Every Combinatorial Polyhedron Can Unfold with Overlap

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Abstract

Ghomi proved that every convex polyhedron could be stretched via an affine transformation so that it has an edge-unfolding to a net [Gho14]. A *net* is a simple planar polygon; in particular, it does not self-overlap. One can view his result as establishing that every combinatorial polyhedron has a realization that allows unfolding to a net.

Joseph Malkevitch asked if the reverse holds (in some sense of "reverse"): Is there a combinatorial polyhedron such that, for every realization, and for every spanning cut-tree, it unfolds to a net? In this note we prove the answer is NO: every combinatorial polyhedron has a realization and a cut-tree that unfolds with overlap.

1 Introduction

Joseph Malkevitch asked¹ whether there is a combinatorial type of a convex polyhedron whose every unfolding results in a net. One could imagine, to use his example, that every realization of a combinatorial cube unfolds without overlap for each of its 384 spanning cut-trees [Tuf11].² The purpose of this note is to prove this is, alas, not true: every combinatorial type can realized and edge-unfolded to overlap: Theorem 1 (Section 5). For an overlapping unfolding of a combinatorial cube, see ahead to Fig. 12.

An implication of Theorem 1, together with [Gho14], is that the resolution of Dürer's Problem [O'R13] must focus on the geometry rather than the combinatorial structure of convex polyhedra.

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 $^{^2\}mathrm{Burnside's}$ Lemma can show that these 384 trees lead to 11 incongruent unfoldings of the cube [GSV19].

2 Proof Outline

We describe the overall proof plan in the form of a multi-step algorithm. We will illustrate the steps with an icosahedron before providing details.

We are given a 3-connected planar graph G, which constitutes the combinatorial type of a convex polyhedron. By Steinitz's theorem, we know G is the 1-skeleton of a convex polyhedron. Initially assume G is triangulated; this assumption will be removed in Section 3.1.

- (1) Select outer face B as base. Initially, any face suffices. Later we will coordinate the choice of B with the choice of the special triangle \triangle .
- (2) Embed *B* as a convex polygon in the plane. Select coordinates for the vertices of *B*, which then pin *B* to the plane. *B* must be convex, but otherwise its shape is arbitrary.
- (3) Apply Tutte's theorem [Tut63] to calculate an equilibrium stress—positive weights on each edge—that, when interpreted as forces, induce an equilibrium at every vertex. This provides explicit coordinates for all vertices interior to B. The result is a Schlegel diagram, with all interior faces convex regions. Fig. 1 illustrates this for the icosahedron.³
- (4) Apply Maxwell-Cremona lifting to P. The Maxwell-Cremona theorem says that any straight-line planar drawing with an equilibrium stress has a polyhedral lifting via a "reciprocal diagram." The details are not needed here;⁴ we only need the resulting lifted polyhedron. An example from [Sch08]

 $^{^3\}mathrm{Here}$ the drawing is approximate, in that I did not explicitly calculate the equilibrium stresses.

 $^{^{4}}$ A good resource on this topic is [RG06].



Figure 1: Icosahedron Schlegel diagram.

shows the lifting of a Schlegel diagram of the dode cahedron: Fig. 2. A lifting of the vertices of the icosahedron in Fig. 1 is shown in Fig. $3.^5$

- (5) Identify special triangle △. This special triangle must satisfy several conditions, which we detail later (Section 3). For now, we select △ = a₁a₂a₃ = 6,8,5 in Fig. 4.
- (6) Scale P horizontally (if necessary). Not needed in icosahedron example.
- (7) Scale P vertically (if necessary). Not needed in icosahedron example.
- (8) Form cut-tree T, including a 'Z'-path around \triangle . We think of a_1 as the root of the spanning tree, which includes the Z-shaped (red) path $a_1a_2a_3a_4$ around \triangle and the adjacent triangle \triangle' sharing edge a_2a_3 . In Fig. 4, the Z vertex indices are 6, 8, 5, 11. The remainder of T is completed arbitrarily.
- (9) Unfold $P \setminus T$.
- (10) Finally, the conditions on \triangle ensure that cutting T unfolds P with overlap along the a_2a_3 edge. See Fig. 5.

 $^{^5\}mathrm{This}$ is again an approximation as I did not calculate the reciprocal diagram.



Figure 2: Maxwell-Cremona lifting to a dodecahedral diagram. [Sch08], by permission of author.



Figure 3: Lifting the vertices of the icosahedron Schlegel diagram in Fig. 1.



Figure 4: Red: face numbers; blue: vertex indices. $\triangle = 5$, $\triangle' = 6$. Z-portion of spanning tree T red; remainder blue.



Figure 5: Close-up views of overlap.

3 Conditions on \triangle

We continue to focus on triangulated polyhedra. In order to guarantee overlap, the special triangle $\Delta = a_1 a_2 a_3$ should satisfy several conditions:

- 1. The angle at a_2 in \triangle must be $\leq \pi/3 = 60^{\circ}$, and the edge a_2a_3 at least as long as a_1a_2 .
- 2. The spanning cut-tree T must contain the Z as previously explained. In addition, no other edge of T is incident to either a_1 or a_2 . In particular, edge a_1a_3 is not cut, so the triangle \triangle rotates as a unit about a_1 .
- 3. The curvatures at a_1 and a_2 must be small. We show below that $< 20^{\circ}$ suffices.
- 4. \triangle should be disjoint from the base B: \triangle and B share no vertices.

This 4th condition might be impossible to satisfy, in which case an additional argument is needed (Section 4). For now we concentrate on the first three conditions.

 \triangle is chosen to be the triangle disjoint from B with the smallest angle α . Clearly $\alpha \leq \pi/3 = 60^{\circ}$. Let $\triangle = a_1a_2a_3$ with a_2 the smallest angle. Chose the labels so that $|a_1a_2| \leq |a_2a_3|$. It will be easy to see that \triangle an equilateral triangle is the "worst case" in that smaller α leads to deeper overlap, and $|a_1a_2| = |a_2a_3|$ suffices for overlap. So we will assume \triangle is an equilateral triangle.

Next, we address the requirement for small curvatures, when the second condition is satisfied: no other edge of T is incident to either a_1 or a_2 . Let ω be the curvature at a_1 and a_2 . Then an elementary calculation shows that $\omega = \frac{1}{9}\pi = 20^{\circ}$ would just barely avoid overlap: see Fig. 6.



Figure 6: Left: $\omega = 20^{\circ}$ avoids overlap. Right: $\omega = 10^{\circ}$ overlaps.

One can view the flattening of a_1 and a_2 when cut as first turning the edge a_2a_3 by ω about a_2 , and then rotating the rigid path $a_1a_2a'_3$ about a_1 by

 ω . For any ω strictly less than 20°, overlap occurs along the a_2a_3 edge. The basic reason this "works" to create overlap is that the cut-path around Δ is not *radially monotone*, a concept introduced in [O'R16] and used in [O'R18] to avoid overlap.

In the unfolded icosahedron in Fig. 4, the angle at a_2 is 59°, and the curvatures ω_1, ω_2 at a_1, a_2 are 2.4° and 8.1° respectively.

If the two curvatures are not less than 20° , then we scale *P* vertically (step (7) of Algorithm 1. As illustrated in Fig. 7, this flattens dihedral angles and reduces vertex curvatures at all but the vertices of base *B*, which increase to compensate the Guass-Bonnet sum of 4π . Clearly we can reduce curvatures as much as desired.



Figure 7: Dihedral angle δ flattens as z-heights scaled: $(1, \frac{1}{2}, \frac{1}{5}) \rightarrow (90^{\circ}, 125^{\circ}, 160^{\circ}).$

3.1 Non-Triangulated Polyhedra

If G and therefore P contains non-triangular faces, then we employ step (6) of Algorithm 1: Scale P horizontally. For example, in the dodecahedron example (Fig. 2), no face has an angle $\alpha \leq \pi/3$. But by horizontal scaling (parallel to the *xy*-plane), we can sharpen any selected face angle, as illustrated in Fig. 8. Then we can identify Δ within that face, and proceed just as in a triangulated polyhedron.



Figure 8: (a) Regular pentagon scaled $\frac{2}{3}$ and $\frac{1}{3}$ horizontally. (b) A triangle with one angle $60^\circ.$

4 No Pair of Disjoint Faces

Finally we focus on the 4th condition that \triangle should be disjoint from the base B. If G contains any two disjoint faces, triangles or k-gon faces with k > 3, we select one as B and the other to yield \triangle . So what remains is those G with no pair of disjoint faces.

For example, a pyramid—*B* plus one vertex *a* (the apex) above *B*—has no pair of disjoint faces. However, note that a pyramid has pairs of faces that share one vertex but not two vertices. It turns out that this suffices to achieve the same structure of overlap. Fig. 9 illustrates why. Here *B* is a triangle $b_1b_2a_3$ and we select $\Delta = a_1a_2a_3$. The small-curvature requirement holds just for a_1, a_2 the start of the Z—the curvature at a_3 could be large (117° in this example) but does not play a role, as the unfolding illustrates. Therefore, if *G* has no pair of disjoint faces, but does have a pair of faces that share a single vertex, we proceed just in Algorithm 1, suitably modified.



Figure 9: (a) B and \triangle share a_3 . $Z = a_1 a_2 a_3 b_2$. (b) Unfolding with overlap.

This leaves the case where there are no two disjoint faces, nor two faces that share a single vertex: every pair of faces shares two or more vertices. If two faces share non-adjacent vertices, they cannot both be convex. So in fact the condition is that each two faces share an edge. Then, it is not difficult to see that G can only be a tetrahedron, as the following argument shows.

Suppose $B = b_1b_2...b_k$ is k-gon. Add one triangle $t_1 = ab_1b_2$; see Fig. 10. A second triangle must share an edge with t_1 , say b_2a , so sharing with B leads to $t_2 = ab_2b_3$. Now a third triangle must share with B, t_1, t_2 . The only uncovered edge of t_1 is b_1a . But $t_3 = ab_1b_k$ does not share an edge with t_2 unless k = 3. In that case we have a tetrahedron.



Figure 10: Every pair of faces shares an edge.

So the only case remaining is a tetrahedron. But it is well known that the thin, nearly flat tetrahedron unfolds with overlap: Fig. 11. And since there is only one tetrahedron combinatorial type, this completes the inventory.



Figure 11: Fig. 28.2 [detail], p.314 in [DO07]: tetrahedron overlap. Blue: exterior. Red: interior.

5 Theorem

We have proved this theorem:

Theorem 1 Any 3-connected planar graph G can be realized as a convex polyhedron P that has a spanning cut-tree T such that the unfolding of $P \setminus T$ overlaps in the plane.

So together with Ghomi's result,⁶ any combinatorial polyhedron type can be realized to unfold and avoid overlap, or realized to unfold with overlap.

Returning to Malkevitch's example of a combinatorial cube, consider Fig. 12. Starting from the standard Schlegel diagram for a cube, horizontal scaling (step (6) of Algorithm 1) is needed to squeeze the top and bottom squares to diamonds, so that the angle at a_2 becomes small, in this case 55°. The lifting leaves the curvatures at a_1, a_2 to be small enough, $6.0^\circ, 6.5^\circ$, so step (7) of Algorithm 1 is not needed.



Figure 12: Unfolding of a combinatorial cube. Diagonals in the left figure are an artifact of the software; all faces are planar quadrilaterals. Base B attached left of b_1b_4 not shown. Vertex coordinates:

(-1, 0, 0.5), (1, 0, 0.5), (0, -2, 0.5), (0, 2, 0.5), (-2, 0, 0), (2, 0, 0), (0, -4, 0), (0, 4, 0)

⁶See [SZ18] for a different proof of [Gho14].

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