Quasigeodesics on the Cube

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Abstract

A quasigeodesic is a curve on the surface of a convex polyhedron that has $\leq \pi$ surface to each side at every point. In contrast, a geodesic has exactly π to each side and so can never pass through a vertex, whereas quasigeodesics can. Although it is known that every convex polyhedron has at least three simple closed quasigeodesics, little else is known. Only tetrahedra have been thoroughly studied.

In this paper we explore the quasigeodesics on a cube, which have not been previously enumerated. We prove that the cube has exactly 15 simple closed quasigeodesics (beyond the three known simple closed geodesics). For the lower bound we detail 15 simple closed quasigeodesics. Our main contribution is establishing a matching upper bound. For general convex polyhedra, there is no known upper bound.

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1 Introduction

1.1 Quasigeodesics

A ***quasigeodesic*** is a curve on the surface of a convex polyhedron that has $\leq \pi$ surface to each side at every point. In contrast, a ***geodesic*** has exactly π to each side. Because a vertex is a point with less than 2π surface, a geodesic can never pass through a vertex. Quasigeodesics can pass through vertices.

Ever since since Poincaré's investigations more than a century ago, closed geodesics have played an important role in the topology of Riemannian manifolds [Ber03, p. 433]. It is a famous 1929 theorem of Lyusternik-Schnirelmann that every smooth genus-0 surface has at least three simple (non-self-intersecting) closed geodesics [LS29]. Pogorelov proved in 1949 a natural analog: Every convex surface has at least three simple closed quasigeodesics [Pog49]. Pogorelov's existence proof does not suggest a way to identify the three quasigeodesics, and it is only recently that finite algorithms have been proposed [DHK20] [CdM24].

Aside from these algorithms, simple closed quasigeodesics have only been systematically studied on tetrahedra. Two results in [OV22] are: (1) On any tetrahedron, there is at least one 1-vertex, one 2-vertex, and one 3-vertex simple closed quasigeodesic. (2) There is an infinite class of tetrahedra that each have at least 34 simple closed quasigeodesics. In contrast to (1), it is known from [DDTY17] that the cube does not have a 1-vertex simple closed quasigeodesic.

Simple closed quasigeodesics play central roles in [HLM+22] and [OV24], and are of interest in their own right. But beyond their existence, much remains unknown. There is no known upper bound on the number of simple closed quasigeodesics on a given polyhedron, and there is an n-vertex polyhedron with $2^{\Omega(n)}$ distinct simple closed quasigeodesics [DO07, Sec. 24.4]. In contrast, it is known that isosceles tetrahedra[1] have arbitrarily long "spiraling" simple closed geodesics [Pro07] [AP18].

In this paper we make a complete inventory of simple closed quasigeodesics on a cube. It was known that there are precisely three simple closed geodesics on the cube. We identify a further 15 simple closed quasigeodesics (up to symmetries), and prove that this list is complete. We consider this proof to be our most significant contribution.

1.2 Three simple closed geodesics

To describe geodesics and quasigeodesics explicitly, we adopt the notation for faces and vertices displayed in Fig. 1. Note that we label vertices in figures by their index i, but refer to them in the text as v_i.

It has long been known that there are precisely three simple closed geodesics on the cube [FF07], displayed in Fig. 2.[2] Note that each of the three geodesics

[1] Also called disphenoids, tetramonohedra, isotetrahedra, and several other names. All faces are congruent acute triangles.

[2] Note these three are not the three from Pogorelov's theorem.

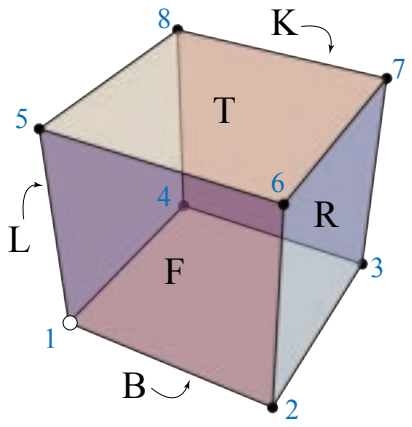

Figure 1: F,R,T,K,L,B = Front, Right, Top, bacK, Left, Bottom. B vertices indexed $1, 2, 3, 4$; T vertices indexed $5, 6, 7, 8$. v_1 is marked white.

can slide within a range, maintaining parallelism. This is because each geodesic lies on a cylinder, with 2π curvature (four vertices, each with $\pi/2$ curvature) to each side.

2 Outline of Argument

We mentioned that simple closed geodesics can spiral around isosceles tetrahedra. A simple closed quasigeodesic also may spiral around other convex polyhedra, as shown in Fig. 3 below. A central aspect of our proof is to show that quasigeodesics cannot spiral on a cube.

Define a ***geodesic segment*** as a non-self-intersecting vertex-to-vertex geodesic.[3] A simple closed quasigeodesic is composed of a sequence of geodesic segments, satisfying the $\leq \pi$ condition to both sides at each vertex.

An instructive example was identified in [DHK20]: a long box with a spiraling simple closed quasigeodesic. See Fig. 3. Each of the four marked vertices has π angle to one side and $\pi/2$ to the other side. Since there is freedom to partition the $3\pi/2$ surface angle differently (while maintaining $\leq \pi$ to each side), the number of spiraling simple closed quasigeodesics of a long box grows with the length of the long side of the box. A crucial property of spiraling is that some geodesic segment re-enters its initial face. For example, the blue geodesic segment from v_1 to v_2 in the figure starts on the long front-side face and later re-enters that face. We will prove that this cannot happen on a cube: a geodesic segment cannot return to its initial face, and in fact, cannot cross any face more than once.

[3] In some literature, a geodesic segment is a shortest path between its endpoints. In this paper, our geodesic segments may or may not be shortest.

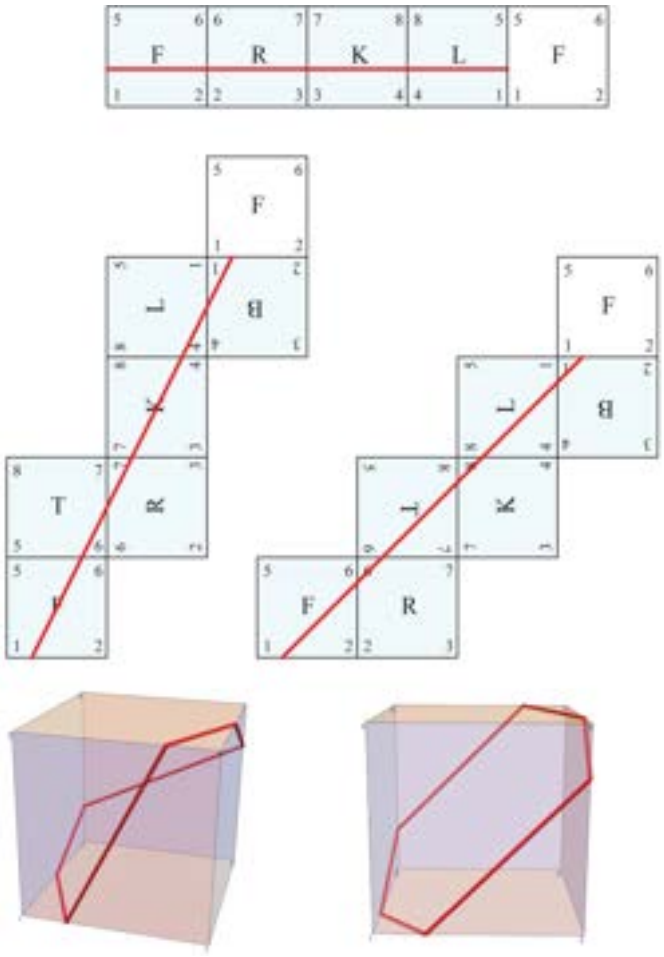

Figure 2: The three simple closed geodesics on a cube. The first is an equatorial band. The other two are as depicted.

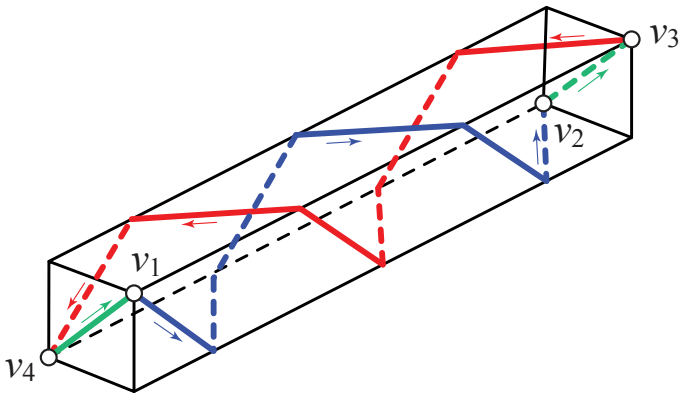

Figure 3: $(v_1, v_2, v_3, v_3, v_4)$ is a simple closed quasigeodesic. Based on Fig. 2 in [DHK20].

3 Fifteen Simple Closed Quasigeodesics

Here is our main result:

Theorem 1 *There are exactly* 15 *simple closed quasigeodesics on the cube (beyond the three simple closed geodesics noted above). These are displayed in Fig. 4 and described in Table 1.*

As our sole focus in the remainder is on "simple closed quasigeodesics," we often simplify that term to ***quasigeos***.

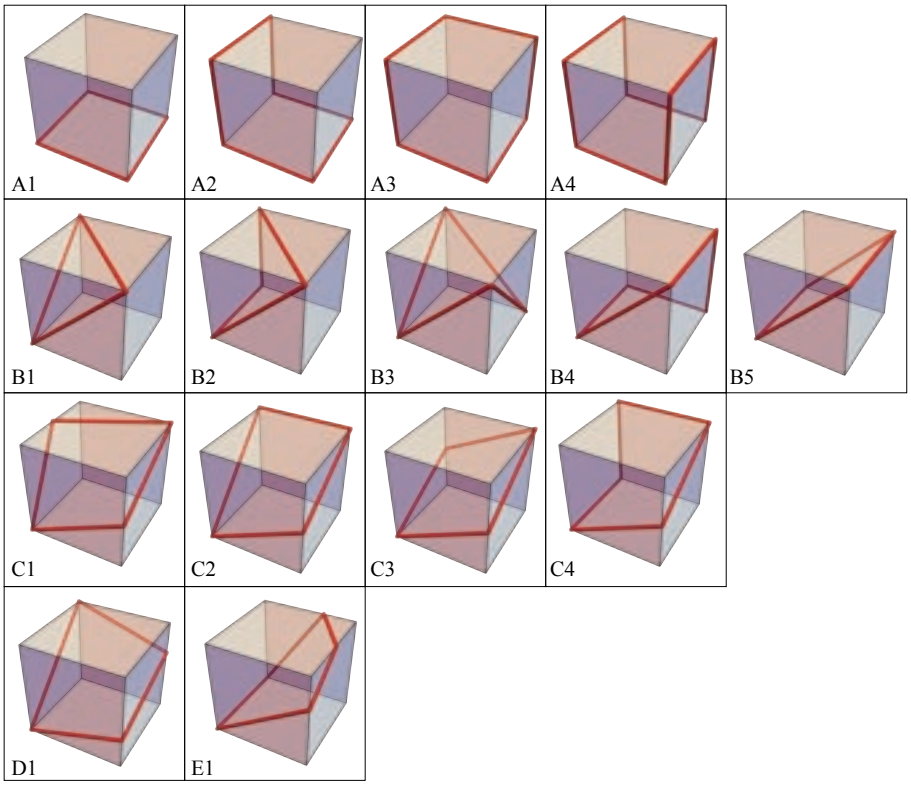

Figure 4: The 15 simple closed quasigeodesics.

A_i:	Four quasigeos using only 0/1 segments (each of length 1).
B_i:	Five quasigeos using at least one 1/1 segment (length $\sqrt{2}$), and none longer.
C_i:	Four quasigeos using at least one 1/2 segment (length $\sqrt{5}$), and none longer.
D_1:	One quasigeo using a single 1/3 (length $\sqrt{10}$), and none longer.
E_1:	One quasigeo using a single 2/3 (length $\sqrt{13}$).

Table 1: Description of the five categories of quasigeos.

The quasigeos are listed in order of the length of the geodesic segments comprising them, as described in Table 4. We identify a geodesic segment by its slope y/x, i.e., vertically up y units and rightward horizontally x units within the natural coordinate system of its starting face.

4 Five Slopes

Our approach is to analyze a geodesic segment based on the angle α it makes in its starting face. Consider a geodesic segment that does not follow an edge of

the cube. Then it enters the interior of a face and makes an angle in the range $(0, \pi/4]$ with one edge of the face. We express this as a slope in the range $(0, 1]$. We first rule out some slopes in this range because the geodesic segment revisits the first face and intersects itself there. We rule out further slopes by finding intersections between two geodesic segments. This reduces the possible slopes to a finite set, which allows a combinatorial enumeration of all simple closed quasigeodesics.

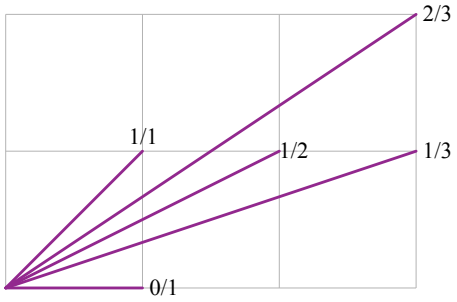

Figure 5: The five possible distinct slopes.

Lemma 1 *A geodesic segment that is a component of a simple closed quasigeodesic on the cube can only have one of the five slopes shown in Fig. 5:* $0/1, 1/3, 1/2, 2/3, 1/1$.

Corollary 1 *A geodesic segment that is a component of a simple closed quasigeodesic on the cube does not cross any face more than once.*

We prove the lemma by partitioning the rest of the slope range $(0, 1]$ into the following seven ranges:

- Case 1. $(0/1, 1/4]$
- Case 2. $(1/4, 1/3)$
- Case 3. $(1/3, 2/5)$
- Case 4. $[2/5, 1/2)$
- Case 5. $(1/2, 2/3)$
- Case 6. $(2/3, 3/4)$
- Case 7. $[3/4, 1/1)$

Fig. 6 shows the seven cases, and Fig. 7 shows how each range progresses on the unfolded surface of the cube. Each case has a (pink) ***F-cone*** with angle θ at v_1. From Fig. 7 we immediately obtain:

Claim 1 *No geodesic segment is possible in Cases 2, 3, and 6 because the segment revisits the starting face and intersects itself there. (We note that the crossing is at right angles, a known constraint [FF07].)*

The remaining four cases are possible for a single geodesic segment, but not for a geodesic segment that is part of a quasigeo.

Claim 2 *Consider a geodesic segment g that is a component of a simple closed quasigeodesic on the cube, and that falls into Case 1, 4, 5, or 7. Then g intersects another segment of the quasigeodesic.*

Proof: We find an intersection point by following the quasigeo backwards from v_1, the starting vertex of g. Let g' be the geodesic segment before g. We trace g' backwards from its terminus at v_1.

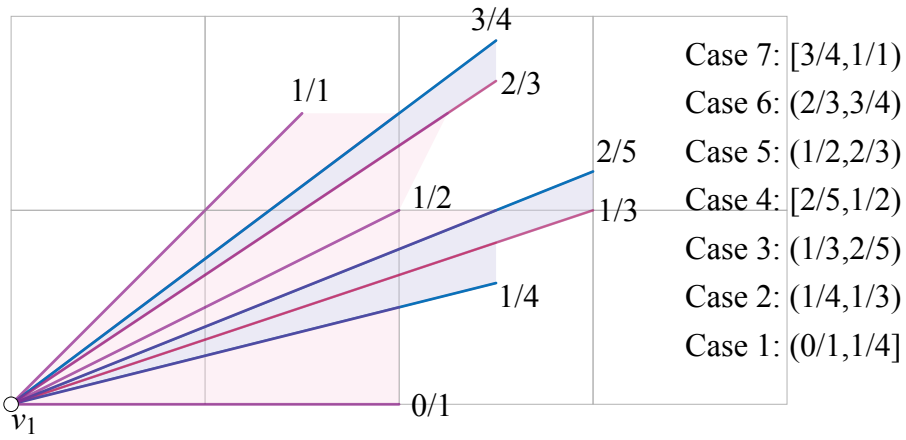

Figure 6: The seven slope ranges. Cases 2, 3, and 6 (in blue) are ruled out in Claim 1, and Cases 1, 4, 5, and 7 (in pink) are ruled out in Claim 2, leaving only the five slopes (in purple) allowed in Lemma 1.

Case 1. We focus on Case 1 in Fig. 9. The 2D unfolding of that case is shown on the 3D cube in Fig. 8. Cases 4, 5 and 7 will follow the same general scheme as does Case 1.

View g as directed crossing faces F_1, R, K, L in order. In Case 1, g has slope in $(0/1, 1/4]$ and lies within the pink F-cone of angle θ as illustrated. We now show that g cannot be part of a quasigeo, by analyzing the possibilities for the previous geodesic segment g'.

Because the angle between g and g' at v_1 must be $\leq \pi$, g' must leave v_1 in a $\theta + \pi/2$ cone that extends counterclockwise from edge v_1v_5. This cone is open along edge v_1v_5 and closed on its other boundary. See vertex v_1 in Face F_2 in the figure. We partition the cone into three possibilities:

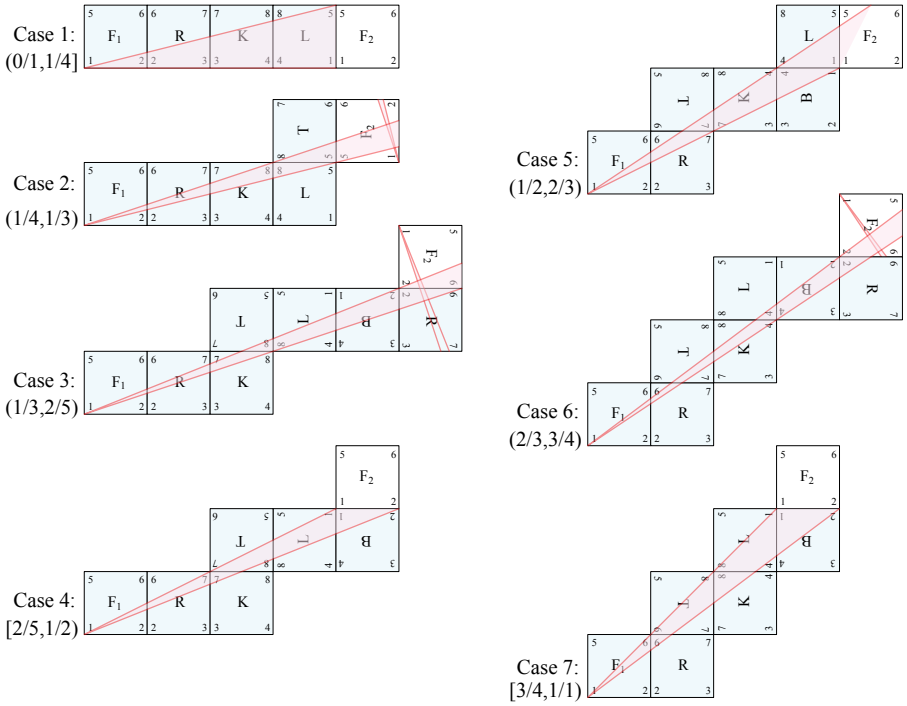

Figure 7: The seven slope cases, showing the range of slopes (in pink) progressing across the faces of the cube. The geodesic segment starts in face F_1 and revisits the starting face, marked F_2 (in white).

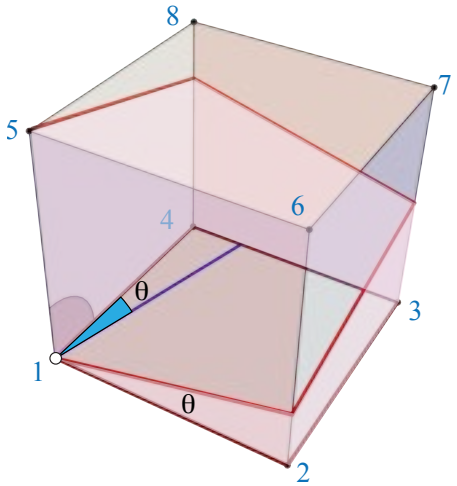

Figure 8: The Case 1 F-cone in 3D. $\theta = \arctan(1/4)$. Cf. Fig. 9.

(1) g' lies strictly within the quarter-circle on face L at v_1 (counterclockwise between edges v_1v_5 and v_1v_4). Then g' crosses g no matter where g and g' lie in their respective cones.

(2) g' lies in the cone of angle θ counterclockwise of edge v_1v_4. This cone (colored blue in Fig. 8) is open along the edge v_1v_4 and closed on its other boundary. Then g' wraps clockwise around v_4 by $\pi/2$, and crosses g in face K.

(3) g' follows the edge v_1v_4. Then g' hits vertex v_4 and ends there. Let g'' be the next geodesic segment. Then g'' leaves v_4 in face K in the closed quarter-circle bounded by edges v_4v_8 and v_4v_3. Any g'' in this cone intersects g unless g'' follows the edge v_4v_3. Repeating this argument, we either find an intersection with g, or we eventually follow the edge v_2v_1—but then the angle with g at v_1 is too sharp for a quasigeodesic.

So we obtain a quasigeo violation for every g inside or on the upper boundary of the F-cone in Case 1.

The argument for the remaining cases proceeds similarly, presented below somewhat more concisely.

Case 4. Again the F-cone has angle θ at v_1 in F_1, and g' must leave v_1 at F_2 in a $\theta + \pi/2$ cone.

(1) g' lies strictly within the quarter-circle on faces B and L. Then g' crosses g no matter where they lie in their cones.

(2) g' lies in the cone of angle θ strictly clockwise of the upper boundary of the F-cone. Then g' wraps counterclockwise about v_7 and crosses g in face R.

(3) g' follows the upper F-cone edge (slope 1/2). Then g' hits v_7. As in Case 1, repeating the argument, the next geodesic segment g'' leaves the quarter-circle similarly anchored on v_7 and either crosses g in the F-cone, or hits v_1 at an angle too sharp for a quasigeodesic.

Case 5.

(1) g' lies strictly within the quarter-circle on faces B and L. Then g' crosses g no matter where they lie in their cones.

(2) g' lies in the cone of angle θ strictly counterclockwise of the lower boundary of the F-cone. Then g' wraps clockwise about v_7 and crosses g in face R or T.

(3) g' follows the lower F-cone edge (slope 1/2). Then g' hits v_7. Repeating the arguments of the previous cases, the next geodesic segment g'' leaves the quarter-circle anchored on v_7 and either crosses g in the F-cone, or hits v_1 at an angle too sharp for a quasigeodesic.

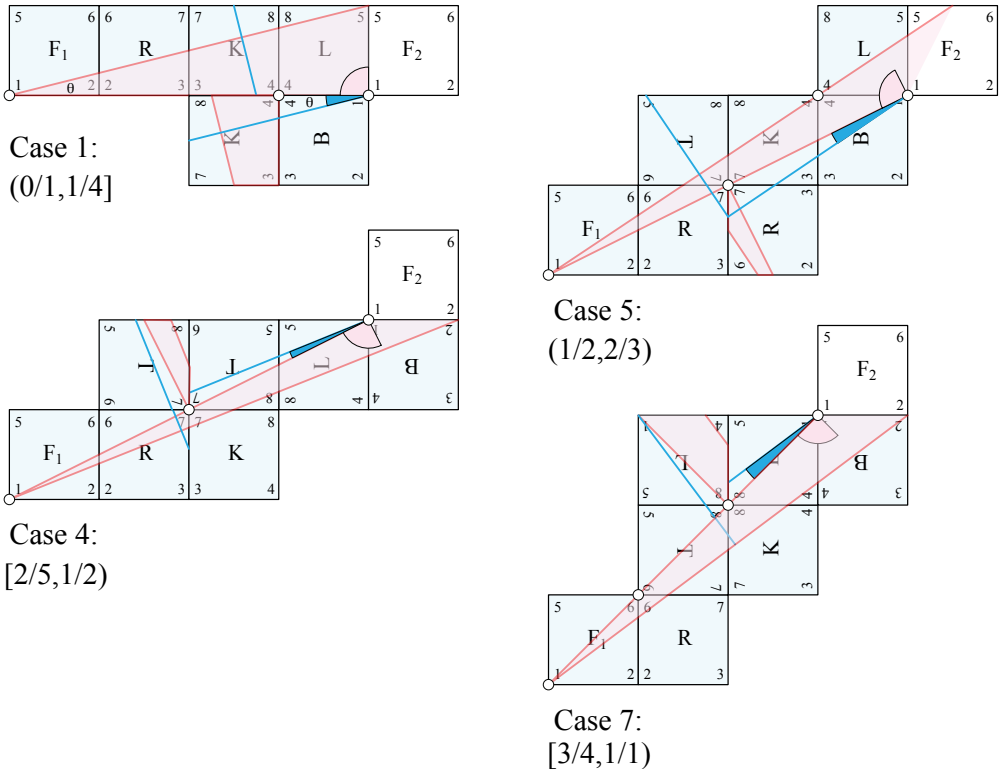

Figure 9: The geodesic segment g in the F-cone is crossed by g', either if starting backwards in the quarter-circle, or starting as much as θ beyond (blue angle and segment.), where θ is the F-cone angle at v_1.

Case 7.

(1) g' lies strictly within the quarter-circle on faces B and L. Then g' crosses g no matter where they lie in their cones.

(2) g' lies in the cone of angle θ strictly clockwise of the upper boundary of the F-cone. Then g' wraps counterclockwise about v_8 and crosses g in face L or K.

(3) g' follows the upper F-cone edge (slope 1/1). Then g' hits v_8. We repeat the previous arguments. The next geodesic segment g'' leaves the quarter-circle anchored on v_8 and either crosses g in the F-cone, or hits v_6. Applying the argument again, the next geodesic segment g''' either crosses g or or hits v_1 at an angle too sharp for a quasigeodesic.

This completes the proof of Claim 2. □

Claims 1 and 2 establish that, of the seven cases filling the entire range of slopes (Fig. 6), all but the five identified slopes are impossible, and so prove Lemma 1.

5 Search for Quasigeos

We initially found the 15 quasigeos in Fig. 4 "by hand." To establish that there are no other possibilities, we programmed an exhaustive search based on Lemma 1. We chose to use a DFS search, starting with the longest geodesic segments first, because they maximize pruning. Ordered by lengths, the slopes are $2/3 > 1/3 > 1/2 > 1/1 > 0/1$. Examples of pruning are shown in Fig. 10.

The DFS found 29 quasigeos, and after eliminating the duplicates congruent by a symmetry, exactly the 15 in Fig. 4 remain.[4]

Recall Corollary 1 established that no single geodesic segment part of a cube quasigeo can cross a face more than once. This contrasts with the long box example, Fig. 3. A consequence of the inventory of the 15 quasigeos is that no cube quasigeo can cross a face more than once.

6 Discussion and Open Problems

We have proved Theorem 1 by verifying that the list in Fig. 4 is exhaustive. Below we list several open questions.

- Is there a finite upper bound to the number of simple closed quasigeodesics (that are not geodesics) on a given nondegenerate polyhedron of n vertices? There is no such bound for simple closed geodesics. Nor is there a bound for (degenerate) doubly-covered squares: see Fig. 11.

[4] We have not made our code available, but it is an easy programming exercise to verify our exhaustive search.

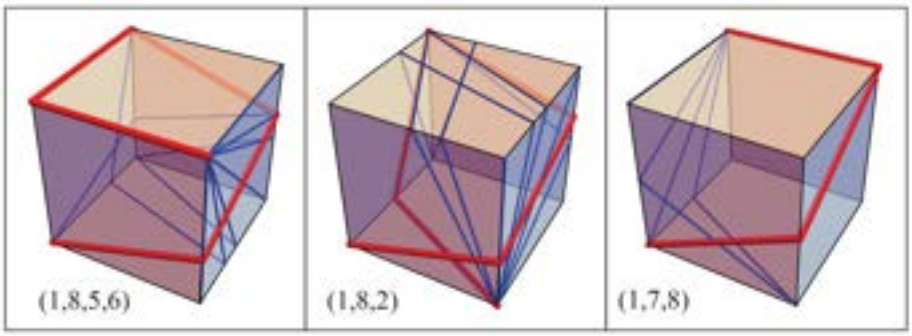

Figure 10: Red: Partial quasigeo, through vertices listed. Blue: All possible next segments based on angle with the previous segment.

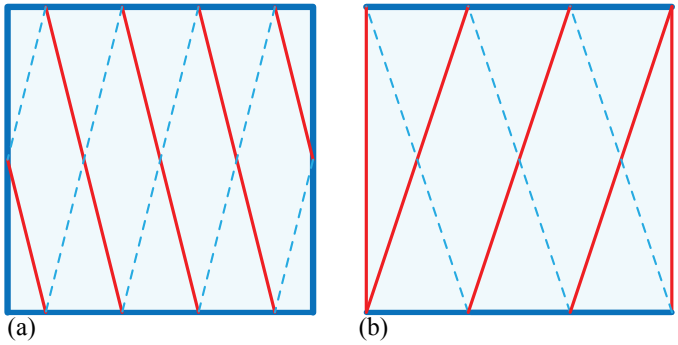

Figure 11: Doubly-covered square. Red segments: front. Blue segments: back. (a) Simple closed geodesic. (b) Simple closed quasigeodesic.

- It was proved in [OV22] that every tetrahedron has a simple closed geodesic or a 1-vertex simple closed quasigeodesic. That the same holds for any convex polyhedron was conjectured in [OV24]. As mentioned, it is known from [DDTY17] that the cube does not have a 1-vertex simple closed quasigeodesic, but it does have simple closed geodesics, so the cube accords with the conjecture. Settling the conjecture either way seems currently out of reach.

- A slightly non-cubical box, $1 \times 1 \times h$ for $h \in (1, 2)$, has a "diamond" 1-vertex simple closed quasigeodesic: see Fig. 12. Characterizing all simple

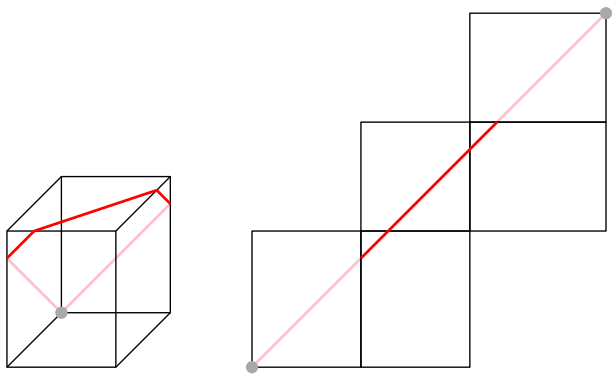

Figure 12: A 1-vertex quasigeo on a $1 \times 1 \times 1\frac{1}{4}$ box.

closed quasigeodesics on boxes is a natural next step.

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