# Every Combinatorial Polyhedron Can Unfold with Overlap 

Joseph O'Rourke*

January 2, 2023


#### Abstract

Ghomi proved that every convex polyhedron could be stretched via an affine transformation so that it has an edge-unfolding to a net Gho14. A net is a simple planar polygon; in particular, it does not self-overlap. One can view his result as establishing that every combinatorial polyhedron has a realization that allows unfolding to a net.

Joseph Malkevitch asked if the reverse holds (in some sense of "reverse"): Is there a combinatorial polyhedron such that, for every realization, and for every spanning cut-tree, it unfolds to a net? In this note we prove the answer is NO: every combinatorial polyhedron has a realization and a cut-tree that unfolds with overlap.


## 1 Introduction

Joseph Malkevitch asked ${ }^{1}$ whether there is a combinatorial type of a convex polyhedron whose every unfolding results in a net. One could imagine, to use his example, that every realization of a combinatorial cube unfolds without overlap for each of its 384 spanning cut-trees Tuf11, 2 The purpose of this note is to prove this is, alas, not true: every combinatorial type can realized and edge-unfolded to overlap: Theorem 1 (Section 5). For an overlapping unfolding of a combinatorial cube, see ahead to Fig. 12

An implication of Theorem 1] together with [Gho14], is that the resolution of Dürer's Problem O'R13 must focus on the geometry rather than the combinatorial structure of convex polyhedra.

[^0]
## 2 Proof Outline

We describe the overall proof plan in the form of a multi-step algorithm. We will illustrate the steps with an icosahedron before providing details.

```
Algorithm 1: Realizing \(G\) to unfold with overlap.
    Input: A 3-connected planar graph \(G\).
    Output: Polyhedron \(P\) realizing \(G\) and a cut-tree \(T\) that unfolds \(P\)
                with overlap.
    (1) Select outer face \(B\) as base.
    (2) Embed \(B\) as a convex polygon in the plane.
    (3) Apply Tutte's theorem to calculate an equilibrium stress.
    (4) Apply Maxwell-Cremona lifting to \(P\).
    (5) Identify special triangle \(\triangle\).
    (6) Scale \(P\) horizontally (if necessary).
    (7) Scale \(P\) vertically (if necessary).
    (8) Form cut-tree \(T\), including ' \(Z\) ' around \(\triangle\).
    (9) Unfold \(P \backslash T\).
    (10) \(\rightarrow\) Overlap.
```

We are given a 3 -connected planar graph $G$, which constitutes the combinatorial type of a convex polyhedron. By Steinitz's theorem, we know $G$ is the 1 -skeleton of a convex polyhedron. Initially assume $G$ is triangulated; this assumption will be removed in Section 3.1.
(1) Select outer face $B$ as base. Initially, any face suffices. Later we will coordinate the choice of $B$ with the choice of the special triangle $\triangle$.
(2) Embed $B$ as a convex polygon in the plane. Select coordinates for the vertices of $B$, which then pin $B$ to the plane. $B$ must be convex, but otherwise its shape is arbitrary.
(3) Apply Tutte's theorem Tut63 to calculate an equilibrium stress-positive weights on each edge - that, when interpreted as forces, induce an equilibrium at every vertex. This provides explicit coordinates for all vertices interior to $B$. The result is a Schlegel diagram, with all interior faces convex regions. Fig. 1 illustrates this for the icosahedron ${ }^{3}$
(4) Apply Maxwell-Cremona lifting to $P$. The Maxwell-Cremona theorem says that any straight-line planar drawing with an equilibrium stress has a polyhedral lifting via a "reciprocal diagram." The details are not needed here ${ }^{4}$ we only need the resulting lifted polyhedron. An example from [Sch08]

[^1]

Figure 1: Icosahedron Schlegel diagram.
shows the lifting of a Schlegel diagram of the dodecahedron: Fig. 2, A lifting of the vertices of the icosahedron in Fig. 1 is shown in Fig. $3{ }^{5}$
(5) Identify special triangle $\triangle$. This special triangle must satisfy several conditions, which we detail later (Section 3 ). For now, we select $\triangle=a_{1} a_{2} a_{3}=$ $6,8,5$ in Fig. 4
(6) Scale $P$ horizontally (if necessary). Not needed in icosahedron example.
(7) Scale $P$ vertically (if necessary). Not needed in icosahedron example.
(8) Form cut-tree $T$, including a ' $Z$ '-path around $\triangle$. We think of $a_{1}$ as the root of the spanning tree, which includes the Z-shaped (red) path $a_{1} a_{2} a_{3} a_{4}$ around $\triangle$ and the adjacent triangle $\triangle^{\prime}$ sharing edge $a_{2} a_{3}$. In Fig. 4 , the $\mathbf{Z}$ vertex indices are $6,8,5,11$. The remainder of $T$ is completed arbitrarily.
(9) Unfold $P \backslash T$.
(10) Finally, the conditions on $\triangle$ ensure that cutting $T$ unfolds $P$ with overlap along the $a_{2} a_{3}$ edge. See Fig. 5

[^2]

Figure 2: Maxwell-Cremona lifting to a dodecahedral diagram. Sch08, by permission of author.


Figure 3: Lifting the vertices of the icosahedron Schlegel diagram in Fig. 1.


Figure 4: Red: face numbers; blue: vertex indices. $\triangle=5, \triangle^{\prime}=6$. Z-portion of spanning tree $T$ red; remainder blue.


Figure 5: Close-up views of overlap.

## 3 Conditions on $\triangle$

We continue to focus on triangulated polyhedra. In order to guarantee overlap, the special triangle $\triangle=a_{1} a_{2} a_{3}$ should satisfy several conditions:

1. The angle at $a_{2}$ in $\triangle$ must be $\leq \pi / 3=60^{\circ}$, and the edge $a_{2} a_{3}$ at least as long as $a_{1} a_{2}$.
2. The spanning cut-tree $T$ must contain the Z as previously explained. In addition, no other edge of $T$ is incident to either $a_{1}$ or $a_{2}$. In particular, edge $a_{1} a_{3}$ is not cut, so the triangle $\triangle$ rotates as a unit about $a_{1}$.
3. The curvatures at $a_{1}$ and $a_{2}$ must be small. We show below that $<20^{\circ}$ suffices.
4. $\triangle$ should be disjoint from the base $B: \triangle$ and $B$ share no vertices.

This 4th condition might be impossible to satisfy, in which case an additional argument is needed (Section 44). For now we concentrate on the first three conditions.
$\triangle$ is chosen to be the triangle disjoint from $B$ with the smallest angle $\alpha$. Clearly $\alpha \leq \pi / 3=60^{\circ}$. Let $\triangle=a_{1} a_{2} a_{3}$ with $a_{2}$ the smallest angle. Chose the labels so that $\left|a_{1} a_{2}\right| \leq\left|a_{2} a_{3}\right|$. It will be easy to see that $\triangle$ an equilateral triangle is the "worst case" in that smaller $\alpha$ leads to deeper overlap, and $\left|a_{1} a_{2}\right|=\left|a_{2} a_{3}\right|$ suffices for overlap. So we will assume $\triangle$ is an equilateral triangle.

Next, we address the requirement for small curvatures, when the second condition is satisfied: no other edge of $T$ is incident to either $a_{1}$ or $a_{2}$. Let $\omega$ be the curvature at $a_{1}$ and $a_{2}$. Then an elementary calculation shows that $\omega=\frac{1}{9} \pi=20^{\circ}$ would just barely avoid overlap: see Fig. 6


Figure 6: Left: $\omega=20^{\circ}$ avoids overlap. Right: $\omega=10^{\circ}$ overlaps.
One can view the flattening of $a_{1}$ and $a_{2}$ when cut as first turning the edge $a_{2} a_{3}$ by $\omega$ about $a_{2}$, and then rotating the rigid path $a_{1} a_{2} a_{3}^{\prime}$ about $a_{1}$ by
$\omega$. For any $\omega$ strictly less than $20^{\circ}$, overlap occurs along the $a_{2} a_{3}$ edge. The basic reason this "works" to create overlap is that the cut-path around $\triangle$ is not radially monotone, a concept introduced in O'R16 and used in O'R18 to avoid overlap.

In the unfolded icosahedron in Fig. 4 the angle at $a_{2}$ is $59^{\circ}$, and the curvatures $\omega_{1}, \omega_{2}$ at $a_{1}, a_{2}$ are $2.4^{\circ}$ and $8.1^{\circ}$ respectively.

If the two curvatures are not less than $20^{\circ}$, then we scale $P$ vertically (step (7) of Algorithm 1. As illustrated in Fig. 7, this flattens dihedral angles and reduces vertex curvatures at all but the vertices of base $B$, which increase to compensate the Guass-Bonnet sum of $4 \pi$. Clearly we can reduce curvatures as much as desired.


Figure 7: Dihedral angle $\delta$ flattens as $z$-heights scaled: $\left(1, \frac{1}{2}, \frac{1}{5}\right) \rightarrow$ $\left(90^{\circ}, 125^{\circ}, 160^{\circ}\right)$.

### 3.1 Non-Triangulated Polyhedra

If $G$ and therefore $P$ contains non-triangular faces, then we employ step (6) of Algorithm 1] Scale $P$ horizontally. For example, in the dodecahedron example (Fig. 22, no face has an angle $\alpha \leq \pi / 3$. But by horizontal scaling (parallel to the $x y$-plane), we can sharpen any selected face angle, as illustrated in Fig. 8 . Then we can identify $\triangle$ within that face, and proceed just as in a triangulated polyhedron.


Figure 8: (a) Regular pentagon scaled $\frac{2}{3}$ and $\frac{1}{3}$ horizontally. (b) A triangle with one angle $60^{\circ}$.

## 4 No Pair of Disjoint Faces

Finally we focus on the 4 th condition that $\triangle$ should be disjoint from the base $B$. If $G$ contains any two disjoint faces, triangles or $k$-gon faces with $k>3$, we select one as $B$ and the other to yield $\triangle$. So what remains is those $G$ with no pair of disjoint faces.

For example, a pyramid- $B$ plus one vertex $a$ (the apex) above $B$-has no pair of disjoint faces. However, note that a pyramid has pairs of faces that share one vertex but not two vertices. It turns out that this suffices to achieve the same structure of overlap. Fig. 9 illustrates why. Here $B$ is a triangle $b_{1} b_{2} a_{3}$ and we select $\triangle=a_{1} a_{2} a_{3}$. The small-curvature requirement holds just for $a_{1}, a_{2}-$ the start of the Z - the curvature at $a_{3}$ could be large ( $117^{\circ}$ in this example) but does not play a role, as the unfolding illustrates. Therefore, if $G$ has no pair of disjoint faces, but does have a pair of faces that share a single vertex, we proceed just in Algorithm 1, suitably modified.


Figure 9: (a) $B$ and $\triangle$ share $a_{3} . \mathrm{Z}=a_{1} a_{2} a_{3} b_{2}$. (b) Unfolding with overlap.
This leaves the case where there are no two disjoint faces, nor two faces that share a single vertex: every pair of faces shares two or more vertices. If two faces share non-adjacent vertices, they cannot both be convex. So in fact the condition is that each two faces share an edge. Then, it is not difficult to see that $G$ can only be a tetrahedron, as the following argument shows.

Suppose $B=b_{1} b_{2} \ldots b_{k}$ is $k$-gon. Add one triangle $t_{1}=a b_{1} b_{2}$; see Fig. 10 A second triangle must share an edge with $t_{1}$, say $b_{2} a$, so sharing with $B$ leads to $t_{2}=a b_{2} b_{3}$. Now a third triangle must share with $B, t_{1}, t_{2}$. The only uncovered edge of $t_{1}$ is $b_{1} a$. But $t_{3}=a b_{1} b_{k}$ does not share an edge with $t_{2}$ unless $k=3$. In that case we have a tetrahedron.


Figure 10: Every pair of faces shares an edge.

So the only case remaining is a tetrahedron. But it is well known that the thin, nearly flat tetrahedron unfolds with overlap: Fig. 11. And since there is only one tetrahedron combinatorial type, this completes the inventory.


Figure 11: Fig. 28.2 [detail], p. 314 in DO07: tetrahedron overlap. Blue: exterior. Red: interior.

## 5 Theorem

We have proved this theorem:
Theorem 1 Any 3-connected planar graph $G$ can be realized as a convex polyhedron $P$ that has a spanning cut-tree $T$ such that the unfolding of $P \backslash T$ overlaps in the plane.

So together with Ghomi's result $]_{[6]}$ any combinatorial polyhedron type can be realized to unfold and avoid overlap, or realized to unfold with overlap.

Returning to Malkevitch's example of a combinatorial cube, consider Fig. 12 Starting from the standard Schlegel diagram for a cube, horizontal scaling (step (6) of Algorithm (1) is needed to squeeze the top and bottom squares to diamonds, so that the angle at $a_{2}$ becomes small, in this case $55^{\circ}$. The lifting leaves the curvatures at $a_{1}, a_{2}$ to be small enough, $6.0^{\circ}, 6.5^{\circ}$, so step (7) of Algorithm 1 is not needed.


Figure 12: Unfolding of a combinatorial cube. Diagonals in the left figure are an artifact of the software; all faces are planar quadrilaterals. Base $B$ attached left of $b_{1} b_{4}$ not shown. Vertex coordinates:
$(-1,0,0.5),(1,0,0.5),(0,-2,0.5),(0,2,0.5),(-2,0,0),(2,0,0),(0,-4,0),(0,4,0)$

[^3]Acknowledgements. I benefitted from discussions with Richard Mabry and Joseph Malkevitch.

## References

[DO07] Erik D. Demaine and Joseph O'Rourke. Geometric Folding Algorithms: Linkages, Origami, Polyhedra. Cambridge University Press, 2007. http://www.gfalop.org
[Gho14] Mohammad Ghomi. Affine unfoldings of convex polyhedra. Geometry \& Topology, 18(5):3055-3090, 2014.
[GSV19] Richard Goldstone and Robert Suzzi Valli. Unfoldings of the cube. The College Mathematics Journal, 50(3):173-184, 2019.
[O'R13] Joseph O'Rourke. Dürer's problem. In Marjorie Senechal, editor, Shaping Space: Exploring Polyhedra in Nature, Art, and the Geometrical Imagination, pages 77-86. Springer, 2013.
[O'R16] Joseph O'Rourke. Unfolding convex polyhedra via radially monotone cut trees. arXiv:1607.07421, 2016. https://arxiv.org/abs/1607. 07421 .
[O'R18] Joseph O'Rourke. Edge-unfolding nearly flat convex caps. In Proc. Symp. Comput. Geom. (SoCG), volume 99, pages 64:1-64:14. Leibniz Internat. Proc. Informatics, June 2018. Full version: https://arxiv. org/abs/1707.01006.
[RG06] Jürgen Richter-Gebert. Realization Spaces of Polytopes. Springer, 2006.
[Sch08] André Schulz. Lifting planar graphs to realize integral 3-polytopes and topics in pseudo-triangulations. PhD thesis, Univerität Berlin, 2008.
[SZ18] Gözde Sert and Sergio Zamora. On unfoldings of stretched polyhedra. arXiv:1803.09828, 2018. https://arxiv.org/abs/1803.09828.
[Tuf11] Christopher Tuffley. Counting the spanning trees of the 3-cube using edge slides. arXiv:1109.6393, 2011. https://arxiv.org/abs/1109. 6393.
[Tut63] W. T. Tutte. How to draw a graph. Proc. London Mathematical Society, 13(52):743-768, 1963.


[^0]:    *Departments of Computer Science and of Mathematics, Smith College, Northampton, MA 01063, USA. jorourke@smith.edu
    ${ }^{1}$ Personal communication, Dec. 2022
    ${ }^{2}$ Burnside's Lemma can show that these 384 trees lead to 11 incongruent unfoldings of the cube GSV19.

[^1]:    ${ }^{3}$ Here the drawing is approximate, in that I did not explicitly calculate the equilibrium stresses.
    ${ }^{4} \mathrm{~A}$ good resource on this topic is RG06.

[^2]:    ${ }^{5}$ This is again an approximation as I did not calculate the reciprocal diagram.

[^3]:    ${ }^{6}$ See [SZ18] for a different proof of Gho14].

