Sweeping Shapes: Optimal Crumb Cleanup

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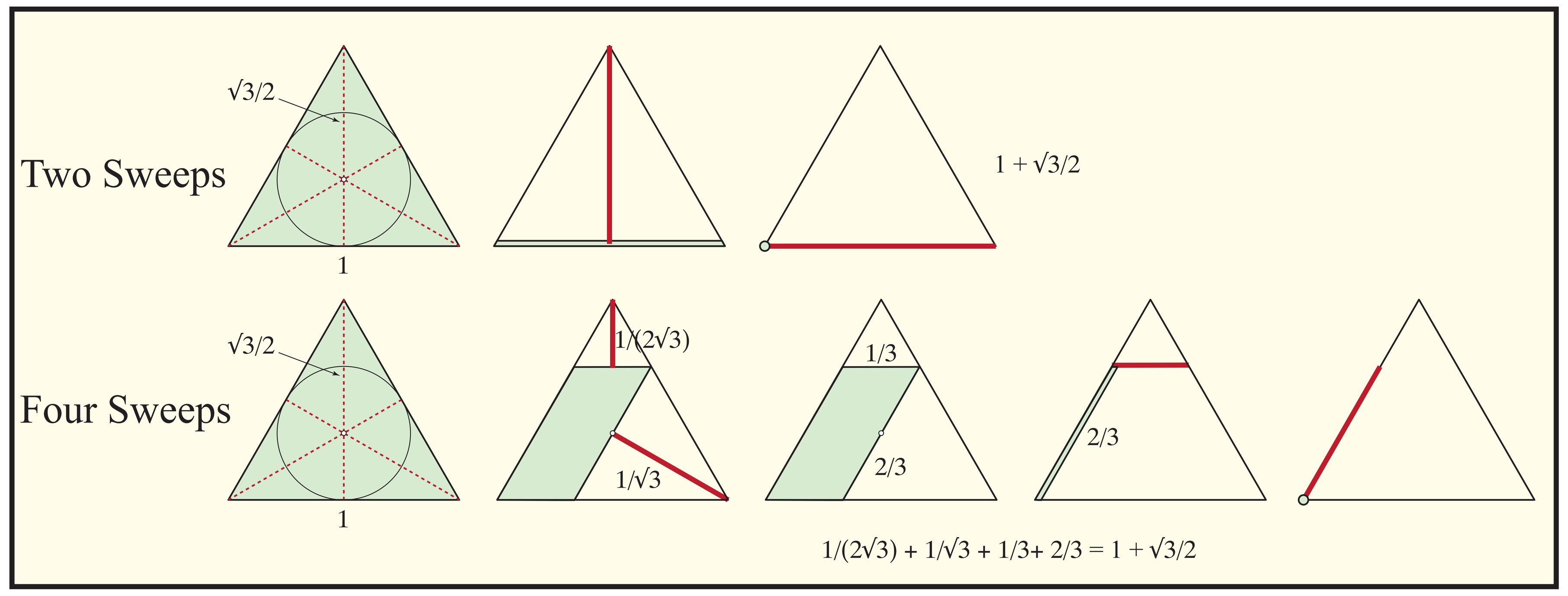
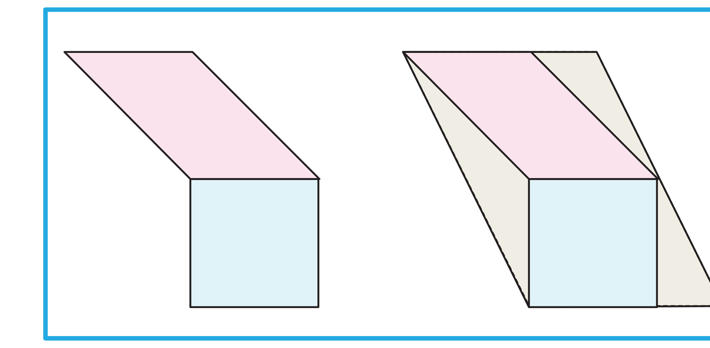
What does it mean to sweep a shape?

Imagine a shape filled with crumbs. Using orthogonal or slanted sweeps, we push all of the crumbs into a single point. The sweep cost is defined as the distance that the sweeper moves.



The best way of sweeping a shape is not necessarily achieved with two sweeps:



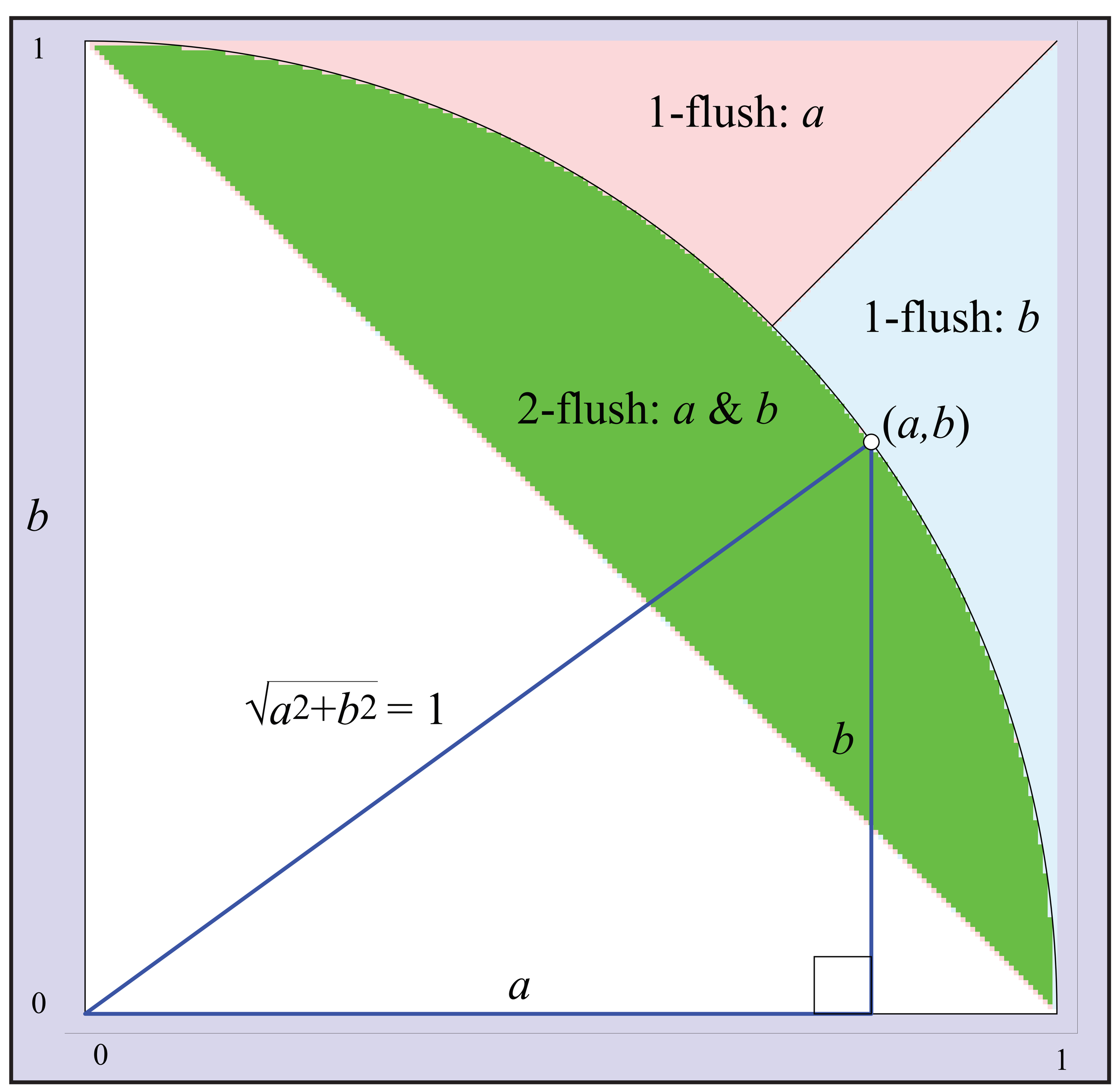
An example requiring three sweeps.

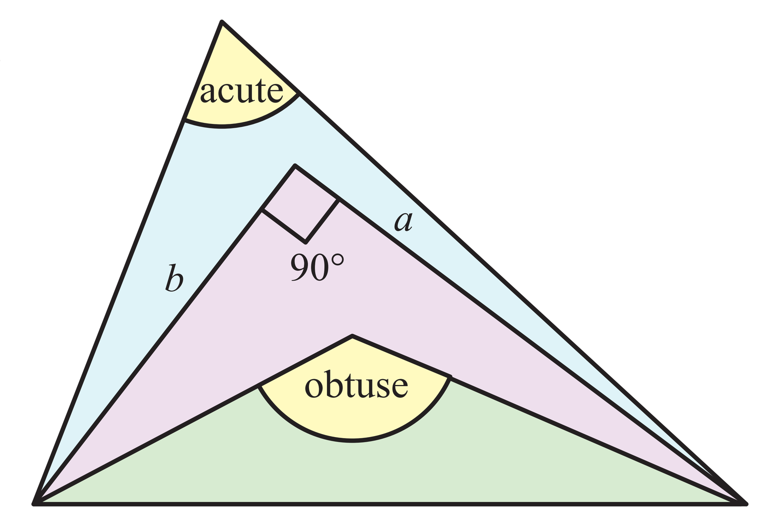
However....

Conjecture: Minimal cost sweeping can be achieved with two sweeps for any convex shape.

Restricting our attention to *two-sweeps* and *triangles*, the minimum sweeping cost is always achieved by enclosing the triangle in a minimum perimeter parallelogram.

One-flush **Lemma:** The minimal perimeter enclosing parallelogram is always flush against at least one edge of the convex hull. [Mitchell and Polishchuk 2006]

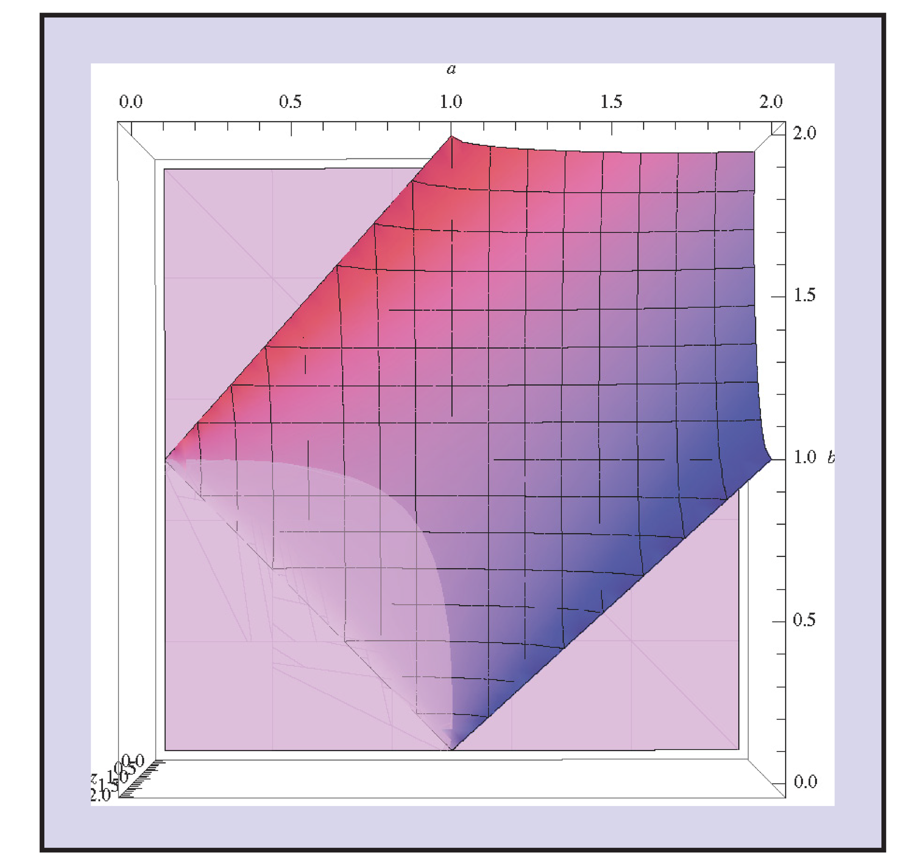




Theorem:

Normalize triangle so that the longest edge=1.
Let θ be the *ab*-apex.
If θ ≥ 90, the min cost sweep is determined by the parallelogram 2-flush against *a* and *b*.
If θ ≤ 90, the min cost sweep is determined by the rectangle 1-flush against the shortest side.

Plot of our flush function, a cubic equation with *b* as a function of *a*.



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Solution2[a_] := 1/3 (-1 + a + (2 2^(1/3) (-2 + a + a^2)) / (20 - 15 a - 3 a^2 - 2 a^3 + 3 Sqrt[3] Sqrt[(-1 + a)^2 (16 + 8 a + 3 a^2)])^(1/3) +
    2^(2/3) (20 - 15 a - 3 a^2 - 2 a^3 + 3 Sqrt[3] Sqrt[(-1 + a)^2 (16 + 8 a + 3 a^2)])^(1/3)) // N;
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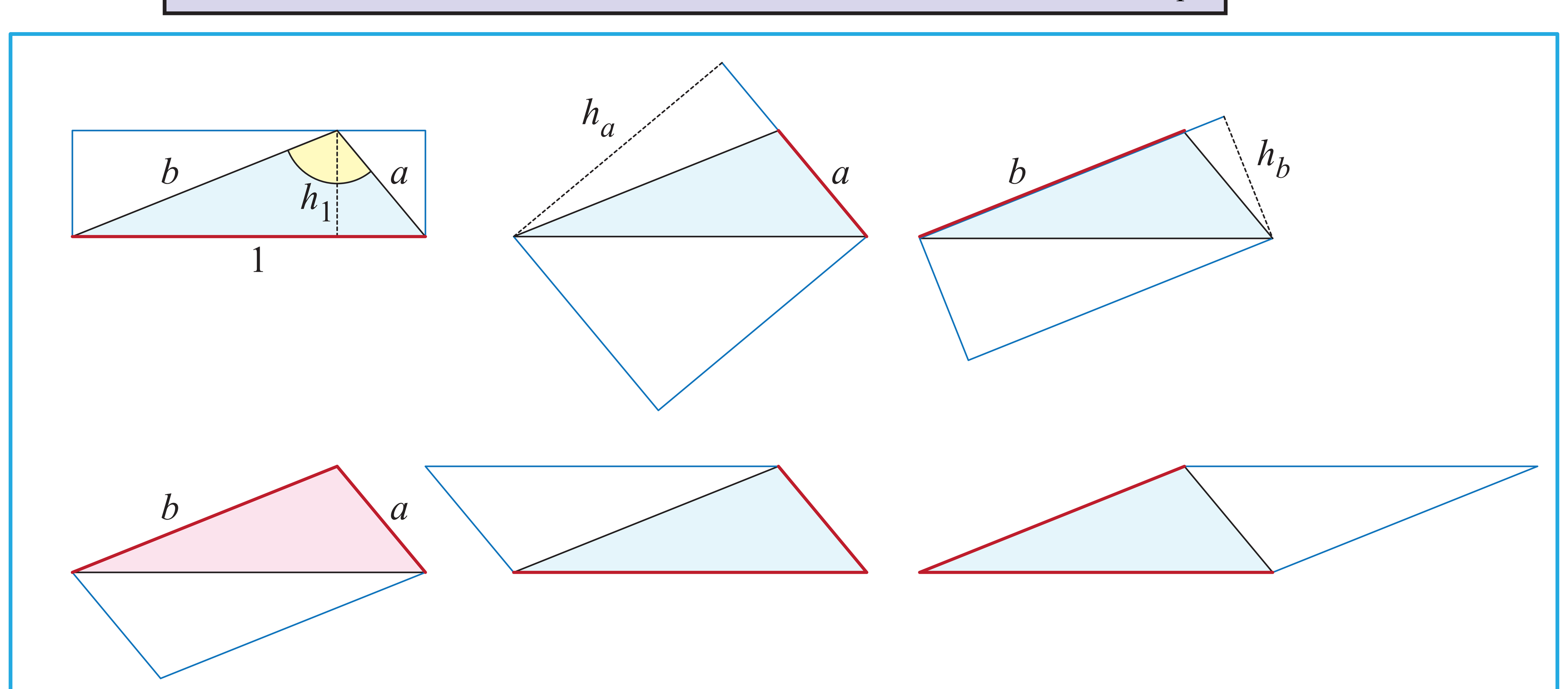
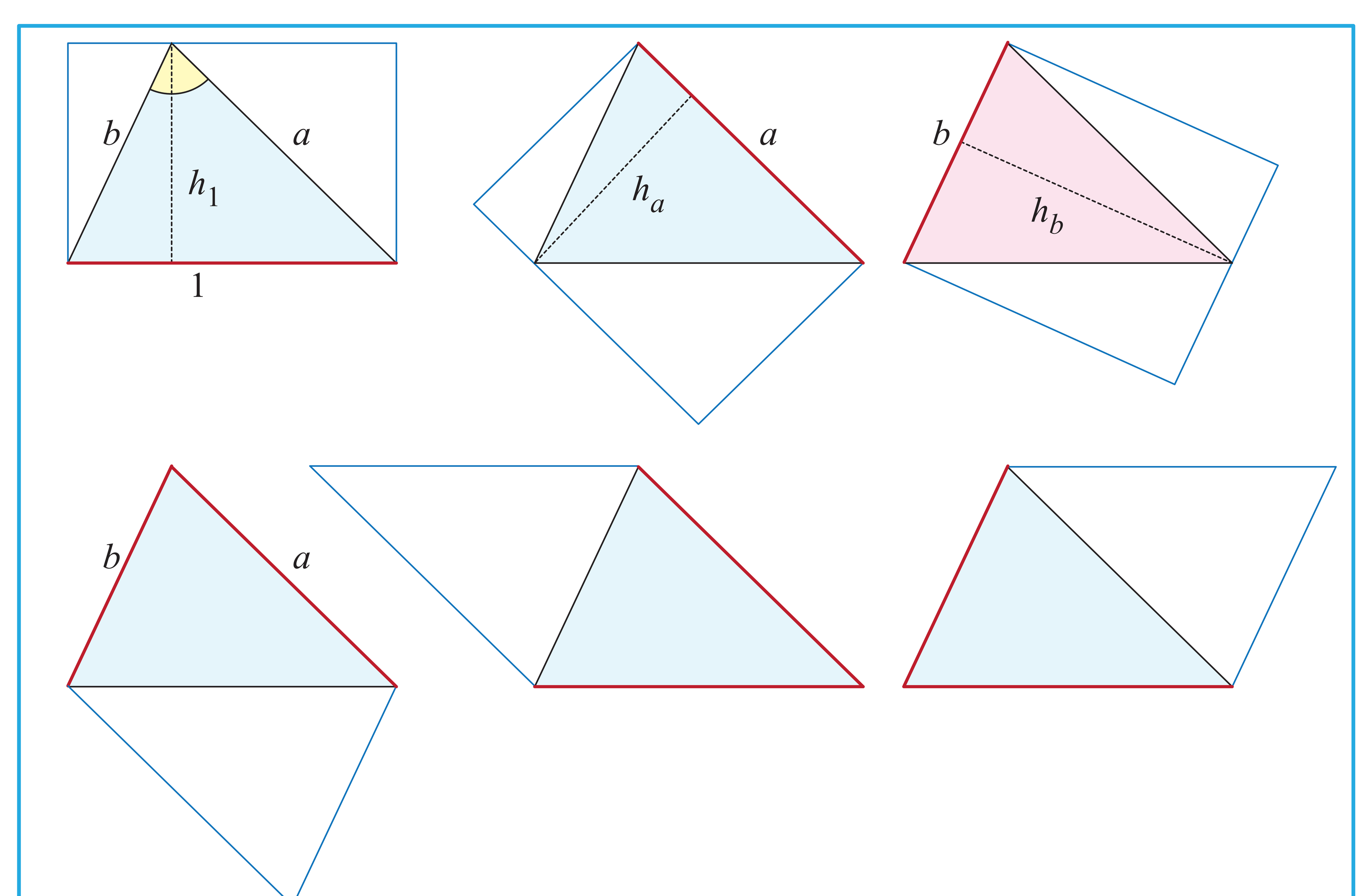
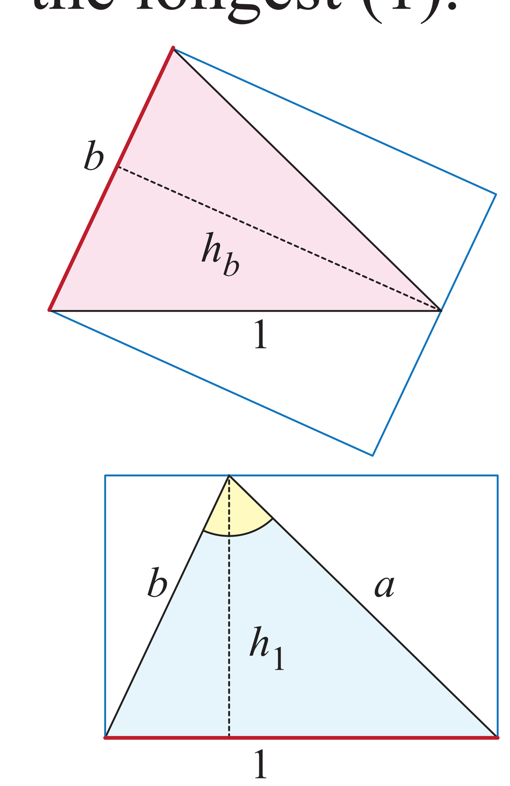


All enclosing parallelograms for obtuse triangles.



All enclosing parallelograms for acute triangles.

Proof that, for acute triangles, the cost of sweeping 1-flush against the shortest side b is less than sweeping 1-flush against the longest (1).



$$h_b < 1$$
$$h_b(1-b) < (1-b)$$
$$h_b - b\,h_b < 1-b$$
$$b \cdot h_b = 1 \cdot h_1$$
$$h_b - h_1 < 1 - b$$
$$h_b + b < 1 + h_1 \quad \square$$