Draining a Polygon –or– Rolling a Ball out of a Polygon

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Abstract

We introduce the problem of draining water (or balls representing water drops) out of a punctured polygon (or a polyhedron) by rotating the shape. For 2D polygons, we obtain combinatorial bounds on the number of holes needed, both for arbitrary polygons and for special classes of polygons. We detail an $O(n^2 \log n)$ algorithm that finds the minimum number of holes needed for a given polygon, and argue that the complexity remains polynomial for polyhedra in 3D. We make a start at characterizing the *1-drainable* shapes, those that only need one hole.

1 Introduction

Imagine a closed polyhedral container P partially filled with water. How many surface point-holes are needed to entirely drain it under the action of gentle rotations of P? It may seem that one hole suffices, but we will show that in fact sometimes $\Omega(n)$ holes are needed for a polyhedron of n vertices. Our focus is on variants of this problem in 2D, with a brief foray in Sec. 5 into 3D. We address the relationship between our problem and injection-filling of polyhedral molds [BvKT98] in Sec. 4.

A second physical model aids the intuition. Let P be a 2D polygon containing a single small ball. Again the question is: How many holes are needed to ensure that the ball, regardless of its initial placement, will escape to the exterior under gentle rotation of P? Here the ball is akin to a single drop of water. We will favor the ball analogy, without forgetting the water analogy.

Models. We consider two models, the (gentle) *Rotation* and the *Tilt* models. In the first, P lies in a vertical xy-plane, and gravity points in the $-y$ direction. The ball B sits initially at some convex vertex v_i; vertices are labeled counterclockwise (ccw). Let us assume that v_i is a local minimum with respect to y, i.e., both v_{i-1} and v_{i+1} are above v_i. Now we are permitted to rotate P in the vertical plane (or equivalently, alter the gravity vector). In the Rotation model, B does not move from v_i until one of the two adjacent edges, say $e_i = v_i v_{i+1}$, turns infinitesimally beyond the horizontal, at which time B rolls down e_i and falls under the influence of gravity until it settles at some other convex vertex v_j. For example, in Fig. 1, B at v_4 rolls ccw

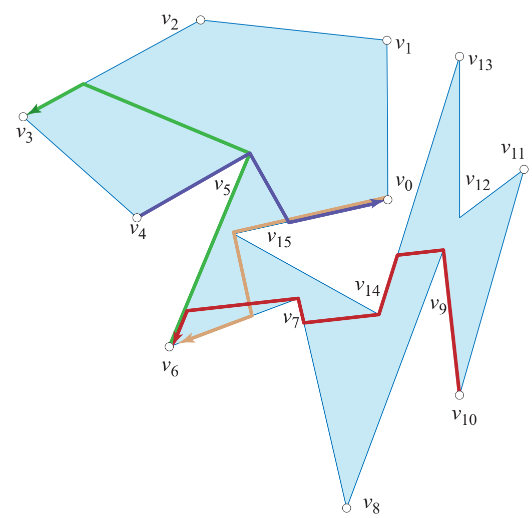

Figure 1: Polygon with several ball paths.

when v_4v_5 is horizontal, falls to edge $v_{15}v_0$, and comes to rest at v_0. Similarly, B at v_{10} rolls clockwise (cw) to v_6 after three falls. Note that all falls are parallel, and (arbitrarily close to) orthogonal to the initiating edge (in the Rotation model). After B falls to an edge, it rolls to the endpoint on the obtuse side of its fall path.

The only difference in the Tilt model is that any gravity vector may be selected. Only vectors between e_{i-1} and e_i will initiate a departure of B from v_i, i.e., the entire wedge is available rather than just the two incident edges. For example, in Fig. 1, B at v_4 rolls to $\{v_0, v_3\}$ in the Rotation model, but can roll to $\{v_0, v_1, v_2, v_3\}$ in the Tilt model. The Rotation model more accurately represents physical reality, for rain drops or for balls. The Tilt model mimics various ball-rolling games

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(e.g., Labyrinth) that permit quickly "tilting" the polygon/maze from the horizontal so that any departure vector from v_i can be achieved. We emphasize that, aside from this departure difference, the models are identical. In particular, inertia is ignored, and rotation while the ball is "in-flight" is forbidden (otherwise we could direct B along any path).

There are two "degenerate" situations that can occur. If B falls exactly orthogonal to an edge e, we arbitrarily say it rolls to the cw endpoint of e. If B falls directly on a vertex, both of whose edges angle down with respect to gravity, we stipulate that it rolls to the cw side.

Questions. Given P, what is the minimum number of point-holes needed to guarantee that any ball, regardless of starting position, may eventually escape from P under some sequence of rotations/tilts? Our main result is that this number can be determined in $O(n^2 \log n)$ time. In terms of combinatorial bounds, we show that some polygons require $\lfloor n/6 \rfloor$ and $\lfloor n/7 \rfloor$ holes (in the Rotation/Tilt models respectively), but $\lceil n/4 \rceil$ holes always suffice. We make a start at characterizing the *1-drainable* polygons, those that only need one hole Finally we argue that the minimum number of holes can be computed for a 3D polyhedron in polynomial time.

2 Traps

We start by exhibiting polygons that need $\Omega(n)$ holes to drain. The basic idea is shown in Fig. 2(a) for the Rotation model. We create *traps* with 5 vertices forming

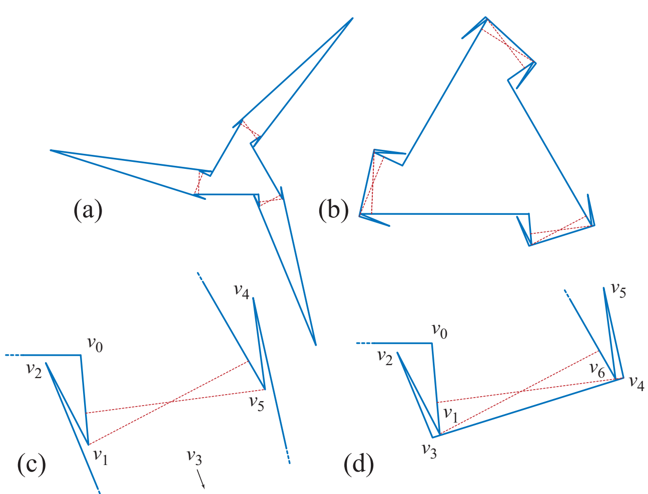

Figure 2: (a) Trap for Rotation model. (b) Trap for Tilt model. (c,d) Details of traps.

an "arrow" shape, connected together around a convex polygonal core so that 6 vertices are needed per trap. A ball in v_4 rolls to fall on edge v_0v_1, but because of the slightly obtuse angle of incidence, rolls to v_2; and symmetrically, v_2 leads to v_4. So there is a cycle (defined precisely in Sec. 3) that "traps" ball between $\{v_2, v_3, v_4\}$ and isolates it from the other two traps. Therefore three holes are required to drain this polygon. In the Tilt model, Fig. 2(a) only needs one hole, because B could roll directly from v_3 through the $v_1{-}v_5$ "gap." However, the polygon in Fig. 2(b) requires 3 holes. Here the range of effective gravity tilt vectors from v_4 is so narrow that the previous analysis holds. These examples establish the necessity half of this theorem:

Theorem 1 (Combinatorial Bounds) *In the Rotation (resp. Tilt) model,* $\lfloor n/6 \rfloor$ *(resp.* $\lfloor n/7 \rfloor$*) holes are sometimes necessary to drain an* n*-vertex polygon.* $\lceil n/4 \rceil$ *holes suffice to drain any polygon.*

Proof. We will show below that a sequence of rotations leads to a cycle (in a graph defined in Sec. 3) of length ≥ 3. So any trap must have at least 3 vertices. In order for this cycle not to include the entire polygon, there must be at least one reflex vertex in a path for an arc of the cycle that leads to a fall that "hides" part of the polygon. So any trap needs at least 4 vertices. This establishes the sufficiency half of the claim. □

Although we believe that at least two reflex vertices are needed in every cycle, we were unable to show that they could not be shared between traps. We nevertheless conjecture that $\lceil n/5 \rceil$ holes suffice. (A variation on Fig. 2(a) permits the formation of two traps with $n = 11$.)

Proposition 2 $\lfloor n/28 \rfloor$ *holes are sometimes necessary to drain an* n*-vertex orthogonal polygon, and* $\lceil n/8 \rceil$ *holes suffice.*

Proof. The example that establishes the lower bound is shown in Fig. 3. A ball at v_1 can only reach v_2 with a narrow range of angles, and that range of angles "skips over" the corridor C. A ball on e_1 can only move under leftward gravity vectors, and so cannot exit from the left "lobe" of the shape. Similar reasoning from e_2 and e_3 shows that a ball anywhere in the trap can never reach C. Putting several such traps together, attached to a common "bus" rectangle, leads to k traps, where $k = n/28$.

The upper bound follows by classifying all convex vertices as one of four types $\{q_0, q_1, q_2, q_3\}$, depending on the quadrant that includes the polygon interior in a neighborhood of the vertex. There are $> n/2$ convex vertices, so there is one quadrant q^* type that occurs $\leq n/8$ times. A ball at any q_i vertex can roll to some q_j vertex, for each $j \neq i$. So piercing each of the q^* vertices suffices to drain P. □

3 The Pin-Ball Graph

Let G be a directed graph whose nodes are the convex vertices of P, with v_i connected to v_j if B can roll in one

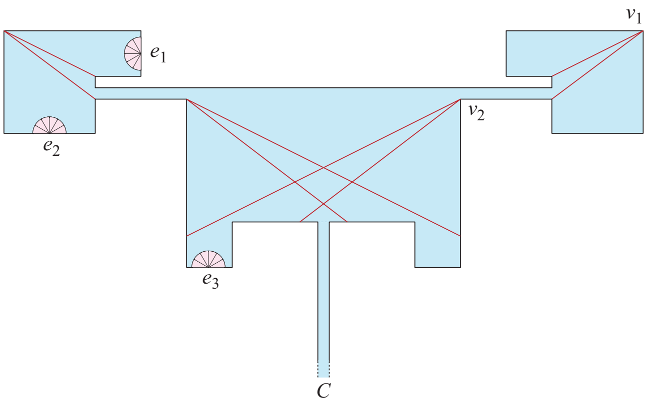

Figure 3: Orthogonal polygon trap of 28 vertices.

"move" from v_i to v_j. Here, a move is a complete path to the local y-minimum v_j, for some fixed orientation of P. We conceptually label the arcs of G with the sequence of vertices and edges along the path $\rho(v_i, v_j)$. Thus, the (v_{10}, v_6) arc in Fig. 1 is labeled $(v_9, e_{13}, v_{14}, e_7, v_7, e_5)$. We use G_R and G_T to distinguish the graphs for the Rotation and Tilt models respectively, and G when the distinction is irrelevant.

We gather together a number of basic properties of G in the following lemma.

Lemma 3 **(G Properties)**

1. *Every node of G_R has out-degree 2; a node of G_T has out-degree at least 2 and at most $O(n)$.*
2. *Both G_R and G_T have $O(n)$ nodes (one per convex vertex). G_R has at most $2n$ arcs, while G_T has $O(n^2)$ arcs, and sometimes $\Omega(n^2)$ arcs.*
3. *Each path label has length $O(n)$ (in either model).*
4. *The total number of path labels on the arcs of G_R is $O(n^2)$, and sometimes $\Omega(n^2)$.*
5. *The total number of labels in G_T is $O(n^3)$, and sometimes $\Omega(n^3)$.*

Proof. We only mention those properties which are not obvious.

1. Assume for the purposes of contradiction that there is a node of G_R that has out-degree 1, i.e., that both cw and ccw departures from v_i lead to the same v_j. A ray from v_i toward v_{i-1} until ∂P is hit bounds a subpolygon on the left of the ray that contains the termination vertex v_j. A second ray from v_i toward v_{i+1} to ∂P is hit bounds a subpolygon on the right of the ray that contains v_j. These two subpolygons only intersect at v_i, which cannot be the destination in either case. (Notice that when an adjacent vertex neighbor of v_i is a convex vertex, the subpolygon degenerates to a digon, i.e., it consists of two coincident edges.)
2. G_T has $\Omega(n^2)$ arcs when P is a convex polygon.
3. A single path travels monotonically with respect to gravity, and so can only touch $O(n)$ edges and vertices.
4. Fig. 4 shows that the total number of path labels for G_R is sometimes $\Omega(n^2)$.

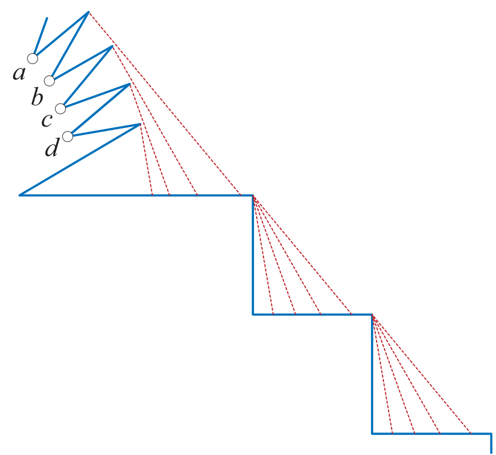

Figure 4: The paths from each of $\Omega(n)$ vertices $a, b, c, d, \ldots$ could touch $\Omega(n)$ edges.

5. A modification of this figure establishes an $\Omega(n^3)$ bound for the total number of labels in G_T.

□

We will see below that G_T can be constructed more efficiently than what the cubic total label size in Lemma 3(5) might indicate.

Noncrossing Paths. A ball path corresponding to one arc of G is a polygonal curve, monotone with respect to gravity $\overrightarrow{g}$. The path is composed of subsegments of polygon edges, as well as *fall* segments, each of which is parallel to $\overrightarrow{g}$ and incident to a reflex vertex. A directed path ρ naturally divides P into a "left half" $L = L(\rho)$ of points left of the traveling direction, and a "right half" $R = R(\rho)$, where L and R are disjoint, and $L \cup R \cup \rho = P$. Two ball paths ρ_1 and ρ_2 (properly) *cross* if ρ_2 contains points in both $L(\rho_1)$ and $R(\rho_1)$. For example, in Fig. 1, $\rho(v_0, v_6)$ crosses $\rho(v_{10}, v_6)$. Let $\overline{L} = L(\rho) \cup \rho$ be the closure of $L(\rho)$, and similarly define $\overline{R}$.

Two paths can only cross at a reflex vertex (as do $\rho(v_4, v_0)$ and $\rho(v_6, v_3)$ in Fig. 1) or on fall segments of each (as do $\rho(v_0, v_6)$ and $\rho(v_{10}, v_6)$).

Lemma 4 (Noncrossing) *Two paths ρ_1 and ρ_2 from the same source vertex v_0 never properly cross (in either model).* See Fig. 5.

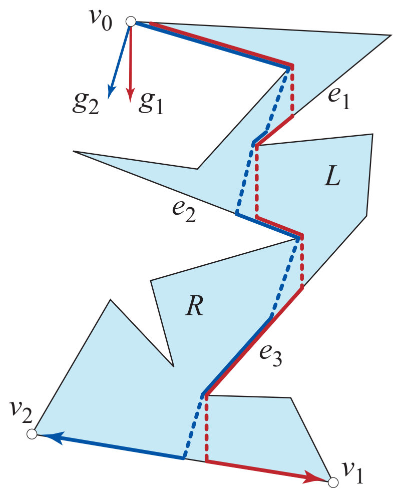

Figure 5: Paths from the same source v_0 do not cross. Fall segments are dashed. L and R indicate polygon "halves" left and right of the directed paths.

Proof. Let $\vec{g_1}$ and $\vec{g_2}$ be the gravity vectors for ρ_1 and ρ_2. Orient P so that $\vec{g_1}$ is vertically downward, as in Fig. 5. Now we argue that $\rho_2 \subset \overline{R_1}$.

We first assume that the angular separation between the two gravity vectors is $\leq \pi$ (as illustrated in the figure). The path ρ_2 is in $\overline{R_1}$ at v_0, and it cannot cross into $L(\rho_1)$ along an edge of P. So it could only cross into $L(\rho_1)$ at either the upper endpoint of a fall segment (a reflex vertex), or in the interior of a fall segment. But both are impossible because $\vec{g_2}$ is pointing right-to-left.

If the angular separation between the two gravity vectors is $> \pi$, then the same result is obtained by reversing the roles of $\vec{g_1}$ and $\vec{g_2}$. □

Lemma 5 (Label Intervals) *In the Tilt model, a particular label λ appears on the arcs of G_T originating at one particular v_i within an interval $[\vec{g_1}, \vec{g_2}]$ of gravity directions.*

Proof. Suppose to the contrary that λ appears on the paths ρ_1 and ρ_2 for $\vec{g_1}$ and $\vec{g_2}$ but not on the path ρ' for $\vec{g'}$, $\vec{g_1} < \vec{g'} < \vec{g_2}$. Then, because ρ' cannot cross ρ_1 by Lemma 4, the element corresponding to λ is inside a region Q between ρ_1 and ρ'. If we let x be the first point at which ρ_1 and ρ' deviate, and let the arcs of G_T induced by $\vec{g_1}$ and $\vec{g_2}$ be (v_i, v_j) and (v_i, v_k) respectively, then Q is bounded by the portion of ρ_1 from x to v_i, the boundary ∂P from v_i to v_j, and the portion of ρ' from x to v_j. Now, for ρ_2 to include λ, it must cross into Q, and since it cannot cross ∂P, it must cross either ρ_1 or ρ', contradicting Lemma 4. □

Cycles and Strongly Connected Components. The directed path from any vertex v_i of G leads to a cycle in G, because every node has at least two outgoing edges by Lemma 3(1). Any maximal cycle in G has length at least 3. Anything less would involve a pair (v_i, v_j) connecting only to each other, which would contradict Lemma 3(1). Note that any pair of convex vertices adjacent on ∂P form a non-maximal cycle of length 2.

A cycle is a particular instance of a *strongly connected component* (SCC) of G, a maximal subset $C \subset G$ in which each node has a directed path to all others.

Define a graph G^* as follows. Let $C_1, C_2, \ldots$ be the SCC's of G. Contract each C_k to a node c_k of G^*, while otherwise maintaining the connectivity of G. Then G^* is a DAG (because all cycles have been contracted).

Lemma 6 (Sinks) *The minimum number m of holes needed to drain P is the number of sinks of G^*.*

Proof. Certainly m holes suffice, for, by the construction of G^*, we can roll any ball to the SCC containing its terminal cycle. Piercing each c_k with one hole ensures that the ball can escape.[1] Leaving any sink unpierced constitutes a trap from which balls in the sink's connected component cannot escape. So m holes are necessary. □

Lemma 7 *The locations of the minimum number m of holes needed to drain P can be found in linear time in the size $|G|$ of G, once G has been constructed.*

Proof. Finding the SCC's of a graph is linear in $|G|$ (via, e.g., two depth-first searches [CLRS01, p. 552ff]). Both contracting to G^*, and finding the sinks of G^*, are again linear in $|G|$. □

Construction of G. Our goal is to construct the unlabeled G. Labels merely represent the paths that realize each arc of G. The example in Fig. 4 seems to require $\Omega(n^2)$ ray-shooting queries in the Rotation model, and as we do not know how to avoid this, our goal becomes an $O(n^2 \log n)$ algorithm. This is straightforward for G_R, so we focus on G_T, which by Lemma 3(5) is potentially cubic.

We first preprocess P for efficient ray-shooting queries, using fractional cascading to support ray shooting in a polygonal chain. This takes $O(n \log n)$ preprocessing time and supports $O(\log n)$ time per query ray [CEG[+]94]. Next we construct the visibility polygon from each vertex of in overall $O(n^2)$ time [JS87]. From these visibility polygons, for each v_i we construct a *gravity diagram D_i*. This partitions all gravity vectors $\vec{g}$ into angular intervals labeled with the next vertex that B will roll to from v_i with tilt $\vec{g}$. For example, Fig. 6(a) shows the gravity diagram for v_4 in Fig. 1. Note that D_i only records the next vertex encountered, not the ultimate destination. We maintain each diagram in a

[1] We should note that, after the ball exits the polygon, it still might be "trapped" in the exterior of P, unable to roll to ∞. We do not pursue this possibility.

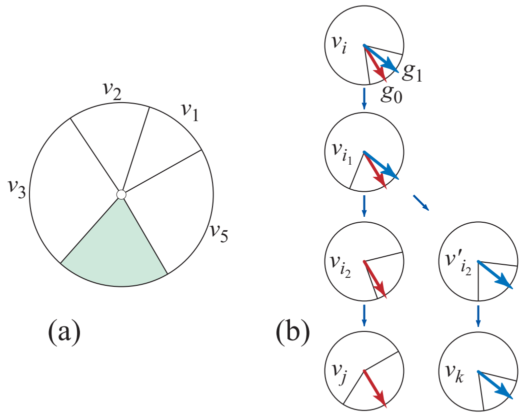

Figure 6: (a) Gravity diagram for v_4 in Fig. 1. (b) Gravity diagrams for paths ρ_0 and ρ_1.

structure that permits any $\overrightarrow{g}$ to be located in $O(\log n)$ time.

We now argue that we can construct all paths with source v_i, and therefore all arcs of G_T leaving v_i, in $O(n \log n)$ time. Compute the path ρ_0 for the cw extreme gravity vector $\overrightarrow{g_0}$ that leaves v_i (perpendicular to $v_i v_{i+1}$). This uses $O(n)$ ray-shooting queries for the fall segments of ρ_0, totaling $O(n \log n)$ time. Let $\rho_0 = (v_i, v_{i_1}, v_{i_2}, \ldots, v_j)$. During its construction, we locate $\overrightarrow{g_0}$ within each diagram D_{i_k}. Now we find the minimum angle between $\overrightarrow{g_0}$ and the next ccw event over all diagrams. This can be done in $O(\log n)$ time using a priority queue. Call the next event $\overrightarrow{g_1}$, and suppose it occurs at v_{i_k} in diagram D_{i_k}. We now construct the path ρ_1 from v_{i_k} onward, until it terminates at a new vertex, or rejoins ρ_0 (recall from Fig. 5 that paths might rejoin, i.e., the suffixes from v_{i_k} are not necessarily disjoint). In our "update" from ρ_0 to ρ_1, let V_0 be the set of vertices lost from ρ_0, and V_1 those gained in ρ_1. The priority queue of minima is updated by deleting those for V_0 and inserting those for V_1. The angular sweep about v_i continues in the same manner until the full gravity vector range about v_i is exhausted. Fig. 6(b) illustrates one step of this process, where v_{i_1} determines the transition event between g_0 and g_1, at which point the path changes from $\rho_0 = (v_i, v_{i_1}, v_{i_2}, v_j)$ to $\rho_1 = (v_i, v_{i_1}, v'_{i_2}, v_k)$.

By Lemma 5, each diagram abandoned in this sweep is never revisited. Thus the number of invocations of the minimum operation to find the next event is $O(n)$, or $O(n \log n)$ overall. Repeating for each v_i we obtain:

Lemma 8 *G can be constructed in $O(n^2 \log n)$ time.*

Theorem 9 *The locations of the minimum number of holes needed to drain P can be found in $O(n^2 \log n)$.*

For orthogonal polygons, all fall segments are (arbitrarily close to) parallel to the two directions determined by the polygon edges, and the situation illustrated in Fig. 4 cannot occur. We leave it as a claim that this permits G_R to be constructed (and the holes located) in $O(n \log n)$ time.

4 1-Drainable Shapes

Define a *k-drainable* polygon as one that can be drained with k holes but not with $k-1$ holes. For example, Fig. 1 is 1-drainable with a hole at v_6. We make a start here at exploring the 1-drainable shapes under each model. Note that these shapes do depend on the model: Fig. 2(a) is 1-drainable in the Tilt model but 3-drainable in the Rotation model.

Our definition of k-drainable polygons is inspired by the k-fillable polygons of [BT94][BvKT98], those mold shapes that can be filled with liquid metal poured into k holes. Despite the apparent inverse relationship between filling and draining, the two concepts are rather different. In particular, there are star-shaped polygons k-drainable in the rotation model (Proposition 13 below), but Theorem 7.2 of [BvKT98] shows that these are all "2-fillable with re-orientation." Also, there are 1-drainable polygons that are k-fillable (with or without reorientation). In Fig. 7, a hole at v_0 suffices to drain each spiral piece, because a ball at, say, v_i, can roll to v_0 (in several moves). However, the tip of each spiral needs to be filled separately, and so the shape is k-fillable.

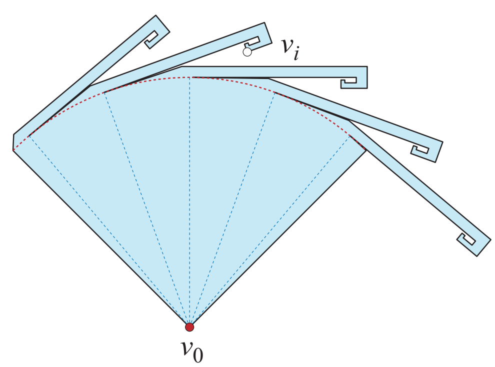

Figure 7: A polygon 1-drainable (in either the Rotation or Tilt model), but k-fillable, with or without reorientation.

Proposition 10 *Monotone polygons are 1-drainable.*

Proof. Let P be a polygon monotone with respect to the y-axis, and select $\overrightarrow{g}$ to point in the $-y$ direction. Then the lowest vertex v is the only local y-minimum, and no ball can stop before arriving at v. So, in either model, a hole at either y-extreme vertex suffices to drain it. □

Let the *ccw roll* from v_i be the roll toward v_{i+1} in the Rotation model, or equivalently, the tilt according to $\overrightarrow{g}$ perpendicular to $v_i v_{i+1}$ in the Tilt model.

Lemma 11 (Kernel) *Let P be star-shaped with kernel K. Then for each arc $(v_i, v_j) \in G$ corresponding to the ccw roll path ρ from v_i, K is in $\overline{L(\rho)}$, i.e., K is on or to the left of ρ.*

Proof. Let $\overrightarrow{g}$ be the gravity vector for the roll, perpendicular to $v_i v_{i+1}$. Then K must be in the closed halfplane H left of $v_i v_{i+1}$, because P is star-shaped from K. On the other hand, $\rho(v_i, v_j)$ is monotonic with respect to $\overrightarrow{g}$, and so lies in the closure of the complementary halfplane $\overline{H}$. Therefore, $v_j \in \overline{H}$, and thus K is left or on the path ρ, and so $K \subset \overline{L(\rho)}$. See Fig. 8. □

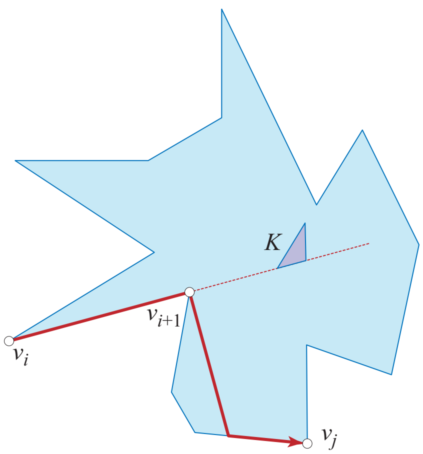

Figure 8: Star-shaped polygon.

A *fan* is a star-shaped polygon whose kernel includes a convex vertex.

Proposition 12 *Fans are 1-drainable.*

Proof. Let $v \in K$ be the vertex in the kernel. Starting at any vertex v_i, let $C = (v_i, \ldots, v_j, v_k, \ldots, v_i)$ be the cycle generated by successive ccw rolls. If $v \notin C$, then there must be an arc (v_j, v_k) of G whose path ρ "skips over" v, which, because these are ccw rolls, must have $v \in R(\rho)$ in the right region for ρ. This contradicts Lemma 11. □

Proposition 12 cannot be extended to star-shaped polygons in the Rotation model:

Proposition 13 *For any $k > 1$, there is a k-drainable star-shaped n-gon in the Rotation model, with $k = \Omega(n)$.*

Proof. Fig. 9 shows the construction for $k = 2$. A ball in "well" 0 rolls ccw to well $2, 4, \ldots$, and cw also to an even numbered well. A ball in well 1 rolls ccw and cw to odd numbered wells. Thus, two holes are needed when there are an even number of wells. The construction can be generalized to any $k > 1$. □

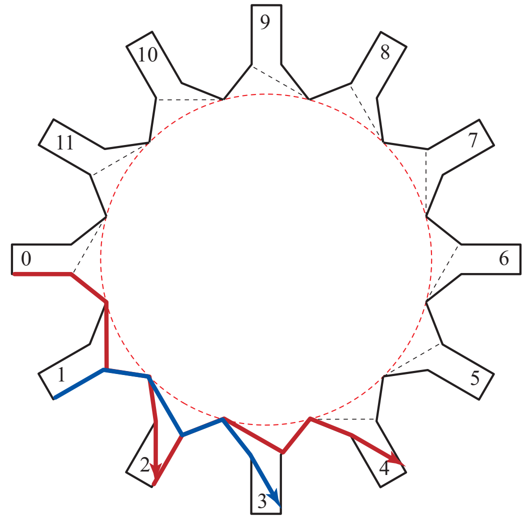

Figure 9: A star-shaped polygon that requires two holes to drain.

5 3D

Define the Tilt model for 3D polyhedra to permit departure from a vertex v at a direction vector lying in any of the faces of P incident to v. We do not see how to mimic the efficient construction of G previously described, so we content ourselves with showing that it can be accomplished in polynomial time: $O(n^7 \log n)$.

For the start vertex v_0, define the gravity diagram D_0 to partition the unit sphere S into regions that indicate the next edge crossed by the ball. Now let e_i be some edge. We define a gravity diagram D_i over the Cartesian product of all the possible crossing positions on e_i, with all the possible approach vectors. This is, topologically, a segment crossed with a circle—a cylinder. We partition this diagram into regions each of which identifies the next edge to be encountered by the ball. Each D_i has complexity $O(n^2)$.

Overlay all of the $O(n)$ gravity edge diagrams. This can be viewed as an arrangement of $O(n^3)$ arcs on a cylinder, and so partitions it into $O(n^6)$ regions. Any gravity vector within a particular region r leads to the same path and therefore arc of G. For each region, compute the path as before, using $O(n \log n)$ time. This will construct G in $O(n^7 \log n)$ time. From here on the logic of Lemmas 6 and 7 applies as before.

6 Open Problems

1. Can the upper bound of $\lceil n/4 \rceil$ in Theorem 1 be improved?

2. Are star-shaped polygons 1-drainable in the Tilt model? More generally, characterize 1-drainable polygons.

3. Suppose the ball B has finite radius r. Perhaps

a retraction of P by r would suffice to lead to a similar algorithm to that in Sec. 3.[2]

4. We phrased our problem as draining water/balls to the exterior of P, and noted in the proof of Lemma 6 that this is not the same as draining to infinity. What is the analog of our combinatorial bounds (Theorem 1) for the draining-to-∞ model?
5. Suppose m balls are present in P at the start, and P is k-drainable. What is the computational complexity of finding an optimal schedule of rotations, say, in terms of the total absolute angle turn, or in terms of the number of angular reversals?

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[2] We owe this idea to a referee.