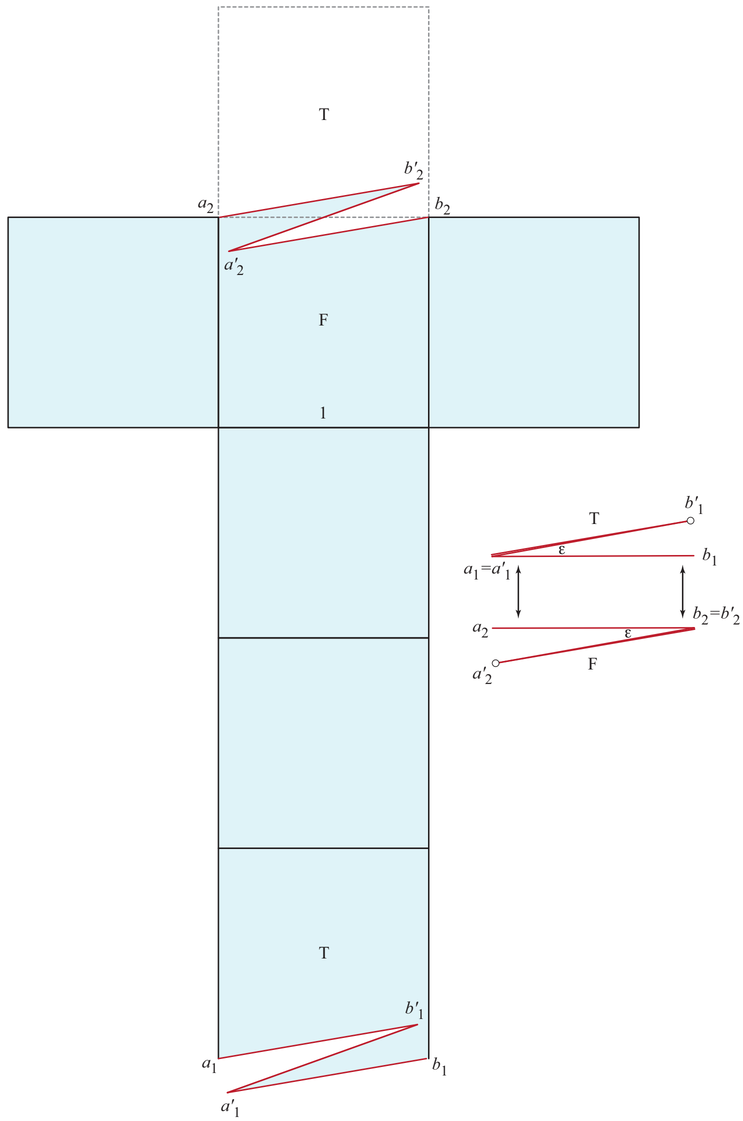

Figure 2: Unfolding of a unit cube. The cut edge ab is replaced by $Z = ab'a'b$. The unfolding P' is illustrated. The insert shows the gluing in the vicinity of ab in the refolding to $\mathcal{P}'$.

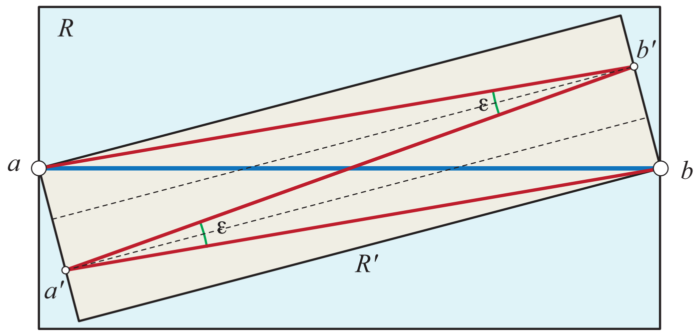

Figure 3: Construction of zig-zag path Z.

This gluing produces new vertices near a' and b', each of curvature $\kappa(a') = \kappa(b') = \varepsilon$. An extra ε of surface angle is glued to both a and b, so their curvatures each decrease by ε (and so maintain the Gauss-Bonnet sum of 4π). By the choice of ε, these curvatures remain positive. Alexandrov's theorem is satisfied everywhere: the curvatures at a, b, a', b' are all positive, and the lengths of the two halves of the new $a'_1 b'_1 = a_2 b_2$ edge are the same (and note this length is not the original length of ab on $\mathcal{P}$, but rather the side-length of the isosceles triangles: $|ab'| = |b'a'| = |a'b|$). So this refolding corresponds to some polyhedron $\mathcal{P}'$. It is different from $\mathcal{P}$ because it has two more vertices at a' and b' (vertices at a and b remain with some positive curvature by our choice of ε). □

Putting Lemmas 2 and 3 together yields the claim:

Theorem 1 *Every polyhedron has an unfolding that refolds to a different polyhedron, i.e., no polyhedron is refold-rigid.*

4 Many (Semi-)Regular Polyhedra are Edge-Unfold Rigid

Our results on edge-unfold rigidity rely on this theorem:

Theorem 2 *Let $\theta_{\min}$ be the smallest angle of any face of $\mathcal{P}$, and let $\kappa_{\max}$ be the largest curvature at any vertex of $\mathcal{P}$. If $\theta_{\min} > \kappa_{\max}$, then $\mathcal{P}$ is edge-unfold rigid.*

Proof: Let T be an edge-unfold cut tree for $\mathcal{P}$, and P the resulting unfolded polygon. No angle on the boundary of P can be smaller than $\theta_{\min}$. Let x be a leaf node of T and y the parent of x. The exterior angle at x in the unfolding P is at most $\kappa_{\max}$. Because every internal angle of P is at least $\theta_{\min}$, which is larger than $\kappa_{\max}$, no point of P can be glued into x, leaving the only option to "zip" together the two cut edges deriving from $xy \in T$. Let $T' = T \setminus xy$

be the cut tree remaining after this partial gluing, and P' the partially reglued manifold.

If T' is not empty, it is a tree, with at least two leaves, one of which might be y (if x was the only child of y). Any leaf $z \in T'$ corresponds to some vertex $v \in \mathcal{P}$, with all but one incident edge already glued. Because P' has not gained any new angles beyond those available in P, we have returned to the same situation: no angle of P' is small enough to fit into the angle gap at z, which is at most $\kappa_{\max}$ at any v. Thus again the edge of T' incident to z must be zipped in the gluing. Continuing in this manner, we see that T may only be reglued by exactly identifying every cut-edge pair, reproducing $\mathcal{P}$. □

Corollary 1 *The regular and semi-regular solids that satisfy Theorem 2, marked with "✓" in Table 4, are all edge-refold rigid, and the three marked "✗" are edge-unfold transformers.*

Note there are "close calls" listed in the table, where $\theta_{\min} = \kappa_{\max} = \frac{1}{3}$ and no conclusion can be drawn via Theorem 2. We knew that the icosahedron is not edge-refold rigid from the zipping found in [O'R10] (and other zippings not reported there), and using similar techniques we found one zipping of one of the 261 unfoldings of the truncated tetrahedron (Fig. 4), and one zipping of one of the 6912 unfoldings of the cuboctahedron (Fig. 5).

Corollary 2 *Any polyhedron $\mathcal{P}$ that satisfies Theorem 2, may be "perturbed" by moving its vertices slightly to create an uncountable number of edge-refold rigid polyhedra.*

Proof: Let $\mathcal{P}$ have unit face normal $\hat{n}_i$ for face f_i, and let d_i be the distance to f_i from an origin fixed inside $\mathcal{P}$. The perturbation is achieved by displacing each face normal slightly, changing $\hat{n}_i$ to $\hat{n}'_i$. Let $\mathcal{P}'$ be the intersection of halfspaces with normals $\hat{n}'_i$, each the same distance d_i from the origin. It is clear that with small enough perturbation, $\mathcal{P}'$ is incongruent to but has the same combinatorial structure as $\mathcal{P}$: If f_i is a k-gon, the corresponding face f'_i of $\mathcal{P}'$ is also a k-gon; and the vertices of $\mathcal{P}'$ have the same number and type and arrangement of incident faces as do the corresponding vertices of $\mathcal{P}$.

Because the inequality $\theta_{\min} > \kappa_{\max}$ is strict for $\mathcal{P}$, it is clear that we can choose the perturbation to be small enough so that the corresponding inequality $\theta'_{\min} > \kappa'_{\max}$ holds for $\mathcal{P}'$. Then Theorem 2 yields the claim. □

Note that we have not attempted an explicit construction of the perturbed $\mathcal{P}'$. It suffices for our purposes to just show they exist in uncountable numbers.

5 Tetrahedra are edge-unfold transformers

The goal of this section is to prove this theorem:

Theorem 3 *Every tetrahedron may be edge-unfolded and refolded to a different polyhedron.*

Polyhedron Name	θ_{min}	κ_{max}	$\theta_{\text{min}} > \kappa_{\text{max}}$
Dodecahedron	$\frac{3}{5}$	$\frac{1}{5}$	✓
Icosahedron	$\frac{1}{3}$	$\frac{1}{3}$	✗
Trunc. Tetrahedron	$\frac{1}{3}$	$\frac{1}{3}$	✗
Cuboctahedron	$\frac{1}{3}$	$\frac{1}{3}$	✗
Trunc. Cube	$\frac{1}{3}$	$\frac{1}{6}$	✓
Rhombicuboctahedron	$\frac{1}{3}$	$\frac{1}{6}$	✓
Trunc. Cuboctahedron	$\frac{1}{2}$	$\frac{1}{12}$	✓
Snub Cube	$\frac{1}{3}$	$\frac{1}{6}$	✓
Icosidodecahedron	$\frac{1}{3}$	$\frac{2}{15}$	✓
Trunc. Dodecahedron	$\frac{1}{3}$	$\frac{1}{15}$	✓
Trunc. Icosahedron	$\frac{3}{5}$	$\frac{1}{15}$	✓
Rhomb- icosidodecahedron	$\frac{1}{3}$	$\frac{1}{15}$	✓
Trunc. Icosidodecahedron	$\frac{1}{2}$	$\frac{1}{30}$	✓
Snub Dodecahedron	$\frac{1}{3}$	$\frac{1}{15}$	✓
Pseudo- rhombicuboctahedron	$\frac{1}{3}$	$\frac{1}{6}$	✓

Table 1: Inventory of minimum face angles and maximum vertex curvatures, for selected regular and semi-regular polyhedra. All angles expressed in units of π.

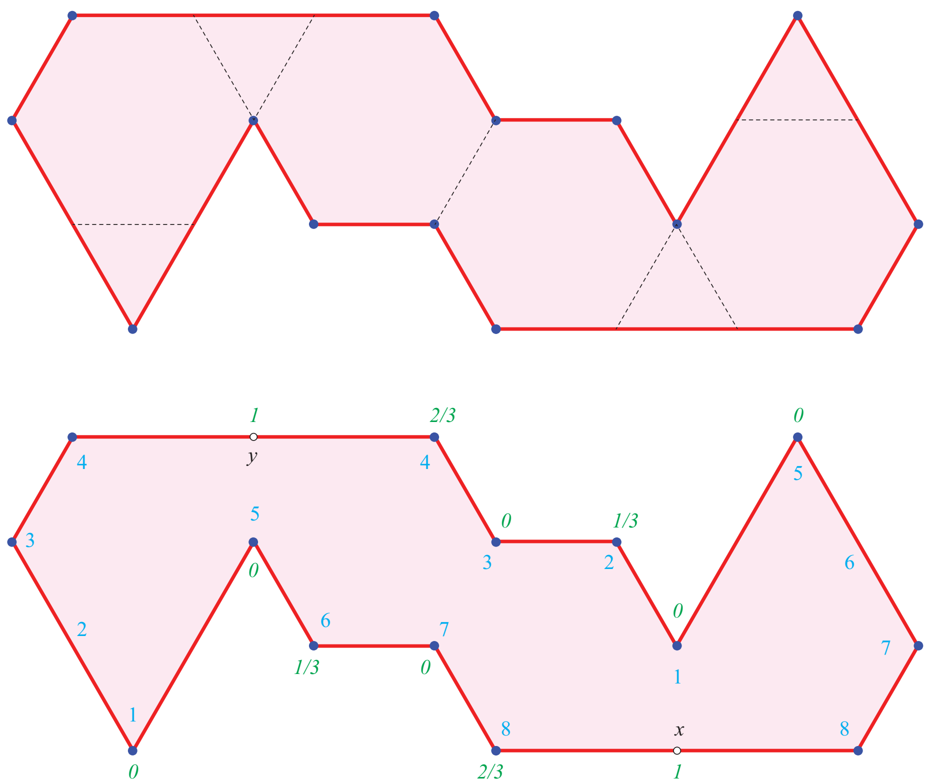

Figure 4: Top: An unfolding of the truncated tetrahedron. Bottom: An alternative Alexandrov gluing is obtained by matching the (blue) vertex numbers, with fold points at x and y. The result is a 6-vertex polyhedron, with two vertices each of curvatures $\{\frac{1}{3}, \frac{2}{3}, 1\}$ (marked in green).

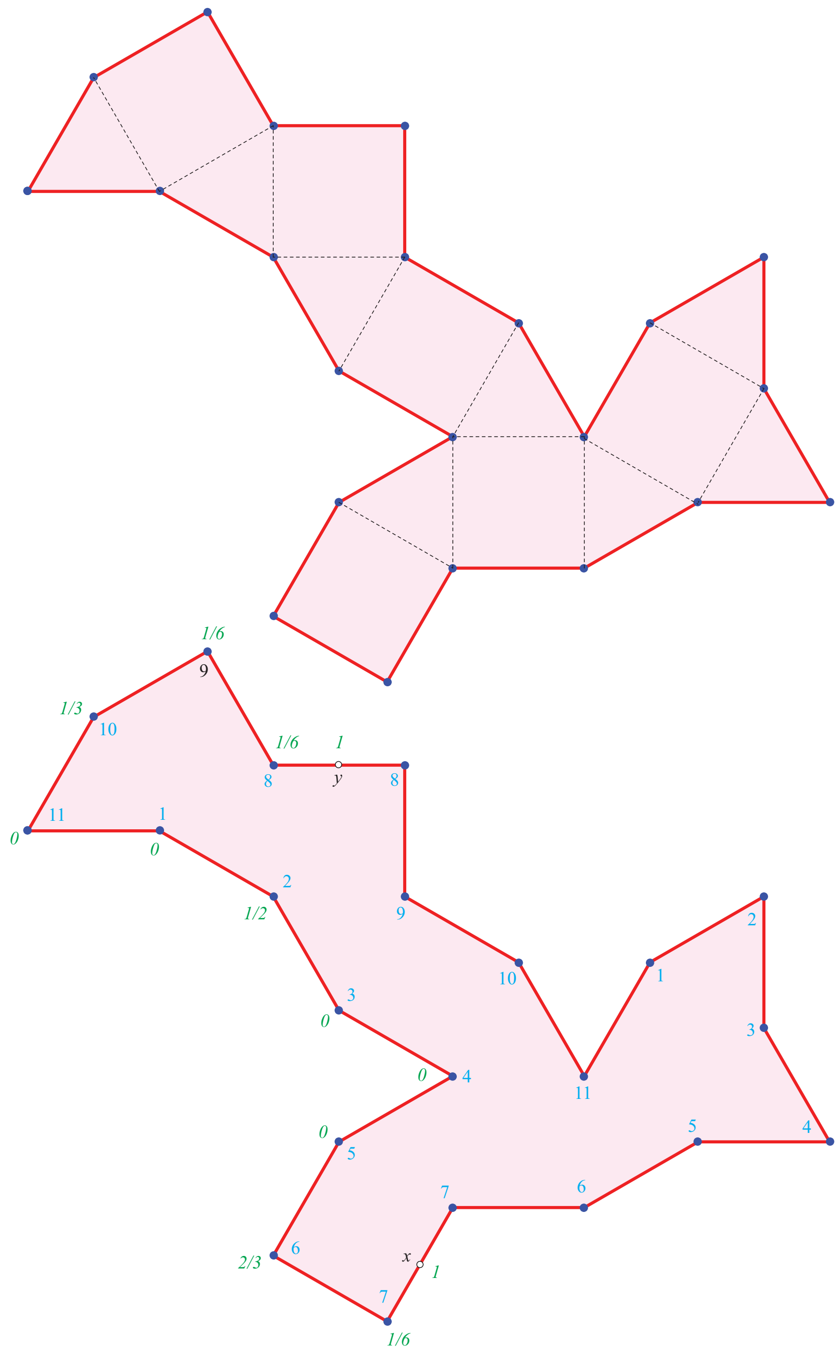

Figure 5: Top: An unfolding of the cuboctahedron. Bottom: Matching the vertices by (blue) vertex number, and creating fold points at x and y, results in a different Alexandrov gluing. The resulting polyhedron has 8 vertices, whose curvatures are marked in green: $\{\frac{1}{2}, \frac{2}{3}, 3 \times \frac{1}{6}, \frac{1}{3}, 2 \times 1\}$.

There are 16 distinct edge unfoldings of a tetrahedron $\mathcal{T}$. The spanning cut trees that determine these unfoldings fall into just two combinatorial types: The cut tree is a star, a Y-shaped "trident" with three leaves, or the cut tree is a path of three edges. There are 4 different tridents, and $2 \cdot \binom{4}{2} = 12$ different paths. In all these unfoldings, the polygon P that constitutes the unfolded surface is a hexagon: the three cut edges becomes three pairs of equal-length edges of the hexagon. Our goal is to show that, for any $\mathcal{T}$, at least one of the 16 unfoldings P may be refolded to a polyhedron $\mathcal{P}'$ not congruent to $\mathcal{T}$.

5.1 Classification of Tetrahedra

It will again be convenient henceforth to represent angles and curvatures in units of π. We will use "corner" for each vertex of $\mathcal{T}$, and "vertex" for the vertices of P.

Definition 4 *Call a corner v_i of $\mathcal{T}$ with $\kappa_i \geq 1$* convex*, and one with $\kappa_i < 1$* reflex*. A convex corner is "sharp" and, if a leaf of the cut tree, unfolds as a convex vertex of the unfolding P; a reflex corner is "flat" and unfolds when a leaf to a reflex vertex of P.*

Label the corners v_i, $i = 1, 2, 3, 4$ so that

$$2 > \kappa_1 \geq \kappa_2 \geq \kappa_3 \geq \kappa_4 > 0 \ .$$

Let α_i be the total face angle incident to v_i on $\mathcal{T}$; so $\alpha_i = 2 - \kappa_i$.

We classify tetrahedra into four classes.

1. π-tetrahedra: $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = 1$.
2. 1r-tetrahedra: $\kappa_1 \geq \kappa_2 \geq \kappa_3 \geq 1 > \kappa_4$: only v_4 is reflex, $\{v_1, v_2, v_3\}$ are convex.
3. 2r-tetrahedra: $\kappa_1 \geq \kappa_2 \geq 1 > \kappa_3 \geq \kappa_4$: $\{v_3, v_4\}$ are reflex, $\{v_1, v_2\}$ are convex.
4. 3r-tetrahedra: $\kappa_1 \geq 1 > \kappa_2 \geq \kappa_3 \geq \kappa_4$: $\{v_2, v_3, v_4\}$ are reflex, only v_1 is convex.

5.2 Lemmas for each class of tetrahedra

Lemma 4 *Every π-tetrahedron $\mathcal{T}$ may be refolded to an incongruent $\mathcal{P}'$.*

Proof: Cut open $\mathcal{T}$ via a trident. The result is a triangle, because the three leaves open to π angle each. By Theorem 25.1.4 of [DO07], every convex polygon folds to an uncountable number of distinct convex polyhedra. □

Lemma 5 *Every 1r-tetrahedron $\mathcal{T}$ may be refolded to an incongruent $\mathcal{P}'$.*

Proof: Recall that only v_4 is reflex. Cut open $\mathcal{T}$ via a trident whose root is at v_4. This results in three images of v_4 in P, each convex in P because each is one of the three face angles incident to v_4 on $\mathcal{T}$. The three leaves of the cut tree are at convex corners of $\mathcal{T}$. Thus P is a convex hexagon, and we again apply Theorem 25.1.4 to obtain $\mathcal{P}'$. □

For 2r- and 3r-tetrahedra, we can no longer guarantee a convex unfolding. Instead we show there is an unfolding that has only one reflex vertex, and then argue that a particular refolding leads to a different polyhedron.

Lemma 6 *Every 2r-tetrahedron $\mathcal{T}$ may be refolded to an incongruent $\mathcal{P}'$.*

Proof: Recall that v_3 and v_4 are reflex, with v_4 being "more reflex," i.e., it unfolds to a smaller exterior angle κ_4. Cut open $\mathcal{T}$ with a trident rooted at v_4. This results in a hexagon P with one reflex vertex, with exterior angle κ_3. By Gauss-Bonnet, we have

$$(\kappa_1 + \kappa_2) + (\kappa_3 + \kappa_4) = 4 \ .$$

Knowing that $\kappa_3 \geq \kappa_4$, if we substitute κ_3 for κ_4 we obtain

$$(\kappa_1 + \kappa_2) + 2\kappa_3 \geq 4 \ .$$

Rephrasing κ_1 and κ_2 in terms of their corresponding α_i values yields

$$(2 - \alpha_1) + (2 - \alpha_2) + 2\kappa_3 \geq 4 \ ,$$

$$\kappa_3 \geq \tfrac{1}{2}(\alpha_1 + \alpha_2) \ .$$

Because $\kappa_1 \geq \kappa_2$, $\alpha_1 \leq \alpha_2$, and substituting the smaller for the larger, we obtain

$$\kappa_3 \geq \tfrac{1}{2}(2\alpha_1) = \alpha_1 \ .$$

This means that the exterior angle κ_3 at the one reflex vertex of P is at least as large as the smaller of the two convex angles produced by leaves of the cut tree at the convex corners v_1 and v_2.

Abstractly, the hexagon P may be viewed as in Fig. 6. We glue the convex α_1 angle into the reflex κ_3 angle, which fits because $\alpha_1 \leq \kappa_3$. Now to either side of that gluing, only convex vertices remain. So these two "loops" can be glued together, zipped closed, necessarily forming an Alexandrov gluing, and thus some convex polyhedron $\mathcal{P}'$.

A concrete example is shown in Fig. 7.

Now we argue that $\mathcal{P}' \neq \mathcal{T}$. Recall that v_4 is the flattest corner of $\mathcal{T}$. $\mathcal{P}'$ has a corner with curvature $\kappa_3 - \alpha_1$ resulting from the gluing $v_1 \rightarrow v_3$. If this is flatter than v_4 but not completely flat, then we have distinguished $\mathcal{P}'$ from $\mathcal{T}$. In other words, if $0 < \kappa_3 - \alpha_1 < \kappa_4$, then we are finished. We next analyze the case $\kappa_3 = \alpha_1$ (i.e., $\kappa_1 + \kappa_3 = 2$), violating the $0 < \kappa_3 - \alpha_1$ inequality, and show that leads to a contradiction.

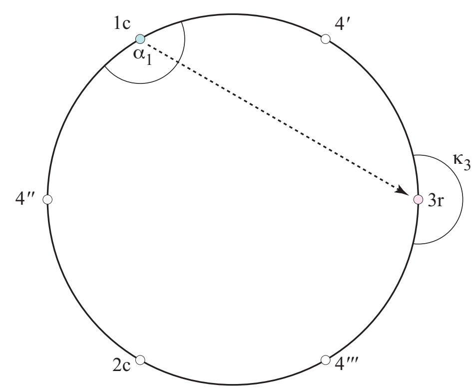

Figure 6: The indices indicate from which corner of $\mathcal{T}$ the vertices of P derive: corner v_4 has three images, the one reflex vertex of P with exterior angle κ_3 derives from corner v_3, and the convex corner v_1 leads to an internal angle α_1.

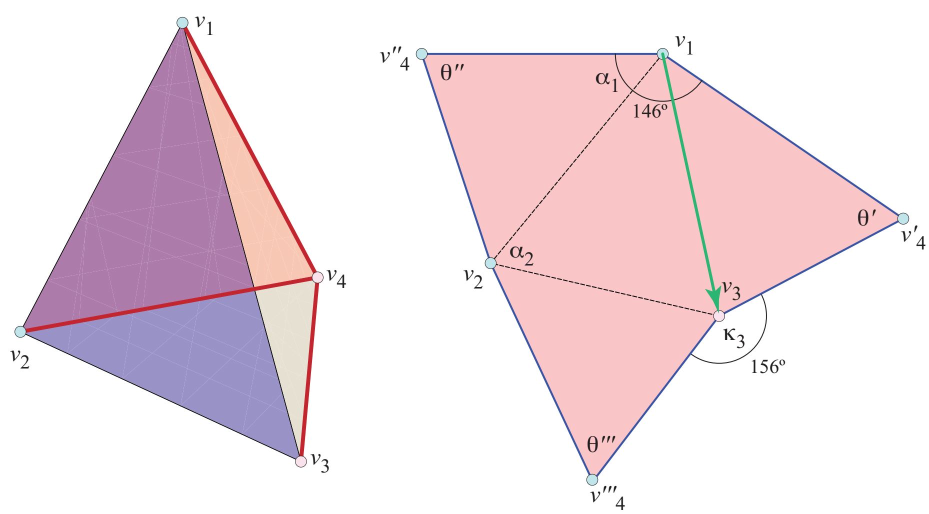

Figure 7: A 2r-tetrahedron (v_1, v_2 convex, v_3, v_4 reflex), cut open with a trident rooted at v_4, producing a hexagon with one reflex vertex with exterior angle κ_3. The proof of Lemma 6 shows that the convex angle α_1 derived from v_1 fits within κ_3. (Note that α_2 does not fit within κ_3.)

Note that $\kappa_1 + \kappa_3 = 2$ implies that $\kappa_2 + \kappa_4 = 2$, and because $\kappa_1 \geq \kappa_2 > \kappa_3 \geq \kappa_4$, we can conclude that $\kappa_3 = \kappa_4$, and then $\kappa_1 = \kappa_2$, and so

$$\alpha_1 = \alpha_2 = \kappa_3 = \kappa_4 \neq 1 \; . \tag{1}$$

Now, unless various edges lengths of P match, $\mathcal{P}'$ will have one or more vertices of curvature 1, violating Eq. 1. In particular, both $|v_1 v_4'| = |v_3 v_4'|$ and

$$|v_1 v_4''| + |v_4'' v_2| = |v_3 v_4'''| + |v_4''' v_2| \tag{2}$$

must hold. Let $\theta', \theta'', \theta'''$ be the inner angles of P at v_4', v_4'', v_4''' respectively. When the above length relationships hold, the vertices of $\mathcal{P}'$ have curvatures $\{2-\theta', 1-\theta'', 1-\theta'', \kappa_2\}$. If we glue v_2 to v_3 instead of v_1 to v_3 (recall $\alpha_1 = \alpha_2$) and obtain the same $\mathcal{P}'$, then $\theta' = \theta'''$. The Gauss-Bonnet theorem implies that

$$2 - \theta' = \kappa_1 = \kappa_2$$

and

$$1 - \theta'' = 1 - \theta''' = \kappa_3 = \kappa_4 \; .$$

So $\theta' = \theta'' = \theta''' = \frac{1}{2}$ and $\alpha_2 = \frac{1}{2}$, and P must have a shape angularly similar to that illustrated in Fig. 8. But P can clearly not satisfy Eq. 2. This contradiction

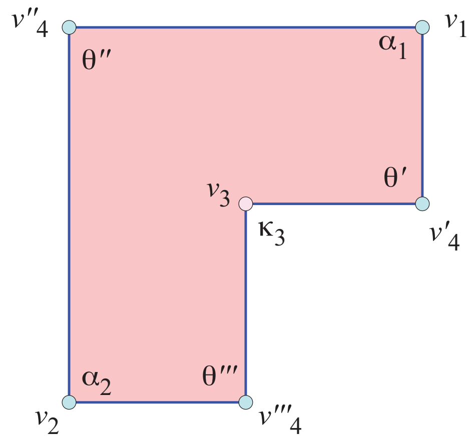

Figure 8: P when $\kappa_1 = \kappa_2 = \frac{3}{2} > \kappa_3 = \kappa_4 = \frac{1}{2}$.

shows that the case $\kappa_3 = \alpha_1$ cannot occur.

We are left with the case $\kappa_3 - \alpha_1 \geq \kappa_4$, which can occur in $\mathcal{T}$, e.g., with curvatures $(\frac{3}{2}, 1+\varepsilon, 1-\frac{1}{2}\varepsilon, \frac{1}{2}-\frac{1}{2}\varepsilon)$. The gluing of v_1 to v_3 produces a reflex corner of $\mathcal{P}'$ with curvature $\kappa_3 - (2-\kappa_1)$, which, if $\mathcal{P}' = \mathcal{T}$, must be either equal to κ_3 or to κ_4. If it equals κ_3, then $\kappa_1 = 2$, not possible in a tetrahedron. So we must have $\kappa_3 - (2-\kappa_1) = \kappa_4$, i.e., $\kappa_1 + \kappa_3 = 2 + \kappa_4$, which in turn implies (by Gauss-Bonnet) that $\kappa_2 + 2\kappa_4 = 2$.

Now we examine the gluing of the two hexagon edges v_1v_4' and $v_4'v_3$. If those two edge lengths are equal, then we produce a convex corner at v_4' of curvature $2-\theta'$, which must be either κ_1 or κ_2. Now concentrate on the gluing of the remaining four hexagon edges; they must produce one convex and one reflex corner, the latter necessarily κ_3 (because we've already accounted for κ_4). It must be that v_2 is the midpoint of the four edges, for otherwise we produce too many vertices. So the gluing at v_2 accounts for κ_2. Finally, the gluing of v_4'' to v_4''' must produce κ_4, which is a contradiction because κ_4 derives in $\mathcal{T}$ from the gluing of all three of $\{v_4', v_4'', v_4'''\}$.

So the edge lengths in the v_4' portion of the hexagon are not equal, $|v_3v_4'| \neq |v_4'v_1|$, and instead those edges produce a reflex corner of curvature $2-(1+\theta') = \kappa_3$, and a convex corner of curvature $\kappa_2 = 1$.

So now we have accounted for three corners of $\mathcal{P}'$, leaving only κ_1 if $\mathcal{P}' = \mathcal{T}$. But this one vertex must arise from the gluing of the four remaining edges of the hexagon. This can only result in just one vertex if v_2 is the midpoint of those four edges, $\kappa_2 = \kappa_1$, and $\theta'' + \theta''' = 2$.

Knowing that $\kappa_1 = \kappa_2 = 1$ and $\kappa_1 + \kappa_3 = 2 + \kappa_4$ leads to $\kappa_3 = 1 + \kappa_4 > 1$, a contradiction because $1 > \kappa_3$ in a 2r-tetrahedron. This exhausts the last case and concludes the proof. □

For a 3r-tetrahedron, unfoldings based on a trident cut tree generally lead to two reflex vertices, rather than just one, so the strategy just used in Lemma 6 cannot be followed. Instead we will use a path cut tree, but this in general could lead to two or three reflex vertices in the hexagon. We will show, however, that there is a path cut tree that (a) leads to just one reflex vertex, and (b) such that some convex vertex of the hexagon fits inside the exterior angle at that reflex vertex.

For a 3r-tetrahedron $\mathcal{T}$, let the three reflex corners $\triangle v_2v_3v_4$ form the base of $\mathcal{T}$, and view the convex corner v_1 as the apex above the plane containing the base. We are going to cut $\mathcal{T}$ via a path starting at v_1 and then curling around two edges of $\triangle v_2v_3v_4$. We want to select the path so that both internal base corners v_i on the path split the total angle there into two convex pieces. Because the angle at the base is always convex, we first aim to show that the sum of the other two face angles incident to one of the v_i is at most 1, i.e., convex. Later we will tackle the other intermediate vertex.

Let α_i' be the portion of the face angle α_i incident to v_i that excludes the base angle.

Lemma 7 *With the labeling conventions just described, for at least one of the base angles $\{a_2', a_3', a_4'\}$, we must have $\alpha_i' \leq 1$.*

Proof: If the claim of the lemma fails to hold, then $\alpha_2' + \alpha_3' + \alpha_4' > 3$. Because the total angle in $\triangle v_2v_3v_4$ is 1, we have $\alpha_2 + \alpha_3 + \alpha_4 > 4$. Replacing α_i with $2 - \kappa_i$ leads to $\kappa_2 + \kappa_3 + \kappa_4 < 2$, which, by Gauss-Bonnet implies that $\kappa_1 > 2$. This contradiction establishes the lemma. □

Lemma 8 *Every 3r-tetrahedron $\mathcal{T}$ may be refolded to an incongruent $\mathcal{P}'$.*

Proof: Recall that only v_1 is convex. We will now let v_2 be the base corner of T that satisfies Lemma 7. (Note this means that no longer do the corner labels correspond to size of reflexivity, in our classification listing earlier.) Cut open $\mathcal{T}$ via a path that starts at v_1, and which passes through v_2. So the path is either (v_1, v_3, v_2, v_4) or (v_1, v_4, v_2, v_3). So the other leaf is either at v_3 or v_4. Because of the choice of v_2, we know the cut through v_2 results in two convex image vertices of the hexagon P. Now we turn to the other internal corner of the cut path, which is either v_4 or v_3. The goal is to show that one of these also results in two convex image vertices of P.

Let β_3, β_4 be the angles of the base face $\triangle v_2 v_3 v_4$ at v_3 and v_4 respectively. Let γ_3, γ_4 be angles of the lateral face $\triangle v_1 v_3 v_4$ at v_3 and v_4 respectively. See Fig. 9. Both of the two possible cut paths are thwarted from splitting the angles

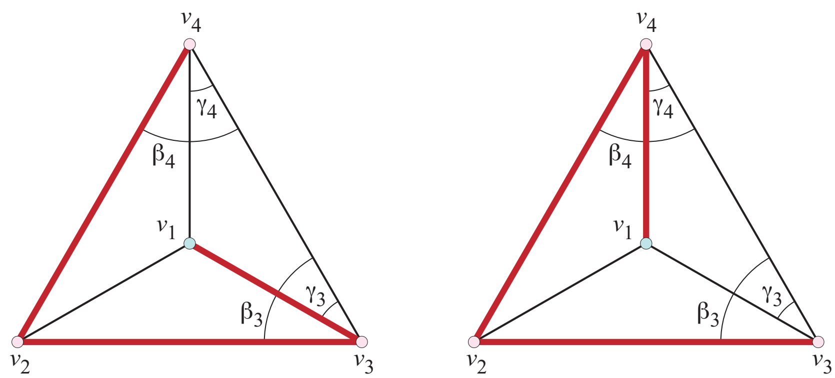

Figure 9: The two cut paths from v_1 passing through v_2.

at v_3 and v_4 into convex-convex images of P if both

$$\beta_3 + \gamma_3 > 1 \text{ and } \beta_4 + \gamma_4 > 1 ,$$

for then either alternative leads to a reflex vertex of P. But note first that $\beta_3 + \beta_4 < 1$ because they are two angles of a triangle, and for the same reason $\gamma_3 + \gamma_4 < 1$. If thwarted, then

$$(\beta_3 + \beta_4) + (\gamma_3 + \gamma_4) > 2 ,$$

a contradiction to the triangle relationships. Therefore one of the two cut paths leads to a hexagon with just one reflex vertex, that made by the leaf node. Henceforth let the leaf corner be v_4; so the cut path is (v_1, v_3, v_2, v_4).

Because each of the three reflex corners satisfies $\kappa_i < 1$, a small κ_4 implies a large κ_1. For example, if $\kappa_4 < \frac{1}{2}$, then $\kappa_2 + \kappa_3 + \kappa_4 < \frac{5}{2}$ and so $\kappa_1 > \frac{3}{2}$. More precisely, from

$$\kappa_1 + (\kappa_2 + \kappa_3 + \kappa_4) = 4$$

we obtain $\kappa_1 > 2 - \kappa_4$. Thus, the smaller the exterior angle κ_4 at the reflex vertex, the sharper must be corner v_1. Indeed, recalling that $\kappa_1 = 2 - \alpha_1$, we see that $\kappa_4 > \alpha_1$: the exterior angle is larger than the sum of the face angles incident to v_1. But then the convex vertex of P that is the image of the leaf at v_1 fits inside the reflex angle. Now we argue that it is not possible for $\mathcal{P}'$ under this gluing to be congruent to $\mathcal{T}$.

Generically, $\mathcal{P}'$ has seven vertices, so if $\mathcal{P}' = \mathcal{T}$, there must be several simultaneous degeneracies. Let us fix, without loss of generality, the cut path (v_1, v_3, v_2, v_4), the left path in Fig. 9. As can be seen in Fig. 10, even with

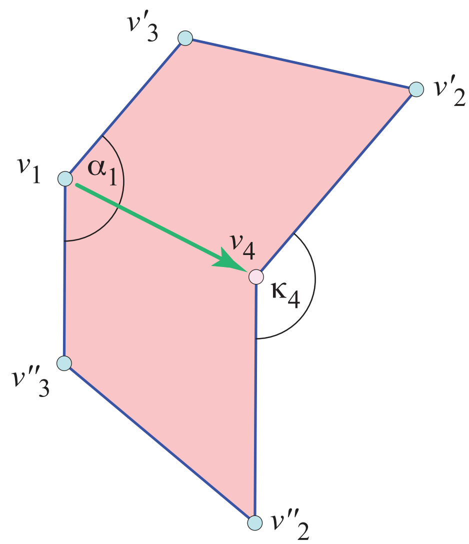

Figure 10: The hexagon P resulting from the cut path (v_1, v_3, v_2, v_4).

degeneracies, each half of the hexagon (on either side of the $v_1 \to v_4$ gluing) necessarily produces at least two vertices, because even if, e.g., v_2' glues to v_3', some curvature remains there because both are convex vertices of P. So it must be that $\kappa_4 - \alpha_1 = 0$ to avoid $\mathcal{P}'$ having at least five vertices; so $\kappa_1 + \kappa_4 = 2$, and therefore $\kappa_2 + \kappa_3 = 2$. But $\kappa_2 < 1$ and $\kappa_3 < 1$ because both are reflex corners, a contradiction. □

Theorem 3 is therefore established by Lemmas 4, 5, 6, and 8, covering the four exhaustive classes of tetrahedra.

6 Open Problems

Our work so far just scratches the surface of a potentially rich topic. Here we list some questions suggested by our investigations.

1. Star unfoldings (e.g., [DO07, Sec. 24.3]) are natural candidates for rigidity. Is it the case that almost every star unfolding of (almost?) every polyhedron is refold-rigid?

2. Which (if either) of the following is true? (a) Almost all polyhedra are edge-unfold rigid. (b) Almost all polyhedra are edge-unfold transformers. Our evidence so far does not clearly support either claim.

3. Characterize the polygons P that can fold in two different ways (have two different Alexandrov gluings) to produce the exact same polyhedron $\mathcal{P}$. We have only sporadic examples of this phenomenon (among the foldings of the Latin cross). Understanding this more fully would permit simplifying the ad hoc arguments that $\mathcal{P}' \neq \mathcal{T}$ in Lemmas 6 and 8.

4. Do our transformer results extend to the situation where the unfoldings are required to avoid overlap? We can extend Lemma 2 to ensure non-overlap, but extending Lemma 3 seems more difficult.

5. One could view an edge-unfold and refold operation as a directed edge between two polyhedra in the space of all convex polyhedra. Theorem 2 and Corollary 2 show neighbors of some (semi-)regular polyhedra have no outgoing edges. What is the connected component structure of this space?

6. A more localized version of the preceding question is to understand the structure of the set of all polyhedra that can be formed from one particular $\mathcal{P}$ by unfolding and refolding. Theorem 1 shows the set has cardinality larger than 1.

7. Various optimization questions suggest themselves, all of which, unfortunately, seem difficult. One suggested by a talk attendee is this natural question: Which refolding of any of the 11 edge-unfoldings of the cube achieves the maximum volume? In particular, is it the cube itself?

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