Edge-Unfolding Medial Axis Polyhedra

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Abstract

It is shown that a convex medial axis polyhedron has two distinct edge unfoldings: cuttings along edges that unfold the surface to a simple planar polygon. One of these unfoldings is a generalization of the point source unfolding, and is easily established to avoid overlap. The other is a novel unfolding that requires a more complex argument to establish nonoverlap, and might generalize.

1 Introduction

Medial Axis Polyhedron. Let P be a convex polygon. The *medial axis* $M = M(P)$, $M \subset P$ is the closure of the locus of the centers of disks in P, each of whose boundary touches ∂P in two or more points. The medial axis is a well-studied construct that applies much beyond convex polygons, but we restrict our attention here to convex P. Then, M is a tree of straight segments whose leaves are the vertices of P. To each point $m \in M$ may be associated the radius $r(m)$ of the maximal disk in P centered on m. Let P lie in the xy-plane, and for each $m \in M$, define a point $p(m) = (m_x, m_y, r(m))$: it is vertically above m at height $z = r(m)$. Finally, define the *medial axis polyhedron* $\mathcal{P}$ for P to be the convex hull of $P \cup \{p(m) : m \in M\}$. See Fig. 1 for an example that we will use throughout. Let $\mathcal{M}$ be the tree of edges of $\mathcal{P}$ that project to M.

The medial axis polyhedron is studied in [PW01, p. 376]. An alternative construction is to define a halfspace through each edge of P that makes an angle of $\pi/4$ with respect to the xy-plane containing P, and includes P. The intersection of these halfspaces with $z \geq 0$ yields $\mathcal{P}$. One property established in [PW01] (for arbitrary piecewise-C^2 closed curves, not just convex polygons) is that the surface over the base is developable, i.e., it can be "developed" without distortion flat to a plane. However, in general developable surfaces develop with overlap. Here we are explicitly seeking a nonoverlapping development via cuttings along edges.

Source Unfolding. The medial axis $M(P)$ is also known as the *cut locus* of ∂P: the closure of the locus of points with more than one distinct shortest path from ∂P. The points $\mathcal{M}$ on $\mathcal{P}$ have the

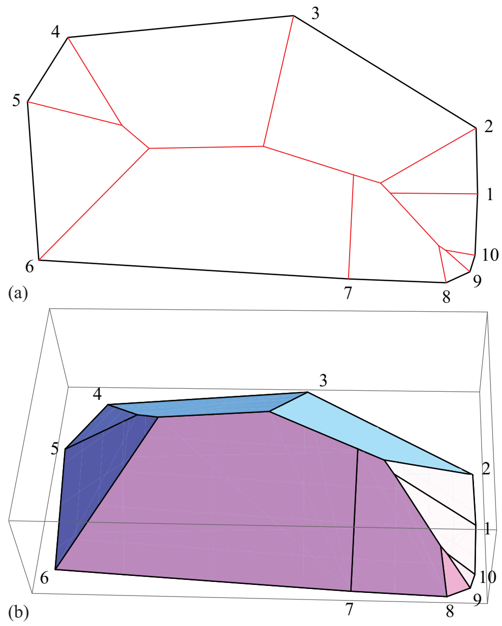

Figure 1: (a) A convex polygon P and its medial axis $M(P)$. (b) The corresponding medial axis polyhedron $\mathcal{P}$.

same property, and so form the cut locus of the base rim ∂P measuring shortest paths on the surface. It is well known that the cutting the cut locus of a "source" point x on a convex polyhedron unfolds the surface to a nonoverlapping unfolding, the *source unfolding* [DO07, p. 359]. Cutting $\mathcal{M}$ on a medial axis polyhedron $\mathcal{P}$ is cutting the cut locus of ∂P, and it is easy to see that this leads to a nonoverlapping unfolding for medial axis polyhedra. For each face f_i incident to a base edge e_i can be viewed as composed of shortest paths to $\mathcal{M}$, each path a segment perpendicular to e_i. Cutting $\mathcal{M}$ permits each face to flip out, rotating about e_i into the xy-plane. The perpendicularity of the shortest segments to e_i and the convexity of P easily guarantee nonoverlap of this unfolding. This is also a special case of a "dome unfolding," which was already known to avoid overlap [DO07, p. 322].

Convex Cap Unfolding. Of more interest is an unfolding that in some sense "squashes" the *convex cap* over P into the plane. Convex caps meet every line orthogonal to P in at most one point. They are an interesting special case to explore the long-unsolved problem of whether or not convex polyhedra always have an edge unfolding. One special case is studied

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in [O'R07]; the work here establishes another special case.

The research in [IOV07a] led to the conjecture[1] that cutting the cut locus of a simple, closed quasigeodesic leads to a nonoverlapping unfolding. As ∂P is such a quasigeodesic, unfolding two medial axis polyhedra glued base-to-base on the same P via the unfolding described in the next section establishes a (very) special case of this conjecture.

2 Unfolding

Unfolding Defined. Let $(v_1, \ldots, v_n)$ be both the 2D vertices of P and the corresponding 3D vertices of $\mathcal{P}$; the context will disambiguate. Let $e_i = v_i v_{i+1}$ be the edges of P (and $\mathcal{P}$), let f_i be the face of $\mathcal{P}$ incident to e_i, and let u_i be the edge of $\mathcal{P}$ incident to v_i and shared between f_{i-1} and f_i. The unfolding U of $\mathcal{P}$ we study is obtained by cutting every edge of $\mathcal{M}$ not incident to a leaf vertex v_i, and cutting u_1, the edge of $\mathcal{M}$ incident to v_1. We ignore the base P for now; it is easily attached later. U consists of the faces $f_1, f_2, \ldots, f_n$ glued together at the shared edges u_i in a sequence. (See ahead to Fig. 2.) We view ∂U as composed of two parts: the *outer shell* constituted by the edges e_i of P, and the *inner path* constituted by images of cut edges of $\mathcal{M}$. We continue to call the vertices of the outer shell $v_1, \ldots, v_n$, with v_1' the second image of v_1.

Let α_i be the angle of P at v_i, and β_i the sum of the two (equal) angles of $\mathcal{P}$ incident to v_i in faces f_{i-1} and f_i. Thus β_i is the angle at v_i in U.

Lemma 1 *The outer shell of ∂U is a convex curve.*

Proof. *Sketch.* Calculation shows that

$$\beta_i = 2\cos^{-1}\left(\frac{\sqrt{2}\cos(\alpha_i/2)}{\sqrt{3-\cos\alpha_i}}\right)$$

and that $\alpha_i < \beta_i < \pi$. $\square$

This ensures that P may be attached to U at any edge e_i and avoid overlap. Henceforth we concentrate on the nonoverlap of U.

Medial Axis Overlay. We close the outer shell of U into a convex region U^* by extending rays from v_2 through v_1, and from v_n through v_1'. If these rays do not meet, then U^* is unbounded. This indeed can occur (roughly, when α_1 is small), but the medial axis is easily defined for unbounded regions.

Define a *cell* of a medial axis $M(P)$ as one of the convex regions into which $M(P)$ partitions P, i.e., closures of the sets $P \setminus M(P)$. The key claim is the following:

[1]Made only in the presentation [IOV07b].

Theorem 2 *Each face f_i of U nests inside a cell of $M(U^*)$.*

We say f_i *nests* inside cell C_i if they share edge e_i and $f_i \subseteq C_i$. Because the cells of $M(U^*)$ partition U^*, this theorem implies nonoverlap of U. See Fig. 2.

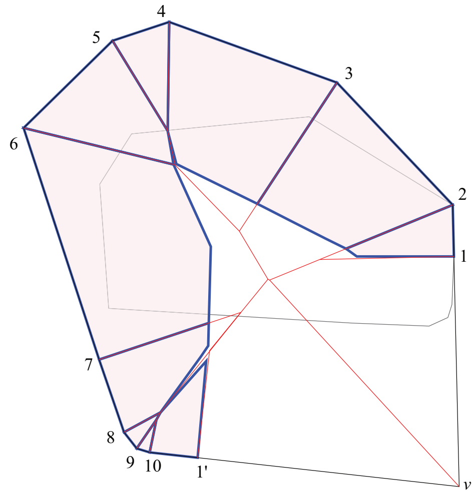

Figure 2: Unfolding U and polygon U^* for $\mathcal{P}$ in Fig. 1(b), overlaid with $M(U^*)$.

3 Inductive Construction

Our proof of Theorem 2 relies on the well-known inductive construction of the medial axis for a convex polygon. $M(P) = M(P_n)$ is constructed by extending a pair of edges e_{i-1} and e_{i+1} to meet at $v_{i,j}$ and "engulf" e_i to create a superset polygon P_{n-1} of one fewer vertex, $(\ldots, v_{i-1}, v_{i,j}, v_{i+2}, \ldots)$. See Fig. 3. We

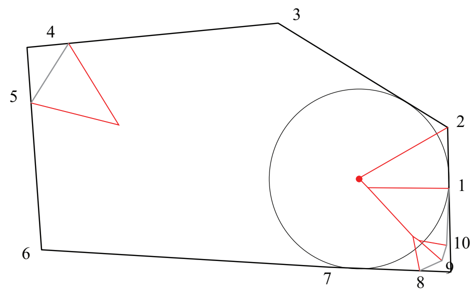

Figure 3: Partial inductive construction of $M(P)$ in Fig. 1(a).

study two unfoldings U_n and U_{n-1} that are based on polygons P_n and P_{n-1} related in just this manner. We will use primes or the subscript $n-1$ to distinguish the elements of U_{n-1} from the corresponding elements of U_n.

Lemma 3 *Let U_n and U_{n-1} be related by removing e_i from P_n, as described above. For $j \notin \{i-1, i, i+1\}$, the cell C_j' of $M(U_{n-1}^*)$ nests inside the corresponding cell C_j of $M(U_n^*)$. For $j \in \{i-1, i+1\}$, the cells nest except for the portion cut away to remove e_i.*

Here by "nests" we mean nests after a rigid movement that places e'_j and e_j into coincidence. With this lemma in hand, it will be straightforward to establish Theorem 2 by induction. We will use Fig. 4 to illustrate the proof. Here $e_i = e_4$ in U_{10} is removed to create U_9.

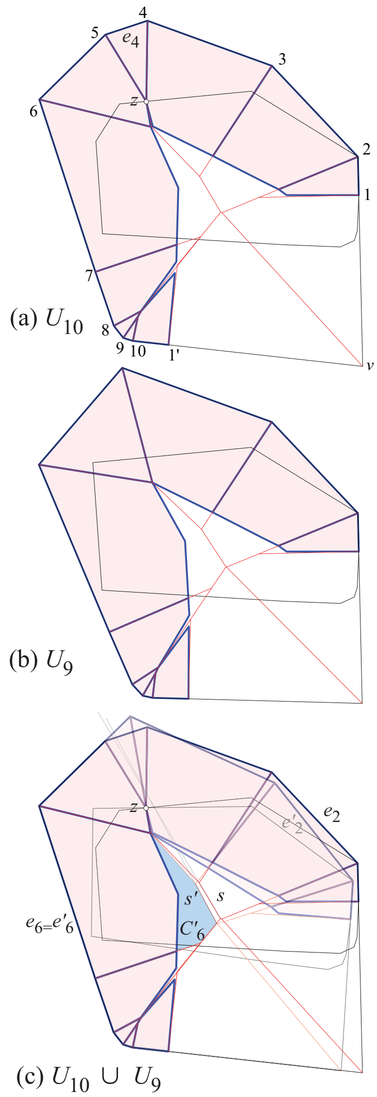

Figure 4: Edge $e_i = e_4$ is engulfed in the $U_{10} \to U_9$ transition. $e_j = e_6$ and $e_k = e_2$. C'_6 enlarges to C_6.

Proof. *Sketch.* If the boundary of the cell C'_j is composed of subsegments of bisectors of edges of U^*_{n-1} all indexed less than $i-1$ or all greater than $i+1$, then $C'_j = C_j$ and there is nothing to prove. In Fig. 4 this holds for $\{C_1, C_7, C_8, C_9, C_{10}\}$. So suppose C'_j's boundary contains a segment s' that is a bisector of e_j and e_k, where i lies between j and k. Let $v_{j,k}$ be the point of intersection of the extensions of these edges, through which the bisector containing s' passes. Let z be the vertex of U_n that is the apex of the triangle eliminated, $\triangle v_i v_{i+1} z$. See Fig. 4(a).

Claim 1. When U_{n-1} is positioned so that e'_j coincides with e_j, then z lies to the same side of a perpendicular line through s' as does $v_{j,k}$. See Fig. 4(c).

The segment s' of C'_j changes to s of C_j by a rotation of e'_k about z to e_k.

Claim 2. The rotation of the bisector of $b' = (e'_j, e'_k)$ containing s' to the bisector $b = (e_j, e_k)$ containing s, with $e'_j = e_j$ fixed, is such that s strictly expands C_j.

These two claims rely on technical lemmas described below. The consequence of Claim 2 (which relies on Claim 1) is that every segment of C'_j moves in such a way as to expand to C_j.

For $j \notin \{i-1, i, i+1\}$, this suffices to show that C'_j nests inside C_j. For $j \in \{i-1, i+1\}$, C'_j in fact does not nest in C_j, because C'_j includes $\triangle z v_i v_{i,j}$ or $\triangle z v_{i,j} v_{i+1}$, not present in C_j. Compare C_3 and C_5 in Figs. 4(a,b). However, $C'_j \setminus \triangle$ does nest in C_j (where $\triangle$ is the appropriate triangle), for the same reason: the segment s' rotates to s about z to enlarge the cell. $\square$

3.1 Technical Lemmas

Lemma 4 *Let s be a segment of the medial axis of a convex polygon P deriving from a maximal disk touching e_j and e_k, whose extensions meet at $v_{j,k}$. Then all points of the medial axis deriving from the portion of ∂P from e_j to $_k$ to the $v_{j,k}$-side is to that same side of any perpendicular line L through s.*

Proof. *Sketch.* See Fig. 5. $\square$

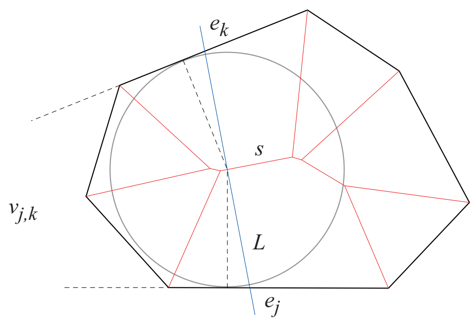

Figure 5: Lemma 4.

Lemma 5 *With $e'_j = e_j$ fixed, let b' and b be the bisectors of e_j with e'_k and e_k respectively, where e_k is a rotation of e'_k about a point z that lies between e_j and b'; see Fig. 6. Then the bisectors meet at a point $q = s' \cap s$ which is left of the line through z perpendicular to b'.*

Proof. *Sketch.* Let e'_k rotate δ about z. If b' forms an angle of θ' with e_j, then after rotation of e_k, bisector b forms an angle $\theta > \theta'$ with e_j. One can show that $\theta = \theta' + \delta/2$.

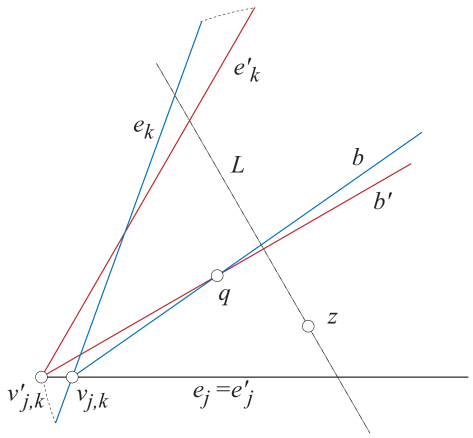

Figure 6: When e'_k rotates to e_k about z, $q = b' \cap b$ is left of L, the perpendicular to b' through z.

Case 1. $v_{j,k}$ lies right of $v'_{j,k}$ as in Fig. 6. As z slides up a fixed L toward b', q moves up b' from $v'_{j,k}$, and $q = z$ at b'. Prior to that, q lies left of L.

Case 2. $v_{j,k}$ lies left of $v'_{j,k}$ on the line containing e_j. Then q falls behind $v'_{j,k}$ on b', well left of L. $\square$

The reason that Lemmas 5 and 4 support the claims of Lemma 3 is as follows. Lemma 4 places z to the "correct" side of the endpoint of s'. Lemma 5 shows that the rotation about z that constitutes the $U_{n-1} \to U_n$ transition causes the bisectors $b \supset s$ and $b' \supset s'$ to meet at a point q even further to the z-side of the endpoint of s'. Thus, s is moved away from s' throughout its length, and so $C_j \supset C'_j$.

Completing the Induction. Consider the construction sequence hinted at in Fig. 3: $P = P_n, P_{n-1}, \ldots, P_3$. Each polygon P_i leads to an unfolding U_i and medial axis $M(U_i^*)$. We know from Lemma 3 the cells of the $M(U_i^*)$ nest. So, starting from face f_j nested in C_j for some U_i, $i \geq 3$, the nesting will continue for all greater i, and thus establish the nesting claimed in Theorem 2. All that remains is establishing the base of this induction.

Lemma 6 *For P_3 a triangle, the three faces f_i of U_3 each nest inside the cell C_i of $M(U_3^*)$.*

Proof. *Sketch.* The apex z of $\mathcal{P}_3$ is equidistant from the three edges of P_3, and therefore z in U_3 is at the center of a circle that touches the three edges of U_3^*. See Fig. 7. $\square$

4 Extensions

Pottmann and Walner consider in [PW01, p. 358ff] the more general polyhedron constructed by slanting planes at some constant angle γ ("constant slope developable surfaces"). Call such a polyhedron $\mathcal{P}(\gamma)$; the medial axis polyhedron is $\mathcal{P}(\pi/4)$. It is not difficult to prove that the projection of $\mathcal{M}$ from $\mathcal{P}(\gamma)$ to the plane of P is independent of γ, i.e., it is always the medial axis $M(P)$. The following additional hypotheses appear to hold, although I have not yet proved them formally:

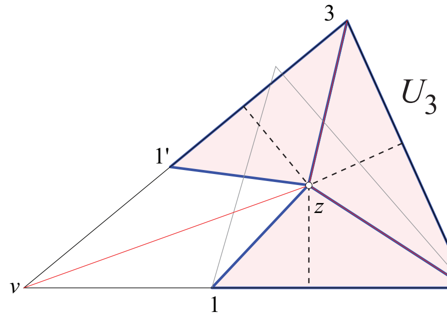

Figure 7: Induction base case: z is a vertex of the medial axis of U_3^*.

1. The main theorem (Theorem 2) holds for $\mathcal{P}(\gamma)$ for any γ and therefore shows all these polyhedra unfold without overlap in the same manner.
2. A polyhedron consisting of $\mathcal{P}(\gamma_1)$ and $\mathcal{P}(\gamma_2)$ glued base-to-base on the same P unfolds by gluing the convex outer shells of the two unfoldings along a common edge.
3. For any given γ, deform $\mathcal{P}(\gamma)$ by driving $\gamma \to 0$ continuously, meanwhile maintaining the original β_i face angles incident to each v_i, and allowing the faces to extend as needed to fill in the gaps at the "cut" edges. When $\gamma = 0$ is reached, the result is the unfolding U^*.

Finally, perhaps the analog of Theorem 2 holds for cutting the cut locus of an arbitrary convex cap, which would establish the quasigeodesic conjecture for convex caps.

References

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