# Conical Existence of Closed Curves on Convex Polyhedra

Joseph O'Rourke* Costin Vîlcu†

February 14, 2011

**Abstract**

Let $C$ be a simple, closed, directed curve on the surface of a convex polyhedron $\mathcal{P}$. We identify several classes of curves $C$ that "live on a cone," in the sense that $C$ and a neighborhood to one side may be isometrically embedded on the surface of a cone $\Lambda$, with the apex $a$ of $\Lambda$ enclosed inside (the image of) $C$; we also prove that each point of $C$ is "visible to" $a$. In particular, we obtain that these curves have non-self-intersecting developments in the plane. Moreover, the curves we identify that live on cones to both sides support a new type of "source unfolding" of the entire surface of $\mathcal{P}$ to one non-overlapping piece, as reported in a companion paper.

## 1 Introduction

Let $\mathcal{P}$ be the surface of a convex polyhedron, and let $C$ be any simple, closed, directed curve on $\mathcal{P}$. In this paper we address the question of which curves $C$ "live on a cone" to either or both sides. We first explain this notion, which is based on neighborhoods of $C$.

**Living on a Cone.** An open region $N_L$ is a *vertex-free neighborhood* of $C$ to its left if its right boundary is $C$, and it contains no vertices of $\mathcal{P}$. In general $C$ will have many vertex-free left neighborhoods, and all will be equivalent for our purposes. We say that $C$ *lives on a cone* to its left if there exists a cone $\Lambda$ and a neighborhood $N_L$ so that $C \cup N_L$ may be embedded isometrically onto $\Lambda$, and encloses the cone apex $a$.

A *cone* is a developable surface with curvature zero everywhere except at one point, its *apex*, which has total incident surface angle, called the *cone angle*, of at most $2\pi$. Throughout, we will consider a cylinder as a cone whose apex

*Department of Computer Science, Smith College, Northampton, MA 01063, USA. orourke@cs.smith.edu.

†Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania. Costin.Vilcu@imar.ro.

is at infinity with cone angle 0, and a plane as a cone with apex angle $2\pi$. We only care about the intrinsic properties of the cone's surface; its shape in $\mathbb{R}^3$ is not relevant for our purposes. So one could view it as having a circular cross section, although we will often flatten it to the plane, in which case it forms a doubly covered triangle with apex angle half the cone angle. Except in special cases, the cone $\Lambda$ is unrelated to any cone that may be formed by extending the faces of $\mathcal{P}$ to the left of $C$.

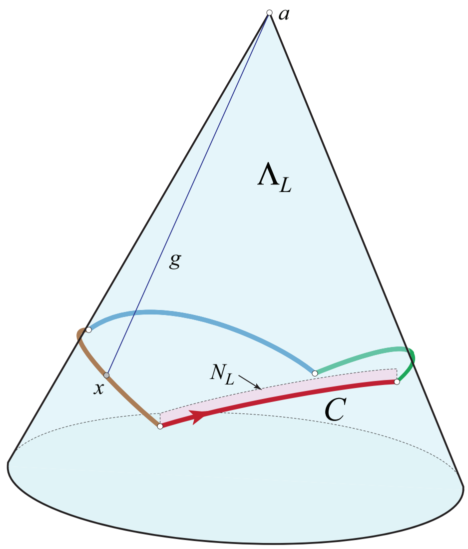


Figure 1: A 4-segment curve $C$ which lives on cone $\Lambda_L$ to its left. A portion of $N_L$ is shown, and a generator $g = ax$ is illustrated.

To say that $C \cup N_L$ embeds isometrically into $\Lambda$ means that we could cut out $C \cup N_L$ and paste it onto $\Lambda$ with no wrinkles or tears: the distance between any two points of $C \cup N_L$ on $\mathcal{P}$ is the same as it is on $\Lambda$. See Figure 1. We say that $C$ lives on a cone to its right if $C \cup N_R$ embeds on the cone, where $N_R$ is a right neighborhood of $C$ such that the cone apex $a$ is inside (the image of) $C$. We will call the cones $\Lambda_L$ and $\Lambda_R$ to the left and right of $C$ when we need to distinguish them. We will see that all four combinatorial possibilities occur: $C$ may not live on a cone to either side, it may live on a cone to one side but not to the other, it may live on different cones to its two sides, or live on the same cone to both sides.

**Motivations.** We have two motivations to study curves that live on a cone, aside from their intrinsic interest. First, every simple, closed curve $C$ on a cone $\Lambda$ may be *developed* on the plane by rolling $\Lambda$ and transferring the "imprint" of $C$ to the plane. This will allow us to strengthen a previous result on simple (i.e., non-self-intersecting) developments of certain curves. Second, for curves $C$ that live on a cone to both sides, our results support a generalization of the "source unfolding" of a polyhedron. Both of these motivations will be detailed further (with references) in Section 7.

**Curve Classes.** To describe our results, we introduce a number of different classes of curves on convex polyhedra, which exhibit different behavior with respect to living on a cone. Altogether, we define eight classes of curves. All our curves $C$ are simple (non-intersecting), closed, directed curves on a convex polyhedron $\mathcal{P}$, and henceforth we will generally drop these qualifications.

For any point $p \in C$, let $L(p)$ be the total surface angle incident to $p$ at the left side of $C$, and $R(p)$ the angle to the right side. $C$ is a *geodesic* if $L(p)=R(p)=\pi$ for every point $p$ on $C$. Generally this is called a *closed geodesic* in the literature. When a geodesic is extended on a surface and later crosses itself, each closed portion generally forms what is known as a *geodesic loop*: $L(p)=R(p)=\pi$ for all but one exceptional *loop point* $x$, at which it may be that $L(x)\neq\pi$ or $R(x)\neq\pi$. (The loop versions of curves are important because they are in general easier to find than "pure" versions.)

Define a curve $C$ to be *convex* (to the left) if the angle to the left is at most $\pi$ at every point $p$: $L(p)\leq\pi$; and say that $C$ is a *convex loop* if this condition holds for all but one exceptional *loop point* $p$, at which $L(p)>\pi$ is allowed.

A curve $C$ is a *quasigeodesic* if it is convex to both sides: $L(p)\leq\pi$ and $R(p)\leq\pi$ for all $p$ on $C$. (This is a notion introduced by Alexandrov to allow geodesic-like curves to pass through vertices of $\mathcal{P}$.) A *quasigeodesic loop* satisfies the same condition except at an exceptional loop point $p$, at which $L(p)\leq\pi$ but $R(p)>\pi$ (or vice versa) is allowed. Thus a quasigeodesic loop is convex to one side and a convex loop to the other side.

Finally, define $C$ to be a *reflex curve*[1] if the angle to one side (we consistently use the right side) is at least $\pi$ at every point $p$: $R(p)\geq\pi$; and say that $C$ is a *reflex loop* if this condition holds for all but an exceptional loop point $p$, at which $R(p)<\pi$.

The eight curve classes are then the four listed in the table below, and their loop variations, which permit violation of the angle conditions at one point: We now describe relations between the classes. Most are obvious, following

| *Curve class* | *Angle condition* |
|---|---|
| geodesic | $L(p) = \pi = R(p)$ |
| quasigeodesic | $L(p) \leq \pi$ and $R(p) \leq \pi$ |
| convex | $L(p) \leq \pi$ |
| reflex | $R(p) \geq \pi$ |

Table 1: Curve classes.

from the definitions. All the non-loop curves are special cases of their loop version: a geodesic is a geodesic loop, etc. A geodesic is a quasigeodesic, and a

[1] We opt for the term "reflex" rather than "concave" for its greater syntactic difference from "convex."

quasigeodesic is convex to both sides. A geodesic loop is a quasigeodesic loop, which is convex to one side and a convex loop to the other side. To explain the relationship between convex and reflex curves, we recall the notion of "discrete curvature," or simply "curvature."

The *curvature* $\omega(p)$ at any point $p \in \mathcal{P}$ is the "angle deficit": $2\pi$ minus the sum of the face angles incident to $p$. The curvature is only nonzero at vertices of $\mathcal{P}$; at each vertex it is positive because $\mathcal{P}$ is convex. The curvature at the apex of a cone is similarly $2\pi$ minus the cone angle.

Define a *corner* of curve $C$ to be any point $p$ at which either $L(p) \neq \pi$ or $R(p) \neq \pi$. Let $c_1, c_2, \ldots, c_m$ be the corners of $C$, which may or may not also be vertices of $\mathcal{P}$. $C$ "turns" at each $c_i$, and is straight at any noncorner point. Let $\alpha_i = L(c_i)$ be the surface angle to the left side at $c_i$, and $\beta_i = R(c_i)$ the angle to the right side. Also let $\omega_i = \omega(c_i)$ to simplify notation. We have $\alpha_i + \beta_i + \omega_i = 2\pi$ by the definition of curvature.

Returning to our discussion of curve classes, a convex curve that passes through no vertices of $\mathcal{P}$ is a reflex curve to the other side, because $\omega_i = 0$ and so $\alpha_i \leq \pi$ implies that $\beta_i \geq \pi$. A convex curve that passes through at most one vertex of $\mathcal{P}$, say at $c_m$, is a reflex loop to the other side, with possibly $\beta_m < \pi$, and is a reflex curve to that side if $\alpha_m + \omega_m \leq \pi$ because then $\beta_m \geq \pi$. The relationship between convex and reflex is symmetric: so a reflex curve that passes through no vertices is convex to the other side, and a reflex curve that passes through one vertex is a convex loop to the other side. The other side of a reflex loop is a convex loop, as will be discussed further in Section 5 (cf. Table 2).

We illustrate some of these concepts in Figure 2: (a) shows an icosahedron, and (b) a cubeoctahedron. For both polyhedra, $\omega(v) = \frac{1}{3}\pi$ for each vertex $v$ of $\mathcal{P}$. The curve illustrated in (a) is convex to both sides, with $\frac{2}{3}\pi$ to one side and $\pi$ to the other at each of its five corners. Thus it is a quasigeodesic. The curve in (b) is convex to one side, with angles

$$\left(\tfrac{5}{6}\pi, \tfrac{5}{6}\pi, \tfrac{1}{2}\pi, \tfrac{5}{6}\pi, \tfrac{5}{6}\pi, \tfrac{1}{2}\pi\right)$$

at its six corners, but because the angles to the other side are (respectively)

$$\left(\tfrac{5}{6}\pi, \tfrac{5}{6}\pi, \tfrac{7}{6}\pi, \tfrac{5}{6}\pi, \tfrac{5}{6}\pi, \tfrac{7}{6}\pi\right)$$

it falls outside our classification system to that side (because it violates convexity at two corners, and reflexivity at four corners).

The main result of this paper is that a convex curve lives on a cone to its convex side, and a reflex loop whose other side is convex lives on a cone to its reflex side. One consequence is that any convex curve (which could be a quasigeodesic) that includes at most one vertex lives on a cone to both sides. We also show that a convex loop might not live on a cone to its convex side.

**Visibility.** An additional property is needed for these cones to support our applications. A *generator* of a cone $\Lambda$ is a half-line starting from the apex $a$ and lying on $\Lambda$. A curve $C$ that lives on $\Lambda$ is *visible* from the apex if every generator

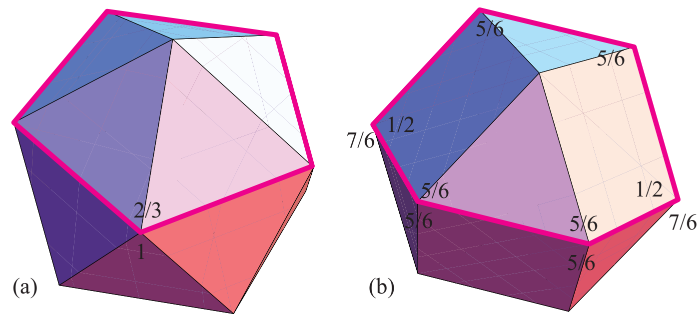


Figure 2: (a) Quasigeodesic curve on a Icosahedron. (b) Convex curve on a Cubeoctahedron. Angles are shown at vertices in units of $\pi$.

meets $C$ at one point.[2] See again Figure 1; Figure 5(a) ahead illustrates a $C$ not visible from $a$. Although it is quite possible for a curve to live on a cone but not be visible from its apex, we establish that, for the classes we identify, $C$ is indeed visible from the apex of the cone on which it lives.

## 2 Preliminary Tools

**The Gauss-Bonnet Theorem.** We will employ this theorem in two forms. The first is that the total curvature of $\mathcal{P}$ is $4\pi$: the sum of $\omega(v)$ for all vertices $v$ of $\mathcal{P}$ is $4\pi$. It will be useful to partition the curvature into three pieces. Let $\Omega_L(C) = \Omega_L$ be the total curvature strictly interior to the region of $\mathcal{P}$ to the left of $C$, $\Omega_R$ the curvature to the right, and $\Omega_C$ the sum of the curvatures on $C$ (which is nonzero only at vertices of $\mathcal{P}$). Then $\Omega_L + \Omega_C + \Omega_R = 4\pi$.

The second form of the Gauss-Bonnet theorem relies on the notion of the "turn" of a curve. Define $\tau_L(c_i) = \tau_i = \pi - \alpha_i$ as the left *turn* of curve $C$ at corner $c_i$, and let $\tau_L(C) = \tau_L$ be the total (left) turn of $C$, i.e., the sum of $\tau_i$ over all corners of $C$. (The turn at noncorner points of $C$ is zero. Note that the curve turn at a point is not directly related to the surface curvature at that point.) Thus a convex curve has nonnegative turn at each corner, and a reflex curve has nonpositive turn at each corner. Then $\tau_L + \Omega_L = 2\pi$, and defining the analogous term to the right of $C$, $\tau_R + \Omega_R = 2\pi$. So, if $C$ is a geodesic, $\tau_L = \tau_R = 0$ and $\Omega_L = \Omega_R = 2\pi$.

**Alexandrov's Gluing Theorem.** In our proofs we use Alexandrov's celebrated theorem [Ale05, Thm. 1, p. 100] that gluing polygons to form a topological sphere in such a way that at most $2\pi$ angle is glued at any point, results in a unique convex polyhedron.

[2] In other terminology, $C$ could be said to be *star-shaped* from $a$.

**Vertex Merging.** We now explain a technique used by Alexandrov, e.g., [Ale05, p. 240]. Consider two vertices $v_1$ and $v_2$ of curvatures $\omega_1$ and $\omega_2$ on $\mathcal{P}$, with $\omega_1 + \omega_2 < 2\pi$, and cut $\mathcal{P}$ along a shortest path $\gamma(v_1, v_2)$ joining $v_1$ to $v_2$. Construct a planar triangle $T = \bar{v}'\bar{v}_1\bar{v}_2$ such that its base $\bar{v}_1\bar{v}_2$ has the same length as $\gamma(v_1, v_2)$, and the base angles are equal to $\frac{1}{2}\omega_1$ and respectively $\frac{1}{2}\omega_2$. Glue two copies of $T$ along the corresponding lateral sides, and further glue the two bases of the copies to the two "banks" of the cut of $\mathcal{P}$ along $\gamma(v_1, v_2)$. By Alexandrov's Gluing Theorem, the result is a convex polyhedral surface $\mathcal{P}'$. On $\mathcal{P}'$, the points $v_1$ and $v_2$ are no longer vertices because exactly the angle deficit at each has been sutured in; they have been replaced by a new vertex $v'$ of curvature $\omega' = \omega_1 + \omega_2$ (preserving the total curvature). Figure 3(a) illustrates this. Here $\gamma(v_1, v_2) = v_1 v_2$ is the top "roof line" of the house-shaped polyhedron $\mathcal{P}$. Because $\omega_1 = \omega_2 = \frac{1}{2}\pi$, $T$ has base angles $\frac{1}{4}\pi$ and apex angle $\frac{1}{2}\pi$. Thus the curvature $\omega'$ at $v'$ is $\pi$. (Other aspects of this figure will be discussed later.)

Note this vertex-merging procedure only works when $\omega_1 + \omega_2 < 2\pi$; otherwise the angle at the apex $\bar{v}'$ of $T$ would be greater than or equal to $\pi$.

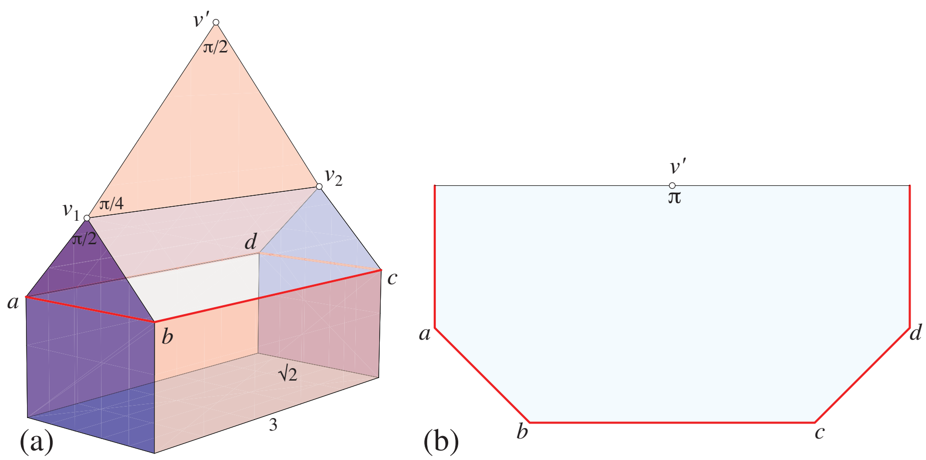


Figure 3: (a) $C = (a, b, c, d)$ is a convex curve with angle $\frac{3}{4}\pi$ to the left at each vertex. The curvature at $v_1$ and at $v_2$ is $\frac{1}{2}\pi$. (b) Cutting along the generator from $v'$ through the midpoint of $ad$ and developing $C$ shows that it lives on a cone with apex angle $\pi$ at $v'$. (Base of $\mathcal{P}$ is $3 \times \sqrt{2}$.)

**Half-Surfaces Notation.** $C$ partitions $\mathcal{P}$ into two *half-surfaces*: $\mathcal{P} \setminus C$. We call the left and right half-surfaces $P_L$ and $P_R$ respectively, or $P$ if the distinction is irrelevant. We view each half-surface as closed, with boundary $C$.

# 3 Convex Curves

We start with convex curves $C$.

**Convexity of Half-Surfaces.** In order to apply vertex merging, we use a lemma to guarantee the existence of a pair to merge. We first remark that it is not the case that every half-surface $P \subset \mathcal{P}$ bounded by a convex curve $C$ is *convex* in the sense that, if $x, y \in P$, then a shortest path $\gamma$ of $\mathcal{P}$ connecting $x$ and $y$ lies in $P$.
*Example.* Let $\mathcal{P}$ be defined as follows. Start with the top half of a regular octahedron, whose four equilateral triangle faces form a pyramid over a square base $abcd$. Flex the pyramid by squeezing $a$ toward $c$ slightly while maintaining the four equilateral triangles, a motion which separates $b$ from $d$. Define $\mathcal{P}$ to be the convex hull of these four moved points $a'b'c'd'$ and the pyramid apex. Let $C = (a', b', c', d')$ and let $P$ be the half-surface including the four equilateral triangles. Then $a'$ and $c'$ are in $P$, but the edge $a'c'$ of $\mathcal{P}$, which is the shortest path connecting those points, is not in $P$: it crosses the "bottom" of $\mathcal{P}$.

Although $P$ may not be convex, $P$ is *relatively convex* in the sense that it is isometric to a convex half-surface: there is some $\mathcal{P}^\#$ and a half-surface $P^\# \subset \mathcal{P}^\#$ such that $P$ is isometric to $P^\#$ and $P^\#$ is convex.

**Lemma 1** *Every half-surface $P \subset \mathcal{P}$ bounded by a convex curve $C$ is relatively convex, i.e., $P$ is isometric to a half-surface that contains a shortest path $\gamma$ between any two of its points $x$ and $y$. More particularly, if neither $x$ nor $y$ is on $C$, then the shortest path $\gamma$ contains no points of $C$. If exactly one of $x$ or $y$ is on $C$, then that is the only point of $\gamma$ on $C$.*

**Proof:** We glue two copies of $P$ along $\partial P = C$. Because $C$ is convex, Alexandrov's Gluing Theorem says the resulting surface is isometric to a unique polyhedral surface, call it $\mathcal{P}^\#$. Because $\mathcal{P}^\#$ has intrinsic symmetry with respect to $C$, a lemma of Alexandrov [Ale05, p. 214] applies to show that the polyhedron $\mathcal{P}^\#$ has a symmetry plane $\Pi$ containing $C$.

Now consider the points $x$ and $y$ in the upper half $P$ of $P^\#$, at or above $\Pi$. If $\gamma$ is a shortest path from $x$ to $y$, then by the symmetry of $\mathcal{P}^\#$, so is its reflection $\gamma'$ in $\Pi$. Because shortest paths on convex surfaces do not branch, $\gamma$ must lie in the closed half-space above $\Pi$, and so lies on $P$.

If neither $x$ nor $y$ are on $C$, they are strictly above $\Pi$, and $\gamma$ must be as well to avoid a shortest-path branch. If, say, $x \in C$ but $y \notin C$, and if $\gamma$ touched $C$ elsewhere, say at $z$, then from $y$ to $x$ we have a shortest path $\gamma$ and another shortest path, composed of the arc of $\gamma$ from $y$ to $z$ and the arc of $\gamma'$ from $z$ to $x$, hence we would have a shortest-path branch at $z$. If both $x$ and $y$ are on $C$, then either $\gamma$ meets $C$ in exactly those two points, or $\gamma \subset C$, for the same reason as above. □

**Lemma 2** *Let $C$ be a convex curve on $\mathcal{P}$, convex to its left. Then $C$ lives on a cone $\Lambda_L$ to its left side, whose apex $a$ has curvature $\Omega_L$.*

**Proof:** By the Gauss-Bonnet theorem, $\tau_L + \Omega_L = 2\pi$. Because $\tau_L \geq 0$ for a convex curve, we must have $\Omega_L \leq 2\pi$. Let $V$ be the set of vertices of the half-surface $P_L$ not on $C$.

Suppose first that $\Omega_L < 2\pi$. If $|V| = 1$, then $P_L$ is a pyramid, which is already a cone. So suppose $|V| \geq 2$, and let $v_1$ and $v_2$ be any two vertices in $V$. Lemma 1 guarantees that a shortest path $\gamma$ between them is in $P_L^{\#}$ and disjoint from $C$. Perform vertex merging along $\gamma$, resulting in a new vertex $v'$ whose curvature is the sum of that of $v_1$ and $v_2$. Note that merging is always possible, because $\omega_1 + \omega_2 \leq \Omega_L < 2\pi$. Also note that $v'$ is not on $C$, by Lemma 1. Let $N_L$ be some small left neighborhood of $C$ in $P_L$. Then $N_L$ is unaffected by the vertex merging: neither $v_1$ nor $v_2$ is in $N_L$ because it is vertex free, and $N_L$ may be chosen narrow enough (by Lemma 1) so that no portion of $\gamma$ is in $N_L$. Replace $V$ by $(V \setminus \{v_1, v_2\}) \cup \{v'\}$.

Continue vertex merging in a like manner between vertices of $V$ until $|V| = 1$, at which point we have $C$ and $N_L$ living on a cone, as claimed.

If $\Omega_L = 2\pi$, then the last step of vertex merging will not succeed. However, we can see that a slight altering of the two glued triangles so that $\Omega_L < 2\pi$ will result in the cone apex approaching infinity, as follows. Cut along a geodesic between the two vertices, say $v_i$ and $v_{i+1}$, and insert double triangles of base angles $\frac{1}{2}\omega_i$ and respectively $\frac{1}{2}\omega_{i+1} - \varepsilon_n$, with $\varepsilon_n > 0$ and $\lim_n \varepsilon_n = 0$. And so in this case, $C$ and $N_L$ live on a cylinder, which we consider a degenerate cone.
□

*Example.* In Figure 3 the two vertices inside $C$, of curvature $\frac{1}{2}\pi$ each, are merged to one of curvature $\pi$, which is then the apex of a cone on which $C$ lives.

*Example.* Figure 4(a) shows an example with three vertices inside $C$. $\mathcal{P}$ is a doubly covered flat pentagon, and $C = (v_4, v_5, v_4)$ is the closed curve consisting of a repetition of the segment $v_4v_5$. $C$ has $\pi$ surface angle at every point to its left, and so is convex. The curvatures at the other vertices are $\omega_1 = \pi$ and $\omega_2 = \omega_3 = \frac{1}{2}\pi$. Thus $\Omega_L = 2\pi$, and the proof of Lemma 2 shows that $C$ lives on a cylinder. Following the proof, merging $v_1$ and $v_2$ removes those vertices and creates a new vertex $v_{12}$ of curvature $\frac{3}{2}\pi$; see (b) of the figure. Finally merging $v_{12}$ with $v_3$ creates a "vertex at infinity" $v_{123}$ of curvature $2\pi$. Thus $C$ lives on a cylinder as claimed. If we first merged $v_2$ and $v_3$ to $v_{23}$, and then $v_{23}$ to $v_1$, the result is exactly the same, although less obviously so.

This last example raises the natural question of whether the cone constructed through vertex merging in Lemma 2 is independent of the order of merging. Indeed the determined cone is unique:

**Lemma 3** *A curve $C$ that lives on a cone $\Lambda$ (say, to its left) uniquely determines that cone.*

**Proof:** Suppose that $C$ lives on two cones $\Lambda$ and $\Lambda'$. We will show that the regions of these two cones bounded by $C$ are isometric. First note that the apex angle of both $\Lambda$ and $\Lambda'$ is $\Omega_L$, the total curvature inside and left of $C$. Let $x \in C$ be a point of $C$ that has a tangent $t$ to one side, and let $x_1$ be a point in the plane and $t_1$ a direction vector from $x_1$. Roll $\Lambda$ in the plane so that $x$ and $t$ coincide with $x_1$ and $t_1$. Continue rolling until $x$ is encountered again; call that point of the plane $x_2$. The resulting positions of $x_1$ and $x_2$ are the same as would be produced by cutting the cone along a generator $ax$.

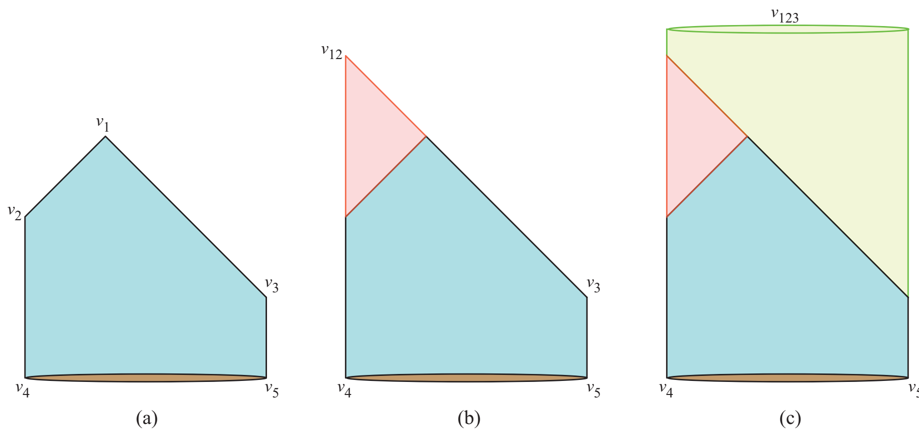


Figure 4: (a) A doubly covered flat pentagon. (b) After merging $v_1$ and $v_2$. (c) After merging $v_{12}$ and $v_3$.

If $x_1 = x_2$, then both $\Lambda$ and $\Lambda'$ are planar and so isometric. So assume $x_1 \neq x_2$. If $\Omega_L \geq \pi$, then the cone angle $\alpha \leq \pi$, as in Figure 5(b). The segment $x_1x_2$ determines two isosceles triangles with apex angle $\alpha$, only one of which can correspond to the left side of $\overline{C}$. Analogously, if $\Omega_L < \pi$, then $x_1x_2$ determines

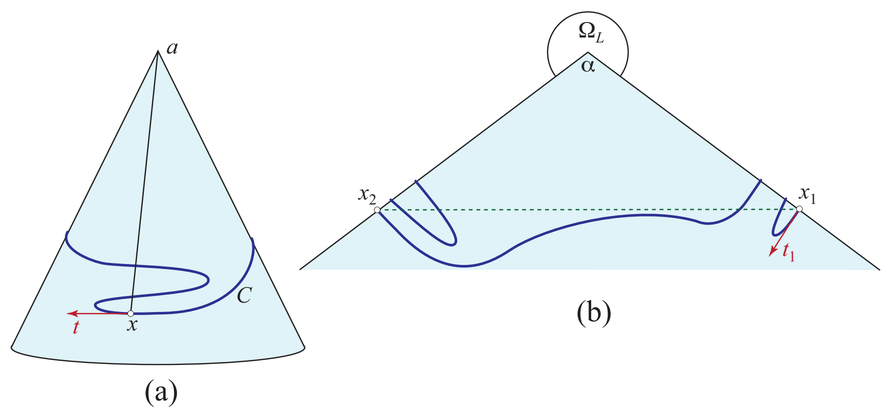


Figure 5: (a) Cone $\Lambda$ on which $C$ lives. (b) Positions of $x_1$ and $x_2$ after cutting open $\Lambda$ along $ax$.

a unique isosceles triangle of apex angle $\Omega_L$, the equal sides of which bound, together with $\overline{C}$, the region of $\Lambda$ to the left of $\overline{C}$. Note that $\overline{C}$ doesn't actually depend on the cones $\Lambda$ and $\Lambda'$, but only on the left neighborhood of $C$ in $P$, and hence this development is the same for $\Lambda$ and $\Lambda'$. So, up to planar isometries, the planar unfolding of the cone supporting $C$ is unique, and thus the cone itself and the position of $C$ on it are unique up to isometries. $\square$

Note that this lemma does not assume that $C$ is convex; rather it holds for any closed curve $C$.

Finally we establish the visibility property mentioned in the introduction.

**Lemma 4** *A convex curve $C$ on $\mathcal{P}$ is visible from the apex $a$ of the unique cone $\Lambda$ on which it lives to its convex side.*

**Proof:** With $C$ directed so that its convex side is its left side, which we may consider its interior, the apex $a$ is inside $C$. Assume there is a cone generator intersecting $C$ twice. Then, rotating the generator around the apex in one direction or the other eventually must reach a generator $ax$ tangent to $C$ at $x$ where $L(x) > \pi$, contradicting convexity. See Figure 6. □

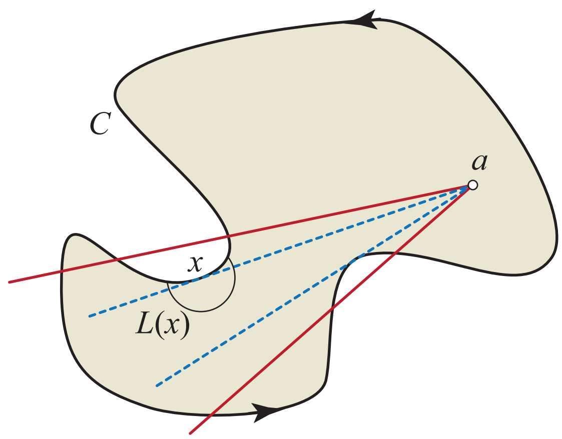


Figure 6: No generator may cross $C$ twice.

This lemma may as well be established with a different proof, whose sketch is as follows. Let $z$ be the closest point of $C$ to $a$. Then $az$ must be orthogonal to $C$ at $p$. Inserting a "curvature triangle" along $az$ with apex angle $\omega(a)$ flattens $P$ to a planar domain with a convex boundary, and visibility from $a$ follows.

We gather the previous three lemmas into a summarizing theorem:

**Theorem 1** *Any curve $C$, convex to its left, lives on a unique cone $\Lambda_L$ to its left side. $\Lambda_L$ has curvature $\Omega_L$ at its apex, and so has apex angle $2\pi - \Omega_L$. Every point of $C$ is visible from the cone apex $a$.*

# 4 Convex Loops

Consider the polyhedron $\mathcal{P}$ shown in Figure 7(a), which is a variation on the example from Figure 3(a). Here $C = (a, b, b', x, c', c, d)$ is a convex loop, with loop point $x$. The cone on which it should live is analogous to Figure 3(b): vertex merging of $v_1$ and $v_2$ again produces the cone apex $v'$ whose curvature

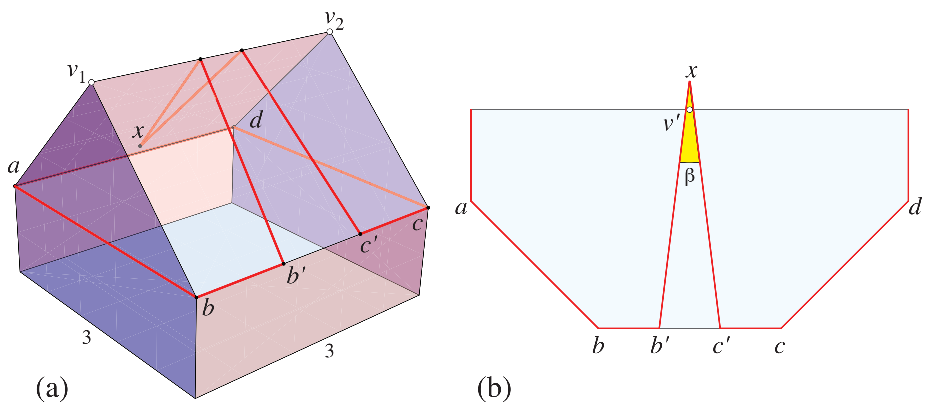


Figure 7: (a) A convex loop $C$ that does not live on a cone. (b) A flattening of the cone on which it should live. (Base of $\mathcal{P}$ is $3 \times 3$.)

is $\pi$. But $C$ does not "fit" on this cone, as Figure 7(b) shows; the apex $a = v'$ is not inside $C$.

We remark that, if the central "spike" $(b', x, c')$ is shortened, it does live on the cone. Even for convex loops that do live on a cone, there are examples that fail to satisfy the visibility property, Lemma 4. Simply shifting the spike in this example to one side of $v'$ blocks visibility to portions of $C$.

# 5 Reflex Curves and Reflex Loops

Recall that, for each corner $c_i$ of a curve $C$, $\alpha_i + \omega_i + \beta_i = 2\pi$, where $\alpha_i$ and $\beta_i$ are the left and right angles at $c_i$ respectively, and $\omega_i$ is the Gaussian curvature at $c_i$. When $C$ is vertex-free, $\omega_i = 0$ at all corners, and the relationships among the curve classes is simple and natural: the other side of a convex curve is reflex, the other side of a reflex curve is convex. The same holds for the loop versions: the other side of a convex loop is a reflex loop (because $\alpha_m \geq \pi$ implies $\beta_m \leq \pi$, where $c_m$ is the loop point), and the other side of a reflex loop is a convex loop. When $C$ includes vertices, the relationships between the curve classes is more complicated. The other side of a convex curve is reflex only if the curvatures at the vertices on $C$ are small enough so that $\alpha_i + \omega_i \leq \pi$; $C$ would still be convex even if it just included those vertices inside. The same holds for convex loops, as summarized in the table below.

On the other hand, the other side of a reflex curve is always convex, because nonzero vertex curvatures only make the other side more convex. The other side of a reflex loop is a convex loop, and it is a convex curve if the curvature at the loop point $c_m$ is large enough to force $\alpha_m \leq \pi$, i.e., if $\beta_m + \omega_m \geq \pi$.

This latter subclass of reflex loops—those whose other side is convex—especially interest us, because any convex curve that includes at most one vertex is a reflex loop of that type. All our results in this section hold for this class of

| *Curve class* | *Other side, and condition* |
|---|---|
| convex | reflex only if $\forall i$, $\alpha_i + \omega_i \le \pi$ |
| convex loop | reflex loop only if $\forall i \neq m$, $\alpha_i + \omega_i \le \pi$ (necessarily, $\beta_m \le \pi$) |
| reflex | convex (always) |
| reflex loop | convex loop (always), and convex if $\beta_m + \omega_m \ge \pi$ |

Table 2: Other-side conditions for curve classes. $m$ indexes the loop-point corner $c_m$ for loop versions.

curves.

**Lemma 5** *Let $C$ be a curve that is either reflex (to its right), or a reflex loop which is convex to the other (left) side, with $\beta_m < \pi$ at the loop point $c_m$. Then $C$ lives on a cone $\Lambda_R$ to its reflex side.*

**Proof:** Again let $c_1, c_2, \ldots, c_m$ be the corners of $C$, with $c_m$ the loop point if $C$ is a reflex loop. Because $C$ is convex to its left, we have $\Omega_L \le 2\pi$. Just as in Lemma 2, merge the vertices strictly in $P_L$ to one vertex $a$. Let $\Lambda_L$ be the cone with apex $a$ on which $C$ now lives. It will simplify subsequent notation to let $\Lambda = \Lambda_L$.

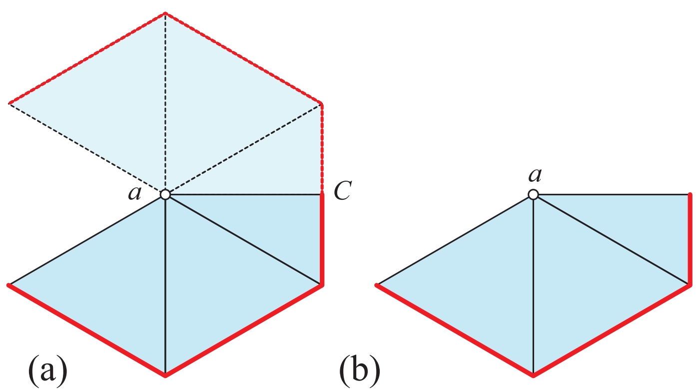


Figure 8: The cone $\Lambda$ for $C$ in Figure 2(a), opened (a) and doubly covered (b).

Let $N_R$ be a (small) right neighborhood of $C$, a neighborhood to the reflex side of $C$. For subsequent subscript embellishment, we use $N$ to represent $N_R$. Its shape is irrelevant to the proof, as long as it is vertex free and its left boundary is $C$.

Join $a$ to each corner $c_i$ by a cone-generator $g_i$ (a ray from $a$ on $\Lambda$). Lemma 4 ensures this is possible. Cut along $g_i$ beyond $c_i$ into $N$. There are choices how to extend $g_i$ beyond $c_i$, but the choice does not matter for our purposes. For example, one could choose a cut that bisects $\beta_i$ at $c_i$. Insert along each cut into $N$ a *curvature triangle*, that is, an isosceles triangle with two sides equal to the cut length, and apex angle $\omega_i$ at $c_i$. (If $c_i$ does not coincide with a vertex of $\mathcal{P}$, then $\omega_i = 0$ and no curvature triangle is inserted.) This flattens the surface at $c_i$, and "fattens" $N$ to $N'$ without altering $C$ or the cone $\Lambda$ up to $C$. Now $N'$ lives on the same cone $\Lambda$ that $C$ and its left neighborhood $N_L$ do.

From now on we view $\Lambda$ and the subsequent cones we will construct as flattened into the plane, producing a doubly covered cone with half the apex angle. (Notice that here "doubly covered" above refers to a neighborhood of the cone apex, and not to the image of the curve $C$.) It is always possible to choose any generator $ax$ for $x \in C$ and flatten so that $ax$ is the leftmost extreme edge of the double cone. We start by selecting $x = c_1$, so that $g_1$ is the leftmost extreme; let $h_1$ be the rightmost extreme edge. We pause to illustrate the construction before proceeding.

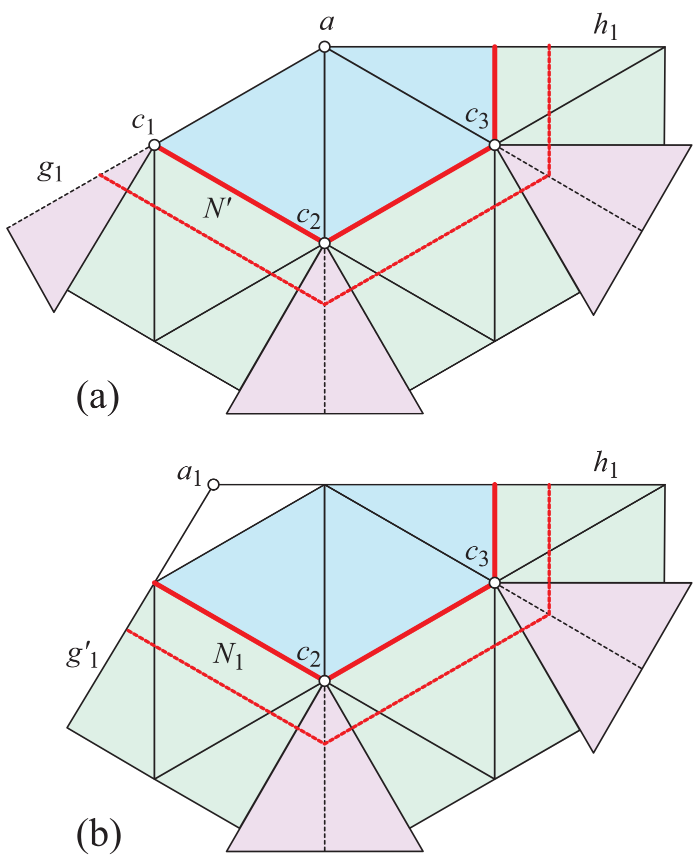


Figure 9: (a) After insertion of curvature triangles, $N'$ lives on $\Lambda$. (b) Removing the doubly covered half curvature triangle at $c_1$ leads to a new cone $\Lambda_1$. (In this and in Figure 10 we display the full icosahedron faces to the right of $C$, although only a small neighborhood is relevant to the proof.)

Let $C$ be the curve on the icosahedron illustrated in Figure 2(a). This curve already lives on the cone $\Lambda$ without any vertex merging. Figure 8(a) shows the five equilateral triangles incident to the apex, and (b) shows the corresponding doubly covered cone. Figure 9(a) illustrates $\Lambda$ after insertion of the curvature triangles, each with apex angle $\omega_i = \frac{1}{3}\pi$. A possible neighborhood $N'$ is outlined.

After insertion of all curvature triangles, we in some sense erase where they were inserted, and just treat $N'$ as a band living on $\Lambda$. Now, with $g_1$ the leftmost extreme, we identify a half-curvature triangle on the front side, matched by a half-curvature triangle on the back side, incident to $c_1$ in $N'$. Each triangle has angle $\frac{1}{2}\omega_1$ at $c_1$. See again Figure 9(a). Now rotate $g_1$ counterclockwise about $c_1$ by $\frac{1}{2}\omega_i$, and cut out the two half-curvature triangles from $N'$, regluing the front to the back along the cut segment. Extend the rotated line $g_1'$ to meet the extension of $h_1$. Their intersection point is the apex $a_1$ of a new (doubly covered) cone $\Lambda_1$, on which neither $a$ nor $c_1$ are vertices. Note that the rotation of $g_1$ effectively removes an angle of measure $\omega_1$ incident to $c_1$ from the $N'$ side, and inserts it on the other side of $C$. See Figure 9(b). Call the new neighborhood $N_1$, and the new convex curve $C_1$. $C_1$ is the same as $C$ except that the angle at $c_1$ is now $\alpha_1 + \omega_1$, which by the assumption of the lemma, is still convex because $\beta_1 \geq \pi$.

Now we argue that $g_1'$ does not intersect $N_1$ other than where it forms the leftmost boundary. For if $g_1'$ intersected $N_1$ elsewhere, then, taking $N_1$ to be smaller and smaller, tending to $C_1$, we conclude that $g_1'$ must intersect $C_1$ at a point other than $c_1$. But this contradicts the fact that either of the two planar images (from the two sides of $\Lambda$) of $C_1$ is convex. Indeed $g_1'$ is a supporting line at $c_1$ to the convex set constituted by $\Lambda_1$ up to $C_1$.

Note that we have effectively merged vertices $c_1$ and $a$ to form $a_1$, in a manner similar to the vertex merging used in Lemma 2. The advantage of the process just described is that it does not rely on having a triangle half-angle no more than $\pi$ at the new cone apex.

Next we eliminate the curvature triangle inserted at $c_2$. Let $g_2$ be the generator from $a_1$ through $c_2$ (again, Lemma 4 applies). Identify a curvature triangle of apex angle $\omega_2$ in $N_1$ bisected by $g_2$; see Figure 10(a). Now reflatten the cone $\Lambda_1$ so that $g_2$ is the left extreme, and let $h_2$ be the right extreme, as in (b) of the figure. Rotate $g_2$ by $\frac{1}{2}\omega_2$ about $c_2$ to produce $g_2'$, cut out the half-curvature triangles on both the front and back of $N_1$, and extend $g_2'$ to meet the extension of $h_2$ at a new apex $a_2$. Now we have a new neighborhood $N_2$, with left boundary the convex curve $C_2$, living on a cone $\Lambda_2$.

We apply this process through $c_1, \ldots, c_{m-1}$. It could happen at some stage that $g_i'$ and the $h_i$ extension meet on the other side of $C_i$, in which case the cone apex is to the reflex side. (Or, they could be parallel and meet "at infinity," which is what occurs with the icosahedron example.) From the assumption of the lemma that $\beta_i \geq \pi$ for $i < m$, $\alpha_i + \omega_i \leq \pi$ and so the curves $C_i$ remain convex throughout the process. So the argument above holds.

For the last, possibly exceptional corner $c_m$, $C_{m-1}$ from the previous step is convex, but the final step could render $C_m$ nonconvex (if $\alpha_m + \omega_m > \pi$). But as there is no further processing, this nonconvexity does not affect the proof. $\square$

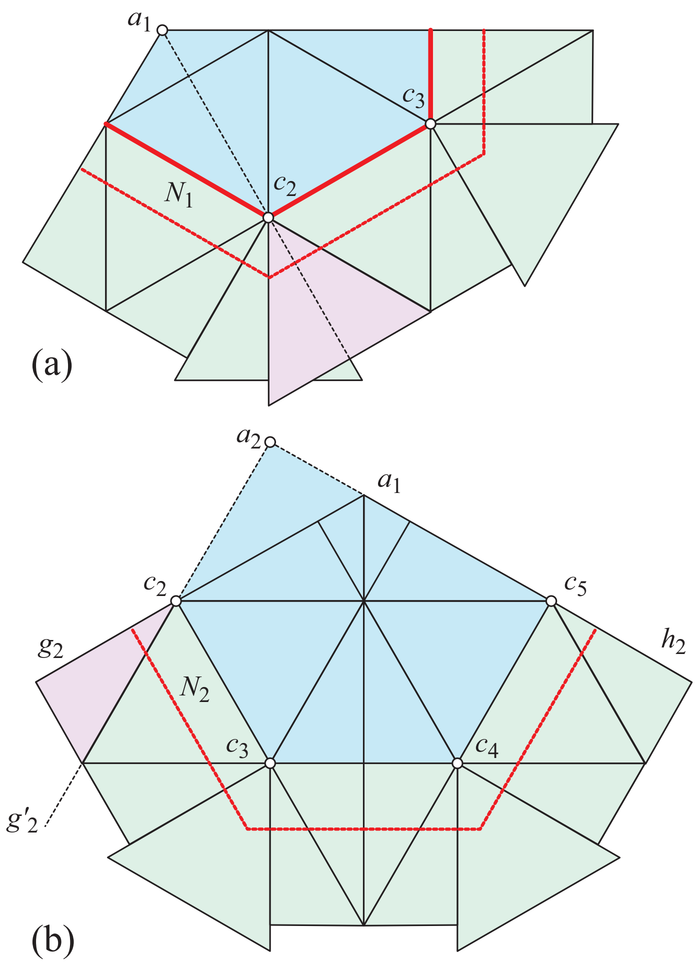


Figure 10: (a) Generator $g_2$ from $a_1$ through $c_2$ into $N_1$. (b) Reoriented so $g_2$ is left extreme.

For the icosahedron example, five insertions of $\frac{1}{3}$ curvature triangles, together with the original $\frac{1}{3}$ curvature at $a$, produces a cylinder. And indeed, $\beta_i = \pi$ for the five $c_i$ corners of $C$, and $C$ forms a circle on a cylinder.

**Lemma 6** *Let $C$ be a curve satisfying the same conditions as for Lemma 5. Then $C$ is visible from the apex $a$ of the cone $\Lambda$ on which it lives to its reflex side.*

**Proof:** Again letting $c_1, \ldots, c_m$ be the corners of $C$, with $c_m$ the possibly exceptional vertex, we know that $\beta_i \geq \pi$ for $i = 1, \ldots, m{-}1$, but it may be that $\beta_m < \pi$. Just as in the proof of Lemma 5, we flatten $\Lambda$ into the plane, this time choosing $c_m$ to lie on the leftmost extreme generator $L_1$of $\Lambda$. Let $b$ be the point of $C$ that lies on the rightmost extreme generator $L_2$ in this flattening. Finally, let $C_u$ be the portion of $C$ on the upper surface of the flattened $\Lambda$, and $C_l$ the portion on the lower surface. See Figure 11. Now that we have

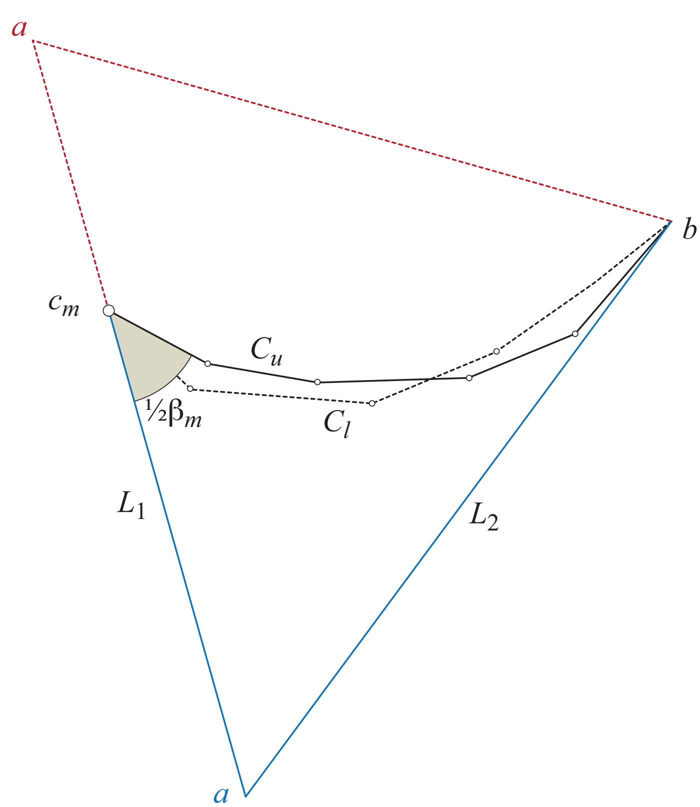


Figure 11: The apex $a$ could lie either to the reflex or to the convex side of $C$.

placed the one anomalous corner on the extreme boundary $L_1$, both $C_u$ and $C_l$ present a uniform aspect to the apex $a$, whether it is to the convex or reflex side of $C$: every corner of $C_u$ and $C_l$ is reflex (or flat) toward the reflex side, and convex (or flat) toward the convex side. In particular, $c_m b \cup C_u$ is a planar convex domain. Each line through $a$ intersects $c_m b$ exactly once, and therefore intersects $C_u$ exactly once; and similarly for $C_l$. □

Just as we observed for convex loops, this visibility lemma does not hold for all reflex loops—the assumption that the other side is convex is essential to the proof.

We summarize this section in a theorem (recall that $\Omega_L + \Omega_C + \Omega_R = 4\pi$).

**Theorem 2** *A curve $C$ that is either reflex (to its right), or a reflex loop which is convex to the other (left) side, lives on a unique cone $\Lambda_R$ to its reflex side. If $\Omega_R > 2\pi$, then the reflex neighborhood $N_R$ is to the unbounded side of $\Lambda_R$, i.e., the apex of $\Lambda_R$ is left of $C$; if $\Omega_R < 2\pi$, then $N_R$ is to the bounded side, i.e., the apex of $\Lambda_R$ is to the right side of $C$. If $\Omega_R = 2\pi$, $C \cup N_R$ lives on a cylinder. In all cases, every point of $C$ is visible from the cone apex $a$.*

**Proof:** The uniqueness follows from Lemma 3. The cone $\Lambda_R$ constructed in the proof of Lemma 5 results in the cone apex to the convex side of $C$ as long as $\Omega_L + \Omega_C \leq 2\pi$, when $\Omega_R \geq 2\pi$. Excluding the cylinder cases, this justifies the claims concerning on which side of $\Lambda_R$ the neighborhood $N_R$ resides. The apex curvature of $\Lambda_R$ is $\min\{\Omega_L + \Omega_C, \Omega_R\}$. □

*Example.* An example of a reflex loop that satisfies the hypotheses of Theorem 2 is shown in Figure 12(a). Here $C$ has five corners, and is convex to one side at each. $C$ passes through only one vertex of the cuboctahedron $\mathcal{P}$, and so it is reflex at the four non-vertex corners to its other side. Corner $c_5$ coincides with a vertex of $\mathcal{P}$, which has curvature $\omega_5 = \frac{1}{3}\pi$. Here $\alpha_5 = \beta_5 = \frac{5}{6}\pi$. Because $\beta_5 < \pi$, $C$ is a reflex loop. We have $\Omega_L = \frac{2}{3}\pi$ because $C$ includes two cuboctahedron vertices, $u$ and $v$ in the figure. $\Omega_C = \omega_5 = \frac{1}{3}\pi$. And therefore $\Omega_R = 3\pi$. The apex curvature of $\Lambda_L$ is $\Omega_L = \frac{2}{3}\pi$, and the apex curvature of $\Lambda_R$ is $\min\{\Omega_L + \Omega_C, \Omega_R\} = \pi$. $N_R$ lives on the unbounded side of this cone, which is shown shaded in Figure 12(b). Note the apex $a$ is left of $C$.

# 6 Summary and Extensions

## 6.1 Summarizing Theorem

Putting Theorems 1 and 2 together, we obtain:

**Theorem 3** *For the following classes of curves $C$ on a convex polyhedron $\mathcal{P}$, we may conclude that $C$ lives on a unique cone to both sides, and is visible from the apex of each cone:*

1. *$C$ is a quasigeodesic (because they are convex to both sides).*
2. *$C$ is convex and passes through no vertices (because then the other side is reflex).*
3. *$C$ is convex and passes through one vertex (because then the other side is a reflex loop whose other side is convex).*

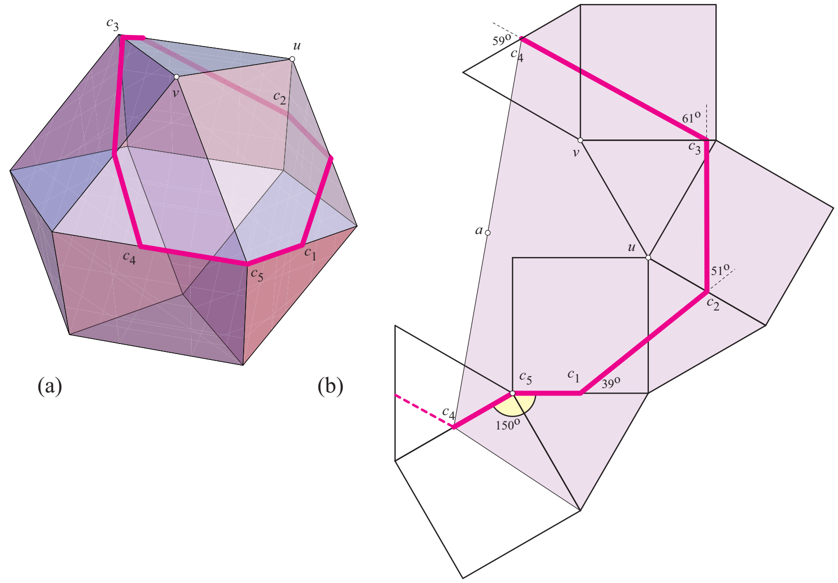


Figure 12: (a) A curve $C$ of five corners passing through one polyhedron vertex. $C$ is a convex to one side, and a reflex loop to the other, with loop point $c_5$, at which $\beta_5 = \frac{5}{6}\pi (= 150°) < \pi$. (b) The cone $\Lambda_R$ with apex $a$ is shaded.

4. *$C$ is convex and passes through several vertices such that, at all but at most one corner $c_i$ of $C$, $\alpha_i + \omega_i \leq \pi$. In this situation, $C$ is a reflex loop to the other side because $\beta_i \geq \pi$ at all but at most one vertex.*

## 6.2 Quasigeodesic Loops

Our extension of the source unfolding of a polyhedron [IOV09] (Section 7.3 below) holds for classes of curves living on a cone to both sides, while our extension of the star unfolding of a polyhedron [IOV10] works for any quasigeodesic loop. It is therefore natural to explore extending Theorem 3 to encompass quasigeodesic loops. Recall that quasigeodesic loops are convex to one side, and convex loops to the other. Despite quasigeodesic loops being very special convex loops, we show by example that there are quasigeodesic loops which fail to satisfy Theorem 3 in that they do not live to a cone to both sides.

The construction is a modification of the example in Figure 7 showing that a convex loop might not live on a cone. In that example, $C$ is a convex loop to the left; we modify the example so that it becomes convex to its right. Let $\mathcal{P}$ be the polyhedron in Figure 7(a). Essentially we will retain $P_L$, the left half of $\mathcal{P}$, and replace $P_R$ with a different surface to produce a new polyhedron $\mathcal{P}^*$. Toward that end, add a new vertex $e$ at the midpoint of edge $ad$ of $\mathcal{P}$. Although we could make $e$ a true vertex with non-zero curvature, it is easiest to see the construction when $\omega(e) = 0$. Let $C^*$ be the new curve, $C^* = (a, b, b', x, c', c, d, e)$, geometrically the same as $C$ but now including $e$ on the path between $a$ and $d$. So $C^*$ is still a convex loop to its left. Let $\beta = \angle b'xc'$ be the convex angle at the loop point $x$.

Now construct a planar convex polygon $Q = (\overline{a}, \overline{b}, \overline{b'}, \overline{x}, \overline{c'}, \overline{c}, \overline{d}, \overline{e})$, each of whose edges has the same length as the corresponding edge of $C^*$—$|\overline{a}\overline{b}| = |ab|$, etc.—and such that $\angle \overline{b'}\overline{x}\overline{c'} = \beta$, matching $\angle b'xc'$. These conditions do not uniquely determine $Q$, but any $Q$ that is convex and has angle $\beta$ at $x$ suffices for the construction. See Figure 13(a).

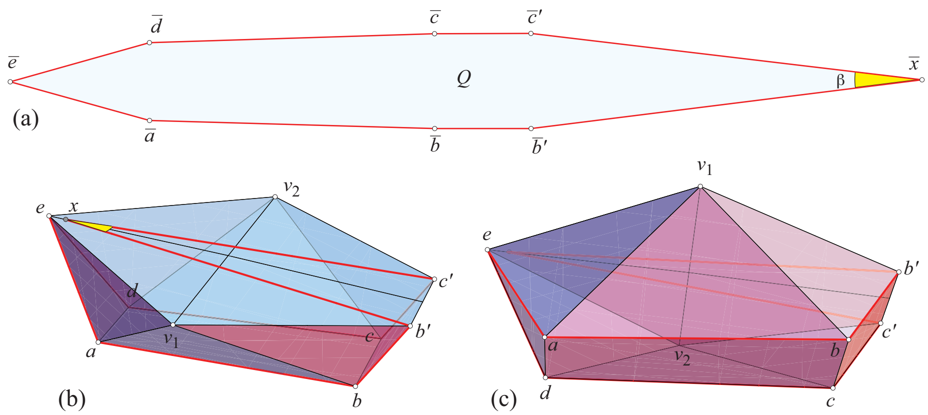


Figure 13: (a) Convex polygon $Q$. (b,c) Two views of $\mathcal{P}^*$. The dihedral angle at the "roof edge" $v_1v_2$ was $\frac{1}{4}\pi$ in Figure 7(a) but is nearly $\pi$ in $\mathcal{P}^*$. (The 3D shape here is only approximate, constructed via ad hoc computations.)

$\mathcal{P}^*$ is now constructed by gluing $P_L$, the top half of $\mathcal{P}$, to $Q$, matching corresponding vertices, $\overline{a}$ to $a$, etc. Alexandrov's Gluing Theorem guarantees that the resulting surface corresponds to a unique convex polyhedron $\mathcal{P}^*$. Figure 13(b,c) shows an approximation to $\mathcal{P}^*$. $C^*$ is a quasigeodesic loop on $\mathcal{P}^*$: a convex loop to the left and convex by construction to the right. $C^*$ lives on a (planar) cone to the right, but does not live on a cone to its left for the same reason that $C$ did not on $\mathcal{P}$: it does not fit.

We have established that convex loops always live on the union of two cones,[3] but we leave that a claim not pursued here.

[3] Very roughly, we cut from the exceptional loop point $x$ via a geodesic to a point $y$ on $C$, yielding two convex curves $C_1$ and $C_2$ sharing $xy$, each of which lives on a cone. (This technique was used in [IOV10].)

# 7 Applications

## 7.1 Development of Curve on Cone

Nonoverlapping development of curves plays a role in unfolding polyhedra without overlap [DO07]. Any result on simple (non-self-intersecting) development of curves may help establishing nonoverlapping surface unfoldings. One of the earliest results in this regard is [OS89], which proved that the left development of a directed, closed convex curve does not self-intersect. The proof used Cauchy's Arm Lemma. The new viewpoint in our current work reproves this result without invoking Cauchy's lemma, and extends it to a wider class of curves.

Every simple, closed curve $C$ drawn on a cone $\Lambda$ and which encloses the apex $a$ of $\Lambda$ may be developed on the plane by rolling $\Lambda$ on that plane. More specifically, select a point $x \in C$ and develop $C$ from $x$ back to $x$ again. We call this curve in the plane $\overline{C}_x$. Once $x$ is selected, the development is unique up to congruence in the plane. There is no distinction between right and left developments of a curve on a cone; that distinction only applies when there is nonzero curvature along $C$, as there may be on the surface of a polyhedron $\mathcal{P}$. If $g$ is a generator of $\Lambda$ that meets $C$ in one point $\{x\} = g \cap C$ —a condition guaranteed by our visibility lemmas (Lemmas 4 and 6) —then $\overline{C}_x$ is non-self-intersecting, because the unrolling of the entire cone is non-overlapping. Thus we obtain from Theorem 3 a broader class of curves on $\mathcal{P}$ that develop without intersection, including reflex loops whose other side is convex.

## 7.2 Overlapping Developments

In general, $\overline{C}_x$ is not congruent to $\overline{C}_y$ when $x \neq y$. We are especially interested in those $C$ for which $\overline{C}_x$ is simple (non-self-intersecting) for every choice of $x$, and we have just identified a class for which this holds. Here we show that there exist $C$ such that $\overline{C}_x$ is nonsimple for every choice of $x$. We provide one specific example, but it can be generalized.

The cone $\Lambda$ has apex angle $\alpha = \frac{3}{4}\pi$; it is shown cut open and flattened in two views in Figure 14(a,b). An open curve $C' = (p_1, p_2, p_3, p_4, p_5)$ is drawn on the cone. Directing $C'$ in that order, it turns left by $\frac{3}{4}\pi$ at $p_2$, $p_3$, and $p_4$. From $p_5$, we loop around the apex $a$ with a segment $S = (p_5, p_6, p_5')$, where $p_5'$ is a point near $p_5$ (not shown in the figure). Finally, we form a simple closed curve on $\Lambda$ by then doubling $C'$ at a slight separation (again not illustrated in the figure), so that from $p_5$ it returns in reverse order along that slightly displaced path to $p_1$ again. Note that $C = C \cup S \cup C'$ is both closed and includes the apex $a$ in its (left) interior.

Now, let $x$ be any point on $C$ from which we will start the development $\overline{C}_x$. Because $C$ is essentially $C' \cup C'$, $x$ must fall in one or the other copy of $C'$, or at their join at $p_1$. Regardless of the location of $x$, at least one of the two copies of $C'$ is unaffected. So $\overline{C}_x$ must include $\overline{C'}$ as a subpath in the plane.

Finally, developing $C'$ reveals that it self-intersects: Figure 14(c). Therefore, $\overline{C}_x$ is not simple for any $x$. Moreover, it is easy to extend this example to force

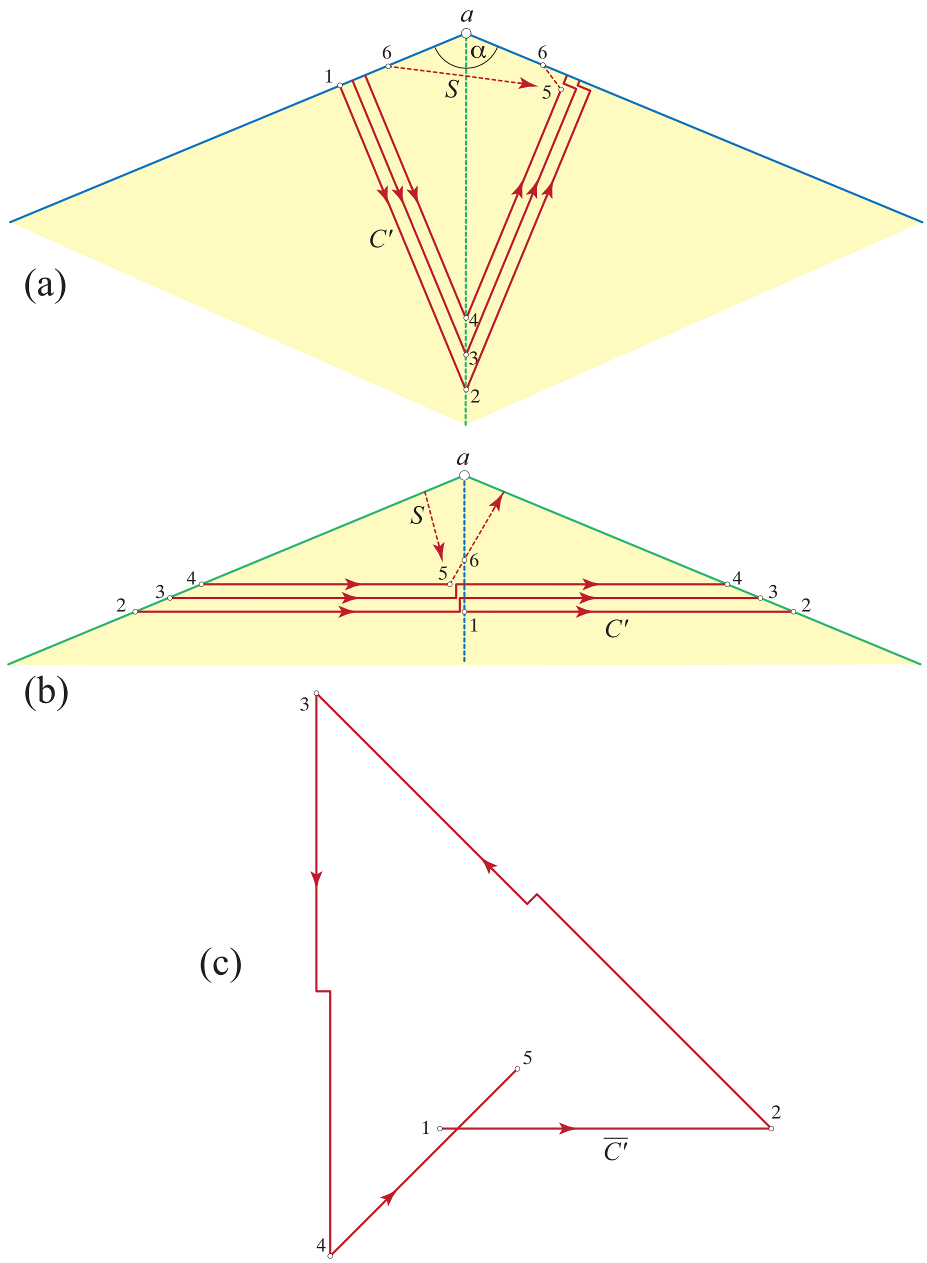


Figure 14: (a) Open curve $C' = (p_1, p_2, p_3, p_4, p_5)$ on cone of angle $\alpha$, with cone opened. (b) A different opening of the same cone and curve. (c) Development of curve $\overline{C'}$ self-intersects.

self-intersection for many values of $\alpha$ and analogous curves. The curve $C'$ was selected only because its development is self-evident.

### 7.3 Source Unfolding

Every point $x$ on the surface of a convex polyhedron $\mathcal{P}$ leads to a nonoverlapping unfolding called the *source unfolding of* $\mathcal{P}$ *with respect to* $x$, obtained by cutting $\mathcal{P}$ along the cut locus of $x$. We can think of this as the *source unfolding with respect to a point* $x$. We have generalized in [IOV09] this unfolding to unfold $\mathcal{P}$ by cutting —roughly speaking— along the cut locus of a simple closed curve $C$ on $\mathcal{P}$. This unfolding is guaranteed to avoid overlap when $C$ lives on a cone to both sides. So it applies in exactly the conditions specified in Theorem 3, and this is a central motivation for our work here.

## 8 Open Problems

We have not completely classified the curves $C$ on a convex polyhedron $\mathcal{P}$ that live on a cone to both sides. Theorem 3 summarizes our results, but they are not comprehensive.

### 8.1 Slice Curves

One particular class we could not settle are the slice curves. A *slice curve* $C$ is the intersection of $\mathcal{P}$ with a plane. Slice curves in general are not convex. The intersection of $\mathcal{P}$ with a plane is a convex polygon in that plane, but the surface angles of $\mathcal{P}$ to either side along $C$ could be greater or smaller than $\pi$ at different points. Slice curves were proved to develop without intersection, to either side, in [O'R03], so they are strong candidates to live on cones. However, we have not been able to prove that they do. We can, however, prove that every convex curve on $\mathcal{P}$ is a slice curve on some $\mathcal{P}'$ (this follows from [Ale05, Thm. 2, p. 231]), and either side of any slice curve on $\mathcal{P}$ is the other side of a convex curve on some $\mathcal{P}'$.

### 8.2 Curve with a Nested Convex Curve

We can extend the class of curves to which Lemma 2 (the convex-curve lemma) applies beyond convex, but the extension is not truly substantive. Let $C$ be a simple closed curve which encloses a convex curve $C'$ such that the region of $\mathcal{P}$ bounded between $C$ and $C'$ contains no vertices. See, e.g., Figure 15. Then the proof of Lemma 2 applies to $C'$ and $C$ lives on the same cone as $C'$.

### 8.3 Cone Curves

We have not obtained a complete classification of the curves on a cone that develop, for every cut point $x$, as simple curves in the plane. It would also be

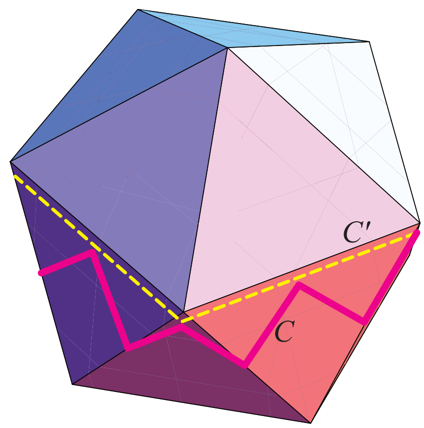


Figure 15: $C'$ is convex (it is a geodesic) and $C$ lives on the same cone (in this case a cylinder) as does $C'$.

interesting to identify the class of curves on cones for which there exists at least one cut-point $x$ that leads to simple development.

**Acknowledgments.** We thank Jin-ichi Itoh, our coauthor on [IOV09] and [IOV10], the papers that motivated this investigation.

## References

[Ale05] Aleksandr D. Alexandrov. *Convex Polyhedra*. Springer-Verlag, Berlin, 2005. Monographs in Mathematics. Translation of the 1950 Russian edition by N. S. Dairbekov, S. S. Kutateladze, and A. B. Sossinsky.

[DO07] Erik D. Demaine and Joseph O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*. Cambridge University Press, July 2007. `http://www.gfalop.org`.

[IOV09] Jin-ichi Itoh, Joseph O'Rourke, and Costin Vîlcu. Source unfoldings of convex polyhedra with respect to certain closed polygonal curves. In *Proc. 25th European Workshop Comput. Geom.*, pages 61–64. EuroCG, March 2009. Full version submitted to a journal, May 2009.

[IOV10] Jin-ichi Itoh, Joseph O'Rourke, and Costin Vîlcu. Star unfolding convex polyhedra via quasigeodesic loops. *Discrete Comput. Geom.*, 44:35–54, 2010.

[O'R03] Joseph O'Rourke. On the development of the intersection of a plane with a polytope. *Comput. Geom. Theory Appl.*, 24(1):3–10, 2003.

[OS89] Joseph O'Rourke and Catherine Schevon. On the development of closed convex curves on 3-polytopes. *J. Geom.*, 13:152–157, 1989.