Computational Geometry Column 52

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Abstract

[Draft, January 27, 2012.] Two art-gallery-like problems of transmitters in polygons are described, and several open problems posed. xxx

I have decided this will be my final *Computational Geometry Column*. It seems not inappropriate to return to my focus at the time I was writing Column #1: art gallery theorems [O'R87]. New life has been injected into that classic field with the guards or lights (in coverage or illumination models respectively) replaced by *transmitters*, point sources of broadcasts that can penetrate walls. Let me first update progress on the "wireless localization problem," described in Column 48 [O'R06].

This problem was introduced by Eppstein et al. in [EGS06]. Given a polygon P of n vertices, one wants to place a minimal number of transmitters, each broadcasting from a fixed point in an arbitrary but fixed angular range, so that the transmitters distinguish the interior of P from its exterior in this sense: a receiver at any point x in the plane can determine if it is inside or outside of P by the "keys" (unique IDs) it receives from the transmitters whose angular ranges cover x. The transmitter broadcasts pass through all walls of P. At the time of [O'R06], only very weak bounds were known for the most unconstrained case: arbitrary location of transmitters and general-position polygons (no vertex collinearities): $\lceil n/2 \rceil$ are needed for convex polygons, and it was known that $n-2$ suffice for all such P. Now both the lower and upper bounds have been improved, to roughly $\frac{2}{3}n$ and $\frac{3}{4}n$. More specifically, Tobias Christ has established bounds of $\lceil (2n-4)/3 \rceil$ and $\lfloor (3n-2)/4 \rfloor$ in his Ph.D. thesis [Chr11], improving on earlier work with collaborators [CHOU10]. For transmitters constrained to be located at polygon vertices, the bounds are roughly $\frac{2}{3}n$ and $\frac{8}{9}n$: $\lfloor 2n/3 \rfloor - 1$ [DFOR07] and $\lfloor (8n-6)/9 \rfloor$, the latter due to Christ and Aurosish Mishra. Finding the actual minimum number of vertex transmitters is known to be NP-hard [CH09], but there is no comparable complexity result for unconstrained transmitter placement: it is as yet unclear if the problem is in NP! See Figure 1 for a hint of why it seems not straightforward. (You may recognize this polygon as inspired by the *Symposium on Computational Geometry* logo!)

As you might imagine, every version of the analogous wireless localization problem in $\mathbb{R}^3$ is open; see [CH09] for a start on a special case.

A second cluster of problems introduced in [FMVU09] and [AFMFP$^+$09] are closer in spirit to the original art gallery problem: place k-transmitters in a polygon P of n vertices

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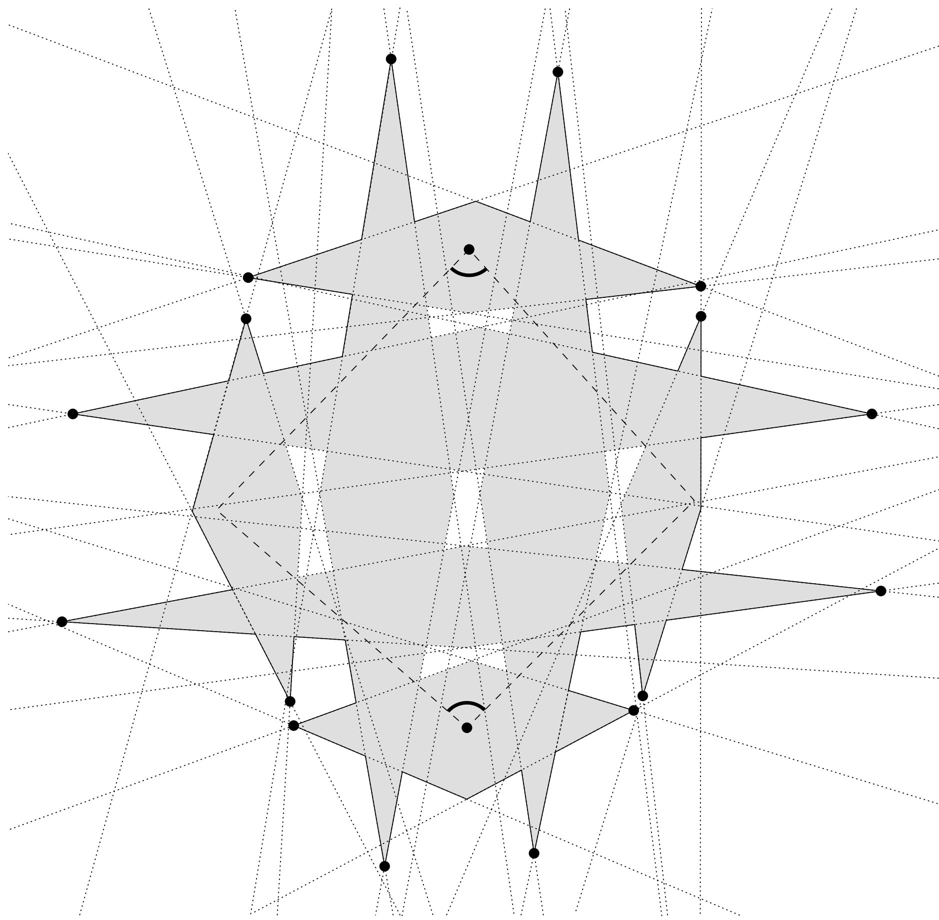

Figure 1: It appears that coverage of this polygon requires transmitters at the two marked locations, which do not lie on the arrangement of lines induced by the polygon edges. Figure 4.10 in [Chr11], used by permission.

to cover every interior point, where a *k-transmitter* is an infinite-range omnidirectional point source with the power to penetrate k walls of P. Transmitters on the boundary of P are considered just barely inside, so they must penetrate one wall to reach the exterior and another to cover an interior point. Thus in this model, only even-k are of interest (for interior coverage). The original art gallery theorem can be viewed as coverage by 0-transmitters.

In [FMVU09] it was shown that any polygon can be covered with one powerful transmitter: $k = \lceil (2n+1)/3 \rceil$, a bound tight up to an additive constant. They also showed that one $\lceil n/3 \rceil$-transmitter suffices for orthogonal polygons. In [AFMFP+09] it was established that in the special case of monotone polygons, $\lceil n/(2k+4) \rceil$ k-transmitters are sometimes necessary, and $\lceil n/(2k) \rceil$ always suffice.

These results concentrate on powerful transmitters. The other end of the spectrum is 2-transmitters, and remarkably the problem is wide open: How many 2-transmitters are sometimes necessary, and how many always suffice, to cover a polygon of n vertices? This was first studied (among other problems) in [BBB+10], which established bounds for special classes of polygons. For example, $\lfloor n/8 \rfloor$ 2-transmitters are sometimes necessary to cover a spiral polygon, and $\lfloor n/4 \rfloor$ always suffice. For general polygons, the only result is that $\lceil n/5 \rceil$ 2-transmitters are sometimes needed. This lower bound is established by the example of Justin Iwerks shown in Figure 2 (improving on the $n/6$ example in [BBB+10]). The only upperbound I know is obtained by ignoring the extra power of the transmitters, when $\lfloor n/3 \rfloor$ suffice—the original art gallery theorem!

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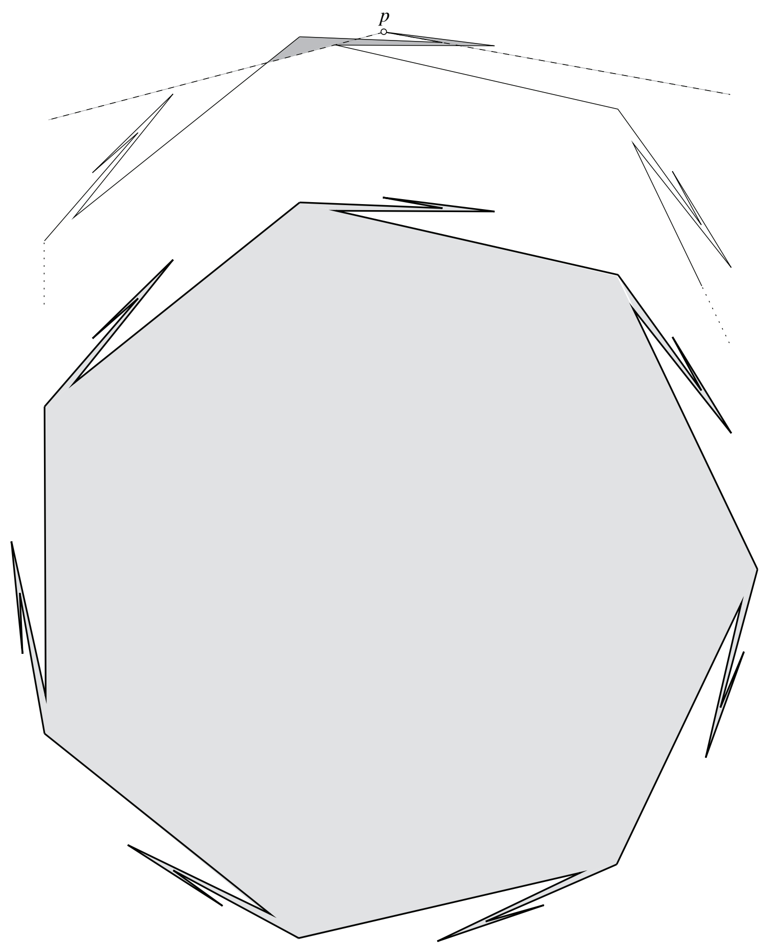

Figure 2: A polygon of n=35 vertices that requires k=7 2-transmitters, one per barb. The inset shows the set of internal points that can reach the barb tip p. Justin Iwerks, by permission.

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