Cut Locus Realizations on Convex Polyhedra

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Abstract

We prove that every positively-weighted tree T can be realized as the cut locus C(x) of a point x on a convex polyhedron P, with T weights matching C(x) lengths. If T has n leaves, P has (in general) n+1 vertices. We show there are in fact a continuum of polyhedra P each realizing T for some $x \in P$. Three main tools in the proof are properties of the star unfolding of P, Alexandrov's gluing theorem, and a cut-locus partition lemma. The construction of P from T is surprisingly simple.

1 Introduction

There is a long tradition of reversing, in some sense, the construction of a graph G from a geometric set. The geometric set may be a point set, a polygon, or a polyhedron, and the graph G could be the Voronoi Diagram, the straight skeleton, or the cut locus, respectively. Reversing would start with, say, the straight skeleton, and reconstruct a polygon with that skeleton. Here we start with the cut locus and construct polyhedra P on which the cut locus is realized for a point $x \in P$. (The cut locus is defined in Section 2.1 below.)

The literature has primarily examined three models for the graph G, often specialized (as here) to trees T:

(1) Unweighted tree: The combinatorial structure of T, without further information.

- (2) Length tree: T with positive edge weights representing Euclidean lengths, and with given circular order of the edges incident to each node of T. Called "ribbon trees" in [CDLR14], and "ordered trees" in [BGP+16].
- (3) Geometric tree: Given by a drawing, i.e., coordinates of nodes, determining lengths and angles.

Our main result is this:

Theorem 1. Given a length tree T of n leaves, we can construct a continuum of star-unfoldings of convex polyhedra P of n+1 vertices, each of which, when folded, realizes T as the cut locus C(x) for a point $x \in P$. Each star-unfolding can be constructed in O(n) time.

Thus, every length tree is isometric to a cut locus on a convex polyhedron.

1.1 Related Results

The computer science literature is extensive, and we cite just a few results:

- Every unweighted tree can be realized as the Voronoi diagram of a set of points in convex position [LM03].
- Every length tree can be realized as the furthest-point Voronoi diagram of a set of points [BGP⁺16].
- Every unweighted tree can be realized as the straight skeleton of a convex polygon, and conditions for length-tree realization are known [CDLR14] [ABH+15] [BGP+16].

In all cases, the reconstruction algorithms are efficient: either O(n) or $O(n \log n)$ for trees of n nodes. Although all these results can be viewed as variations on realizing Voronoi diagrams, and a cut locus is a subgraph of a Voronoi diagram, it appears that prior work does not imply our results.

Our inspiration derives from two results in the convexity literature:

- Every length graph can be realized as a cut locus on a Riemannian surface [IV15]. The result is non-constructive.
- Every unweighted tree can be realized as a cut locus on a doubly covered convex polygon, and length trees can be realized on such polygons when several conditions are satisfied [IV04].

2 Background

In this section we describe the tools needed to prove our main theorem, drawing heavily on our [OV20].

2.1 Cut Locus

The cut locus C(x) of a point x on (the surface of) a convex polyhedron P is the closure of the set of points to which there are more than one shortest path from x. This concept goes back to Poincaré [Poi05], and has been studied algorithmically since [SS86] (under the name "ridge tree"). Some basic properties and terminology:

- C(x) is a tree whose endpoints are vertices of P, and all vertices of P are in C(x).
- Points interior to C(x) of tree-degree 3 or more we will call ramification points.
- The edges of C(x) are geodesic segments on P, geodesic shortest paths between their endpoints [AAOS97].

2.2 Star Unfolding

The star unfolding $S_P(x)$ of P with respect to x is formed by cutting a shortest path from x to every vertex of P [AO92] [AAOS97]. This unfolds to a simple non-overlapping planar polygon $S = S_P(x)$ of 2n vertices: n images x_i of x, and n images of the vertices v_i of P. The connection between the cut locus and the star unfolding is that the image of C(x) in S is the restriction to S of the Voronoi diagram of the images of x [AO92]. See Fig. 1.

2.3 Alexandrov's Gluing Theorem

We rely on Alexandrov's celebrated "Gluing" Theorem [Ale05, p.100].

Theorem AGT. If the boundaries of planar polygons are glued together (by identifying portions of the same length) such that

- (1) The perimeters of all polygons are matched (no gaps, no overlaps).
- (2) The resulting surface is a topological sphere.

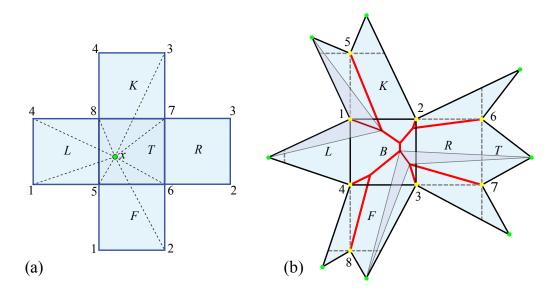


Figure 1: (a) Cut segments to the 8 vertices of a cube from a point x on the top face. T, F, R, K, L, B = Top, Front, Right, Back, Left, Bottom. (b) The star-unfolding from x. The cut locus C(x) (red) is the Voronoi diagram of the 8 images of x (green). Two pairs of fundamental triangles are shaded.

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(3) At most 2π surface angle is glued at any point.

Then the result is isometric to a convex polyhedron P, possibly degenerated to a doubly-covered convex polygon. Moreover, P is unique up to rigid motion and reflection.

The proof of this theorem is nonconstructive, and there remains no effective procedure for constructing the polyhedron guaranteed to exist by this theorem.

2.4 Fundamental Triangles

The following lemma is one key to our proof. See Fig. 1(b).

Lemma 1 (Fundamental Triangles [INV12]). For any point $x \in P$, P can be partitioned into flat triangles whose bases are edges of C(x), and whose lateral edges are geodesic segments from x to the ramification points or leaves of C(x). Moreover, those triangles are isometric to plane triangles, congruent by pairs.

The overall form of our proof of Theorem 1 is to create a star unfolding S by pasting together x_i -apexed fundamental triangles straddling each edge of T, and then applying Alexandrov's theorem to conclude that the folding of S yields a convex polyhedron P which realizes C(x).

2.5 Cut Locus Partition

The last tool we need is a generalization of lemmas in [INV12]. On a polyhedron P, connect a point x to a point $y \in C(x)$ by two geodesic segments γ, γ' . This partitions P into two "half-surface" digons H_1 and H_2 . If we now zip each digon separately closed by joining γ and γ' , AGT leads to two convex polyhedra P_1 and P_2 . The lemma says that the cut locus on P is the "join" of the cut loci on P_i . See Fig. 2.

Lemma 2. Under the above circumstances, the cut locus C(x, P) of x on P is the join of the cut loci on P_i : $C(x, P) = C(x, P_1) \sqcup_y C(x, P_2)$, where \sqcup_y joins the two cut loci at y. And starting instead from P_1 and P_2 , the natural converse holds as well.

Proof. (Sketch. See [OV20] for a formal proof.) All geodesic segments starting at x into H_i remain in H_i , because geodesic segments do not branch. Therefore, H_1 has no influence on $C(x, P_2)$ and H_2 has no influence on $C(x, P_1)$.

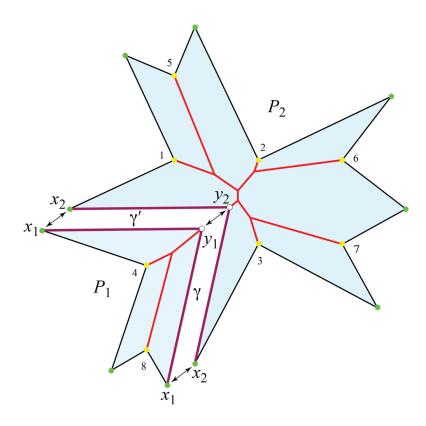


Figure 2: Geodesic segments γ and γ' (purple) connect $x=x_1=x_2$ to $y=y_1=y_2$. P_1 folds to a tetrahedron, and P_2 to an 8-vertex polyhedron, with x and y vertices in each. P_1 and P_2 are cut open along geodesic segments from x_i to y_i and glued together to form P. Based on the cube unfolding in Fig. 1(b).

3 Star-Tree

If T has a node of degree-2, then its incident edges may be merged and their edge-lengths summed. So henceforth we assume T has no nodes of degree-2.

We start with T a star-tree: one central node u of degree-m with edges to nodes u_1, u_2, \ldots, u_m .

A *cone* in the plane is the unbounded region between two rays from its apex. Set $\lambda > L$ to be longer than L, the length of the longest edge of T.

We realize T within a cone of apex angle α , with $0 < \alpha \le 2\pi$.¹ See Fig. 3. Identify points x_1 and x_{m+1} on the cone boundary, with $|ux_i| = \lambda$. Inside the cone, place x images x_2, \ldots, x_m , with each $|ux_i| = \lambda$. Finally place u_i so that uu_i bisects $\angle(ux_i, ux_{i+1})$, $i = 1, \ldots, m$. Chose u_i so that $|uu_i|$ matches T's edge weights. Finally, connect

$$(u, x_1, u_2, x_2, \dots, x_m, u_m, x_{m+1})$$

into a simple polygon.²

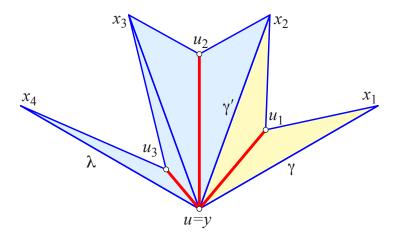


Figure 3: $\alpha = 120^{\circ}$, m = 3, T edge lengths (2,3,1) (red), $\lambda = 4$. In the induction proof of Lemma 3, $\bar{P}(T_1)$ (yellow) is joined to $\bar{P}(T_2)$ (blue).

Before we proceed with the proof, we emphasize that there are several free choices in this construction, illustrated in Fig. 4:

- The angle α at the root is arbitrary.
- The angular distribution of the ux_i segments is arbitrary.
- $\lambda > L$ needs to be "sufficiently large" in a sense we will quantify, but otherwise is arbitrary.

¹Note that we allow $\alpha > \pi$; $\alpha = 2\pi$ represents the whole plane.

²If $\alpha = 2\pi$, $x_1 = x_{m+1}$ and u is interior to the polygon.

In general we will distribute x_i equi-angularly. Choosing $\alpha = 2\pi$ results in a polyhedron of n+1 vertices; for $\alpha < 2\pi$, u is an additional vertex.

We call the described star-T construction a triangle packing.

Lemma 3. A triangle packing of star-T, for sufficiently large λ , is the star-unfolding of a polyhedron P with respect to a point x such that T = C(x).

Proof. The proof is by induction on the number m of edges of T, which is the degree of the root node u. If T is a single edge $e = uu_1$, then folding the twin triangles by creasing e and joining x_1 and x_2 , the two images of x, leads to a doubly covered triangle with x at the corner opposite e. (See the yellow triangles in Fig. 3.) Clearly e = C(x) is the restriction of the Voronoi diagram of x_1 and x_2 , the bisector of x_1 and x_2 . Concerning λ , we need that the angle θ_x incident to x is at most 2π . In this base case, $\theta_1 + \theta_2 \leq 2\pi$ is immediate, so λ is sufficiently large.

Now let n > 1, and partition T into $T_1 \cup_u T_2$ with root degrees m_1 and m_2 respectively, where \cup_u indicates joining the trees at the root u. We will use $\bar{P}(T)$ for the planar triangle packing for T, and P(T) for the folded polyhedron.

We first address λ . In order to apply AGT, we need that θ_x , the sum of the angles at the tips of the triangles—the images of x—is at most 2π .

Let λ be the larger of λ_1 and λ_2 for $\bar{P}(T_1)$ and $\bar{P}(T_2)$ respectively, and stretch the smaller λ_i so that they both share the same λ . Form $\bar{P}(T)$ by joining the two now-compatible triangle packings. Fixing α and the sector angles, it is clear that the angle at each triangle tip decreases monotonically as λ increases. So increase λ as needed so that $\theta_x \leq 2\pi$. Somewhat abusing notation, call these possibly enlarged packings $\bar{P}(T_1)$, $\bar{P}(T_2)$ and $\bar{P}(T)$.

Now we aim to show that $\bar{P}(T)$ is the star-unfolding of a polyhedron with T = C(x). Certainly $\bar{P}(T)$ folds to a polyhedron P, because (a) by construction the edges incident to u_1, \ldots, u_k match in length, and (b) we have ensured that $\theta_x \leq 2\pi$. So Alexandrov's theorem applies. Now identify γ and γ' on P from x to y = u separating the surface into pieces corresponding to $\bar{P}(T_1)$ and $\bar{P}(T_2)$, which fold to P_1 and P_2 respectively. (Refer again to Fig. 3.) By the induction hypothesis, $T_1 = C(x, P_1)$ and $T_2 = C(x, P_2)$. Applying Lemma 2, we have $C(x, P) = C(x, P_1) \sqcup_y C(x, P_2)$, where the two cut loci are joined at y = u. And so indeed $T_1 \cup_u T_2 = T = C(x)$.

In Section 4.1 we will calculate the needed λ explicitly.

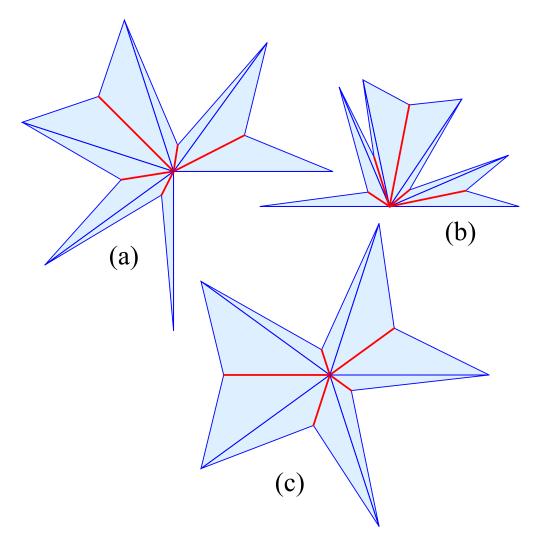


Figure 4: Star-T with edge lengths (3,1,4,2,1). (a) $\alpha=270^\circ$, equiangular $x_i,\ \lambda=6$. (b) $\alpha=180^\circ$, random $x_i,\ \lambda=5$. (c) $\alpha=360^\circ$, equiangular $x_i,\ \lambda=6$.

4 General Length-Trees T

We now generalize the above to arbitrary length-trees T, using the example in Fig. 5 as illustration.

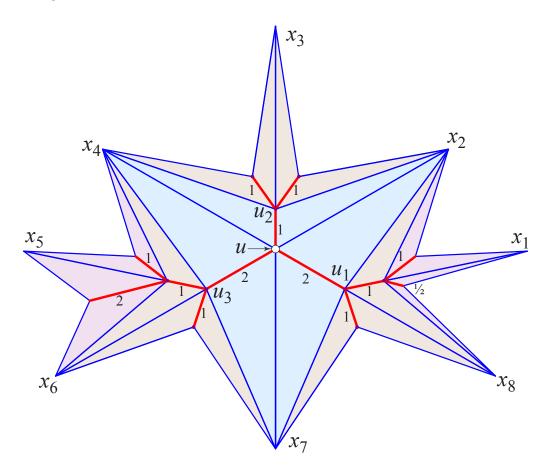


Figure 5: Realization of a length-tree T of height 3, using $\alpha=2\pi$ and $\lambda=L=5$. This polygon folds to a polyhedron of 9 vertices: the 8 leaves of T, and x.

Given T, select any node u to serve as the root. Fix any α , and choose λ to exceed the length L of the longest path from u to a leaf in T. Now create a triangle packing for T as follows.

First create a triangle packing for u and its immediate children u_1, \ldots, u_m , just as previously described. With $\lambda > L$, the external angle at u_i is

strictly less than π , i.e., it forms a "V-shape" there. Call this a cup, $c_i = (x_i, u_i, x_{i+1})$ with α_i external angle at u_i . Let u_i have children $u_{i1}, u_{i2}, \ldots, u_{il}$. So $u_i, (u_{i1}, \ldots, u_{il})$ is a star-tree. Fill in the cup c_i by inserting a triangle packing for this sub-star-tree, with apex at u_i , angle α_i , and λ -length $|u_i x_i|$, the distance from u_i to the tips of the cup.

After filling the c_i cups for all the u_i at level-2 of T, repeat the process with level-3 of T, and so on. Throughout the construction, the locations for x_i remain fixed after their initial placement. And with sufficiently large λ , all the cups form V-shapes.

Note that the triangles incident to an internal node u_i of T (neither the root nor a leaf) leave no gaps: they cover the 2π surrounding u_i .

Lemma 4. A triangle packing for any T, as just described, for sufficiently large λ , is the star-unfolding of a polyhedron P with respect to a point x such that T = C(x).

Proof. The proof is by induction, and parallels the proof of Lemma 3 closely. Consequently, we only sketch the proof.

The base of the induction is a star-graph, settled by Lemma 3. Let T be an arbitrary length-tree, and partition T into two smaller trees T_1 and T_2 sharing the root u, so $T = T_1 \cup_u T_2$. Select an α for T and α_i for T_i , i = 1, 2, so that $\alpha = \alpha_1 + \alpha_2$.

By the induction hypothesis, T_i can be realized in cups of angle α_i . Moreover, $\bar{P}(T_i)$ folds to P_i and $T_i = C(x, P_i)$. Stretch λ_i as needed to allow $\bar{P}(T_1)$ to share λ with $\bar{P}(T_2)$ at u, and stretch again so that $\theta_x \leq 2\pi$.

Form $\bar{P}(T)$ by adjoining $\bar{P}(T_1)$ and $\bar{P}(T_2)$ at u, with cup apex α . Fold $\bar{P}(T)$ to P by AGT. Apply Lemma 2 to conclude $C(x,P) = C(x,P_1) \sqcup_y C(x,P_2)$, where y=u. And so $T_1 \cup_u T_2 = T = C(x)$.

Note that all ramification points of C(x) are flat on P, with 2π incident surface angle. If θ_x is strictly less than 2π , then the source x is a vertex on P. If at the root, $\alpha < 2\pi$, then in addition u is a vertex on P. So P has n, n+1, or n+2 vertices.

Lemmas 3 and 4, together with Lemma 5 (below) prove Theorem 1. The construction of the triangle packing can be achieved in O(n) time: we are given the cyclic ordering of the edges incident to each node, so sorting is not necessary, and the level-by-level packing construction is proportional to the the number of edges.

4.1 Total angle at x

We derive here a sufficient condition on the value of the parameter λ for the root cup to guarantee that $\theta_x \leq 2\pi$.

Lemma 5. If T has m edges, and L is the longest path from the root in T then $\theta_x \leq 2\pi$ when

 $\lambda \geq L \left[1 + \cot \left(\frac{\pi}{m} \right) \right] \ .$

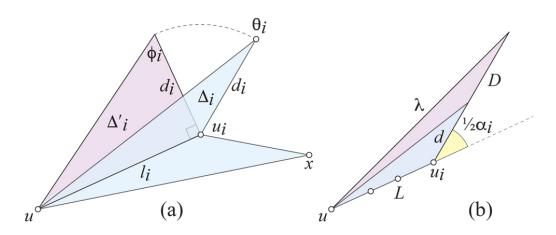


Figure 6: (a) $\theta_i \leq \phi_i$. (b) Increasing λ increases d. L is the length of the longest path in T, D the target bound for $\lambda \geq L + D$.

Proof. First we establish notation, illustrated in Fig. 6(a). As before, u is the root of T, and let u_i be a leaf of T, with edge uu_i shared by twin triangles, one of which is \triangle_i . Let $\ell_i = |uu_i|$ and d_i the distance from the tip of \triangle_i to u_i , and θ_i the angle at that tip, an image of x. Because the fundamental triangles come in pairs, and there are m edges, we have that the total angle at x satisfies $\theta_x = 2 \sum \theta_i$.

Consider now a right triangle Δ_i' having the same base as Δ_i and height d_i , and denote by ϕ_i its angle opposite to the base uu_i . Then $\phi_i = \arctan\left(\frac{\ell_i}{d_i}\right)$ and $\theta_i \leq \phi_i$; see again Fig. 6.

Because arctan is an increasing function, we can obtain an upper bound by replacing ℓ_i with the longest edge length ℓ , and replacing d_i by the shortest

of the xu_j diagonals, call it d: $\ell = \max_i \ell_i$, $d = \min_i d_i$. So $\phi_i \leq \arctan\left(\frac{\ell}{d}\right)$, and therefore

$$\theta_x = 2 \sum \theta_i \le 2 \sum \phi_i \le 2m \arctan\left(\frac{\ell}{d}\right)$$
.

The expression $2m \arctan\left(\frac{\ell}{d}\right)$ decreases as d increases. For it to evaluate to at most 2π , we must have

$$d \geq \ell \cot \left(\frac{\pi}{m}\right)$$
.

The longest path L from root to leaf is at least as long as the longest edge, $L \ge \ell$, so this bound will more than suffice:

$$d \ge L \cot\left(\frac{\pi}{m}\right) = D$$
.

Now we show that if λ is long enough, then $d \geq D$. Consider Fig. 6(b), where L is the longest path from u to a leaf u_i . Let the external angle at u_i be α_i . By definition, there is some index j such that $d = d_j$. The triangle inequality directly implies $\lambda \leq \ell_j + d_j \leq L + d$.

With L and α_i fixed, increasing λ increases d. If we substitute the needed lower bound D for d in the expression (see Fig. 6(b)), then $\lambda \geq L + D$ forces $d \geq D$. Explicitly

$$\lambda \ge L \left[1 + \cot \left(\frac{\pi}{m} \right) \right]$$

suffices to guarantee that $\theta_x \leq 2\pi$.

The bound—approximately $L(1+m/\pi)$ —is far from tight. For example, in Fig. 5, m=13 and L=5 leads to $\lambda \geq 26$, but $\lambda = 5$ (illustrated) leads to $\theta_x \approx 167^\circ$, and so easily suffices.

5 Remarks

One further example is shown in Fig. 7. It is the star unfolding of a polyhedron of 49 vertices, whose resemblance to a fractal suggests there might at a deeper connection. We will only mention that fractals play a role in the folding of specific convex polyhedra in [Ueh20], and fractal cut loci on

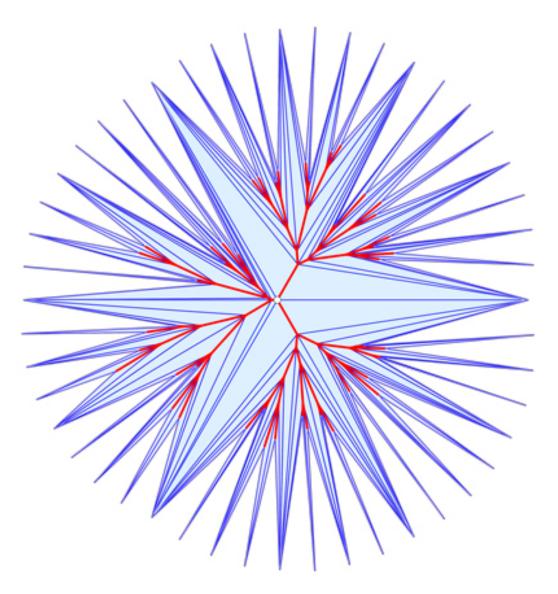


Figure 7: Regular degree-3 tree, random edge lengths, $\alpha=2\pi,\ n=48,$ $\theta_x\approx 317^\circ.$

on k-differentiable Riemannian and Finslerian spheres are shown in [IS16] to exist for any $2 \le k < \infty$.

A natural question is whether geometric trees—drawings embedded in the plane, and so providing angles between adjacent edges—can be realized as cut loci on convex polyhedra. Certainly not all geometric trees are realizable, for there are constraints on the angles: around a ramification point, no angle can exceed π , and the angles must sum to $\leq 2\pi$. And the sum of the curvatures at the u_i and at x must be 4π to satisfy the Gauss-Bonnet theorem.

We leave it as an open problem to characterize those geometric trees that are realizable as the cut locus on a convex polyhedron.

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