

# Convex Polyhedra Realizing Given Face Areas

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## Abstract

Given  $n \geq 4$  positive real numbers, we prove in this note that they are the face areas of a convex polyhedron if and only if the largest number is not more than the sum of the others.

## 1 Introduction

Let  $\mathcal{A} = (A_1, A_2, \dots, A_n)$  be a vector of  $n$  positive real numbers sorted so that  $A_i \geq A_{i+1}$ . The question we address is in this note is:

When does  $\mathcal{A}$  represent the face areas of a convex polyhedron in  $\mathbb{R}^3$ ?

For example, suppose  $\mathcal{A} = (100, 1, 1, 1)$ . It is clear there is no tetrahedron realizing these areas, because the face of area 100 is too large to be “covered” by the three faces of area 1. So  $A_1 \leq \sum_{i>1} A_i$  is an obvious necessary condition. The main result of this note is that this is also a sufficient condition. Enroute to establishing this we connect the question to robot arm linkages and to 3D polygons.

The main tool we use is Minkowski’s 1911 theorem. Here is a version from Alexandrov, who devotes an entire chapter to the theorem and variations in his book [Ale05, Chap. 7, p. 311ff].

**Theorem 1 (Minkowski (a))** *Let  $A_i$  be positive faces areas and  $n_i$  distinct, noncoplanar unit face normals,  $i = 1, \dots, n$ . Then if  $\sum_i A_i n_i = 0$ , there is a closed polyhedron whose faces areas uniquely realize those areas and normals.*

Here, uniqueness is up to translation.

In our situation, we are given the areas  $A_i$ , and the task is to determine if there exist normals  $n_i$  that satisfy Minkowski’s theorem. Although this superficially may seem like a complex problem, we will see it has a simple solution. Although I have not been able to find this result in the literature, it seems likely that it is known, because the proof is not difficult.

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It will be more convenient for our purposes to follow Grünbaum’s (equivalent) formulation of Minkowski’s theorem [Grü03, p. 332] phrased in terms of “fully equilibrated” vectors. Vectors are *equilibrated* if they sum to zero and no two are positively proportional. They are *fully equilibrated* in  $\mathbb{R}^k$  if they in addition span  $\mathbb{R}^k$ .

**Theorem 1 (Minkowski (b))** *Let  $v_i = A_i n_i$  be vectors whose lengths are  $A_i$ ,  $|v_i| = A_i$ , and whose directions are unit normal vectors  $n_i$ ,  $i = 1, \dots, n$ . Then if the vectors are fully equilibrated in  $\mathbb{R}^3$ , there is a unique closed polyhedron  $P$  with faces areas  $A_i$  and normal vectors  $n_i$ .*

Note that for  $n$  vectors to be fully equilibrated in  $\mathbb{R}^3$ ,  $n$  must be at least 4: It requires 3 vectors to span  $\mathbb{R}^3$ , but any three vectors that sum to zero form a triangle and so lie in a plane. Thus 3 vectors cannot both be equilibrated and span  $\mathbb{R}^3$ . Four clearly suffices:  $P$  is a tetrahedron.

## 2 Main Result & Proof

**Theorem 2** *If  $A_1, A_2, \dots, A_n$  are positive real numbers with  $n \geq 4$  and  $A_i \geq A_{i+1}$ , then there is a closed convex polyhedron  $P$  with these  $A_i$  as its face areas if and only if  $A_1 \leq \sum_{i>1} A_i$ . When equality holds, we permit the polyhedron to be flat, i.e., its faces tessellate and doubly cover a planar convex polygon.*

### 2.1 Flat Polyhedra

Let  $F_i$  be the face whose area is  $A_i$ . As previously mentioned, the condition is necessary, because if  $A_1 > \sum_{i>1} A_i$ , then the  $F_1$  face cannot be covered by all the others, so it is not possible to form a closed surface. In the case of equality,  $A_1 = \sum_{i>1} A_i$ , the areas are realized by a flat polyhedron that can be constructed as follows. For  $F_1$ , select a square of side length  $\sqrt{A_1}$ . (Any other convex polygon would serve as well.) This becomes one side of  $P$ . For the other side, partition  $F_1$  into strips of width  $A_i/\sqrt{A_1}$ , so that each strip has area  $A_i$ , and serves as  $F_i$ .

Henceforth assume  $A_1$  is strictly smaller than the sum of the other areas. In this circumstance we can always obtain a non-flat polyhedron  $P$ , with none of the faces  $F_i$  coplanar.

### 2.2 Robot Arms

Let  $\mathcal{C}$  be a polygonal chain (sometimes called a “robot arm” or just “arm”) whose  $n$  links have lengths  $A_1, A_2, \dots, A_n$ . Then it is a corollary to a theorem of Hopcroft, Joseph, and Whitesides [HJW84] that the chain can close (i.e., the “hand” can touch the “shoulder”) iff the longest link is not longer than all the other links together. See, e.g., [O’R98, Thm. 8.6.3, p. 326] or [DO07, Thm. 5.1.2, p. 61]. The similarity to the statement of Theorem 2 should be evident. Our plan is to form a chain  $\mathcal{C}$  from fully equilibrated vectors, and

apply Minkowski’s theorem, Theorem 1(b). This can be accomplished in three stages: (1) arrange that the vectors sum to zero, (2) ensure that none is a positive multiple of another, and (3) ensure that they span  $\mathbb{R}^3$ . We know from the robot-arm theorem that (1) is achievable, but that theorem is an existence theorem.

Satisfying (1) can be accomplished with the “Two Kinks” theorem [O’R98, Thm. 8.6.5, p. 329] [DO07, Thm. 5.1.4, p. 62], which would result in the links arranged to form a triangle (and so summing to 0), with (in general) many vectors aligned. Although it is then not so difficult to break all the alignments and achieve (2), instead we opt for a method that achieves (1) and (2) simultaneously.

### 2.3 Cyclic Polygon

Lay out all the links in a straight line of length  $\sum_i A_i$ . View the links as inscribed in a circle of infinite radius. Now imagine shrinking the radius from  $R = \infty$  down toward  $R = 0$ , maintaining the chain inscribed at all times. Knowing from the Hopcroft et al. theorem that the chain can close, at some radius  $R$  it just barely closes up, and we have a cyclic polygon  $\mathcal{C}$ . See Figure 1.

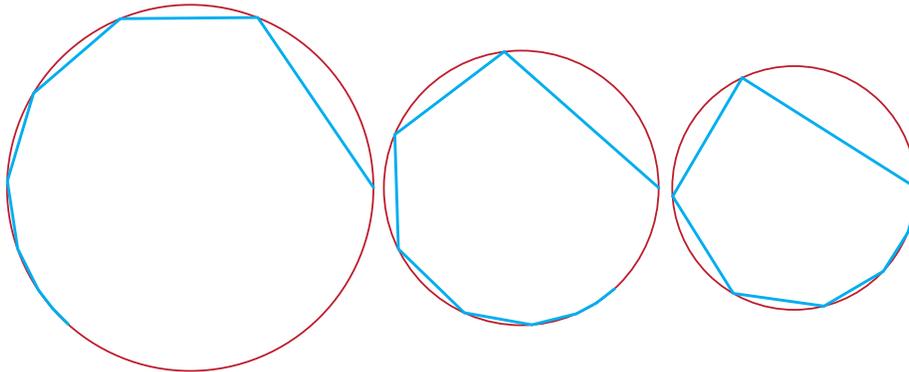


Figure 1: Links of lengths (9, 6, 5, 4, 3, 2, 1) inscribed in circles of radius  $R = 8$ ,  $R = 6$ , and closing at  $R \approx 5.325$ .

Let  $v_1, v_2, \dots, v_n$  be the vectors comprising this cyclic  $\mathcal{C}$ . They form a planar convex polygon connected head to tail, with no two vectors aligned. These vectors are therefore equilibrated, but they do not span  $\mathbb{R}^3$ , so they are not fully equilibrated in  $\mathbb{R}^3$  in Grünbaum’s terminology.

### 2.4 Spanning $\mathbb{R}^3$

Arranging the vectors to span  $\mathbb{R}^3$  is easily accomplished, in many ways. Here is one. Let  $a$  be the base of  $v_1$ , and select some  $k$  in  $(2, 3, \dots, n - 2)$  (so for  $n = 4$  we must have  $k = 2$ ). Let  $b$  be the head of  $v_k$  around the convex

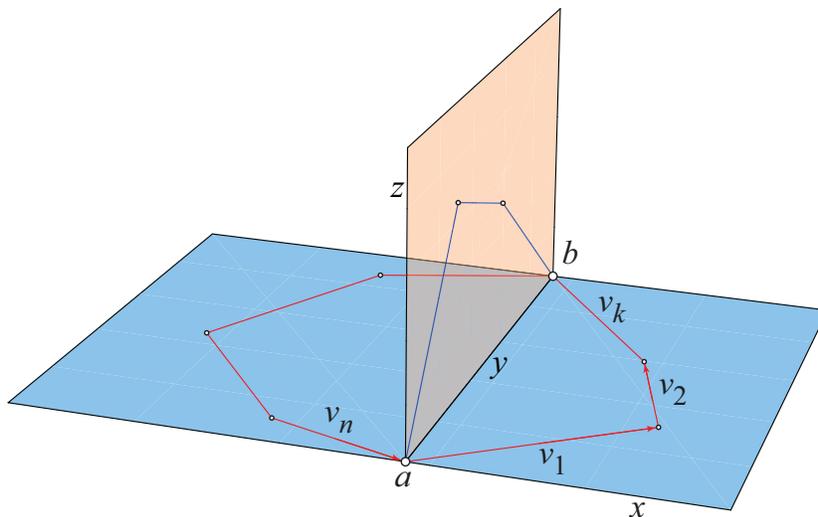


Figure 2: Rotating  $v_1, \dots, v_k$  to the  $yz$ -plane.

polygon. See Figure 2. Rotate the portion (we'll call it "half") of the convex polygon including  $v_1, \dots, v_k$  around the line through  $ab$  until that portion lies in a vertical plane, say, the  $yz$ -plane. Now half the vectors lie in this  $yz$ -plane, and the other half lie in the  $xy$ -plane. Each half contains at least two vectors by our choice of  $k$ . Thus the vectors in the  $xy$ -plane span that plane, and the vectors in the  $yz$ -plane span that plane. Consequently, together they span  $\mathbb{R}^3$ .

## 2.5 Proof Completion

Finally we may apply Minkowski's Theorem, Theorem 1(b) to conclude that there is a closed, convex polyhedron  $P$  whose face areas are the lengths of the vectors  $v_i$ ,  $|v_i| = A_i$ .

## 3 Discussion

1. Theorem 2 still holds for  $n = 2$ , when two given areas  $A_1 = A_2$  are realized by a flat two-face polyhedron. But it does not hold for  $n = 3$  (except in the case of equality,  $A_1 = A_2 + A_3$ , by a flat polyhedron). For  $n = 3$ , the theorem condition is the triangle inequality, and one obtains a triangle with side lengths the three given "areas." One could construct an infinite triangular prism with these face area ratios.
2. The value of  $R$  that closes  $\mathcal{C}$  to a cyclic polygon satisfies this equation, where each term in the sum is the angle at the circle center subtended by

$A_i$ :

$$\sum_i 2 \sin^{-1} \left( \frac{1}{2} A_i / R \right) = 2\pi \quad (1)$$

$R$  can easily be computed numerically; I cannot see a more direct computation.

3. An artifact of the method used to ensure the vectors span  $\mathbb{R}^3$  is that all the normal vectors of the polyhedron  $P$  lie in two orthogonal planes. The resulting polyhedra all have roughly the shape of the intersection of two half-cylinders, as depicted in Figure 3. Instead of spinning half the vectors

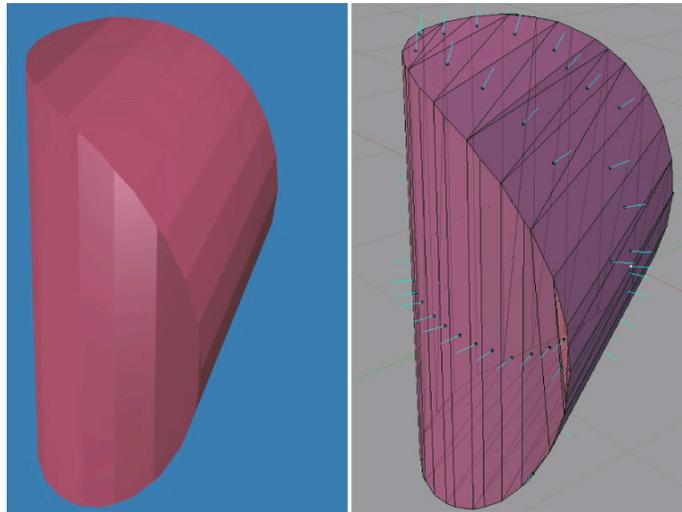


Figure 3: Intersection of two half-cylinders. Right image shows normal vectors.

about  $ab$ , we could spin each pair of successive vectors  $(v_i, v_{i+1})$  about the line containing their sum  $v_i + v_{i+1}$ . Choosing the spins independently would result in a more “balanced” collection of normals  $n_i$ .

4. Along these lines, it would be pleasing to have a natural candidate for a “canonical polyhedron” that realizes given areas.
5. In the 1980s, Little described a constrained optimization procedure that could compute the polyhedron guaranteed by Minkowski’s theorem [Lit85]. Unfortunately I know of no modern implementation.
6. Minkowski’s theorem generalizes to  $\mathbb{R}^d$ , for any  $d \geq 2$ . It is clear that, given facet volumes  $V_1, \dots, V_n$ , the analog of Theorem 2 holds. One only need ensure that the vectors span  $\mathbb{R}^d$  by choosing spins of adjacent vectors about “planes of rotation” that rotate the vectors to span the dimensions beyond  $\mathbb{R}^3$ . I will leave this as a claim requiring further work to justify precisely.

7. The “configuration space” (or “moduli space”) of all the polyhedra that realize a given list of areas is the same as the configuration space of a 3D polygon (closed polygonal chain) with those areas as edge lengths. This configuration space is known to be connected, by a result of Lenhart and Whitesides [LW95] (see also [DO07, Thm. 5.1.9, p. 67]). The space is well studied, e.g. [PT07], [Man08]. For a quadrilateral, the space is topologically a sphere  $S^2$ , but its structure is more complicated for arbitrary  $n$ -gons.

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