Coloring Objects Built From Bricks

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Abstract

We address a question posed by T. Sibley and Stan Wagon. They proved that rhombic Penrose tilings in the plane can be 3-colored, but a key lemma of their proof fails in the natural 3D generalization. In that generalization, an object is built from bricks, each of which is a parellelopiped, and they are glued face-to-face. The question is: How many colors are needed to color the bricks of any such object, with no two face-adjacent bricks receiving the same color?

We have settled a number of questions on the general problem. For arbitrary parellelopiped bricks, we have proven that zonohedral balls are 4-colorable, and 4 colors are sometimes necessary. For orthogonal bricks, we have several results. First, any object built from such bricks is 4-colorable. Moreover, any genus-zero object (a ball) is 2-colorable. Our most complex result is that any object with no "dividing" holes (ones that a plane parallel to one of the coordinate planes is divided into two disconnected pieces by the hole), regardless of its genus, is 2-colorable. We have examples, however, that require 3 colors. We prove that all genus-one objects are 3-colorable, as well as object of higher genus subject to certain restrictions, and we conjecture that any object built from orthogonal bricks is 3-colorable.

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7.3 An orthogonal object object built from bricks constructed by drilling two holes through a cube. (a) Hole A, parallel to y, and hole B, parallel to z, intersect. Cracks a_1 and a_2 of A do not align. (Not all cracks are shown in the figure.) (b) The yz quarter-planes incident to a_1 and a_2 are therefore not coplanar. 29

Chapter 1

Introduction

Our work stems from questions posed by Stan Wagon at the open-problem session [DO03] of the 14th Canadian Conference on Computational Geometry.[1]

Sibley and Wagon noticed that rhombic Penrose tilings were 3-colorable, and proved that any collection of "tidy" parallelograms in the plane (including those Penrose tilings) is 3-colorable [SW00]. In tidy collections, each pair of bricks intersects either in a single point, or a single whole edge of each, or not at all. We will introduce our own terminology for this concept below. Their proof establishes the existence of an "elbow" in any such collection: a parallelogram with at most two neighbors. (Again, we will introduce our own notation below.) This then supports an inductive coloring algorithm.

As reported in [Wag02], attempts to extend this result to three dimensions have failed, largely because the analog of the elblow lemma is false. Wagon, Robertson, and Schweitzer found a genus-7 polyhedron composed of parallelopipeds in which none has degree three or less. Were such an elbow always present, an inductive argument could establish 4-colorability. Without that lemma, the chromatic number of such objects is unclear.

Although we do not settle the general question for 3D, we establish a number of partial results, which we will summarize after setting notation.

[1] Lethbridge, Alberta, Canada, August 2002.

1.1 Notation and Questions

A *parallelopiped* is a hexahedron composed of three pairs of parallel faces, each of which is necessarily a parallelogram. We will use the simpler term *brick* to refer to the same shape. An *orthogonal brick* is one whose (internal) dihedral angles are all $\pi/2$, and whose faces are necessarily rectangles, i.e., it is a rectangular box.

A *ball* is a solid object that is topologically equivalent to (i.e., homeomorphic to) a solid sphere. To emphasize the dimension of the ambient space $\mathbb{R}^d$, we'll call it a *d-ball*; without qualification a "ball" will be understood to be a 3-ball.

A collection of bricks is said to be *properly joined* if each pair of bricks intersects either in a single point, a single whole edge of each, or a single whole face of each.[2] Two bricks in a collection are *adjacent* if they share a single whole face. A face of a brick that has no other brick face joined to it is called *exposed*. Define the *brick graph* of a collection of bricks to have a node for each brick, and an arc for each pair of adjacent bricks. Sometimes this is called the "dual graph" of the collection.

We will say that an object is *built from bricks* if it is a collection of properly joined bricks whose brick graph is connected.

A *k-coloring* of an object built from bricks is a k-coloring of its brick graph, i.e., an assignment of k colors, one per brick, such that every pair of adjacent bricks are assigned different colors.

The primary question on which we are focussing is due to Stan Wagon, posed in a presentation at CCCG August 2002 (in another guise, with different notation):

> Is every ball built from bricks 4-colorable?

Call any brick that has at most degree-3 in the brick graph an (≤ 3)-*brick*. A brick is a *corner* brick if it has a vertex all three of whose incident faces are exposed. A corner brick is a (≤ 3)-brick, but not every (≤ 3)-brick is a corner brick.

Call a collection of objects built from bricks *strongly shellable* if removal of any brick from an object, that does not disconnect the object's brick graph,

[2] This is the three-dimentional version of what Sibley and Wagon call "tidy".

results in another object in the collection. (Balls are not strongly shellable, because removal of a brick could change the genus.) A collection is *shellable* if there is at least one brick whose removal results in another object in the collection.

Lemma 1.1.1 *If, among a strongly shellable class $\mathcal{C}$ of objects built from bricks, every object has at least one* (≤ 3)*-brick, then every object in $\mathcal{C}$ is* 4*-colorable.*

Proof: Let P be an object in $\mathcal{C}$. By hypothesis, P has a (≤ 3)-brick b. Then $P' = P \setminus b$ is also a member of $\mathcal{C}$ because the class is strongly shellable. Inductively 4-color P'. Repace b, using the color not employed among its at most 3 adjacencies. □

This question is of interest independent of the shelling connection:

Does every ball built from bricks have at least one corner brick?

As mentioned before, there exists a genus 7 object built from bricks that has no (≤ 3)-brick [Wag02] (and so no corner brick). Nevertheless, the question is open for balls built from bricks.

1.2 Summary of Results

1. Results for arbitrary parallelopiped bricks.

   (a) Some balls built from bricks require four colors.

   (b) Every zonohedron (a particular type of ball) has a corner brick.

   (c) Every zonohedron is 4-colorable.

2. Results for orthogonal bricks.

   (a) Every ball built from orthogonal bricks has a corner brick.

   (b) Every object built from orthogonal bricks is 4-colorable.

   (c) Some objects built from orthogonal bricks need three colors.

   (d) Every ball built from orthogonal bricks is 2-colorable.

   (e) Every object built from orthogonal bricks with no "dividing" holes is 2-colorable.

(f) Every object built from orthogonal bricks with dividing holes satisfying certain restrictions is 3-colorable.

(g) Every genus-1 object built from orthogonal bricks is 3-colorable.

Chapter 2

Zonohedra

2.1 Introduction

In this section we consider balls built from general parallelopiped bricks. If the bricks are glued together tree-like, more precisely, such that the brick graph is a tree, then it is easy to see the object is 2-colorable: Remove a leaf, color the remaining object inductively, and now place back the leaf using the color opposite its parent. So it is clear that more colors will only be needed when the object has a higher degree of connectivity. It seems intuitively that the highest degree of connectivity comes from building a sphere-like object from the bricks, surrounding each internal brick as much as possible. For balls (genus zero), this seems to be the worst case. And this worst case is captured in an object known as a zonohedron.

A typical zonohedron is shown in Fig. 2.1. A zonohedron is a convex polyhedron all of whose faces are parallelograms. The notion generalizes to arbitrary dimensions, then called a *zonotope*. Zonohedra are natural candidate objects for us, because clearly any object built from parallelopiped bricks will have parallelogram faces. But is the reverse true? Can every zonohedron be built from bricks? Indeed, this is true, although to understand this requires going a bit deeper into the structure of zonohedra.

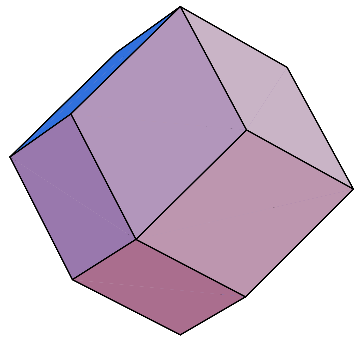

Figure 2.1: A rhombic dodecahedron.

2.2 Structure of Zonohedra

One can generate a zonohedron by starting with n independent vectors (or segments) $e_1, \ldots, e_n$, and taking all the points expressible as scaled versions of these vectors:

$$p = \sum_{i=1}^{n} \alpha_i e_i$$

with $\alpha_i \in [0, 1]$. The convex hull of all such points is a zonohedron, whose edges are all parallel to the n vectors. Coxeter [Cox73, p. 27] calls the n vectors an *n-star*. There is some variation in the literature depending on whether one insists the vectors be independent, in which case all faces are parallelograms, or not, in which case faces could be, e.g., hexagons. The zonohedron in Fig. 2.1 is generated by four segments, no two of which are coplanar.

Two other ways of viewing a zonotope is as the affine projection of a cube from higher dimensions, or as the Minkowski sum of n line segments [Zie94, p. 198ff].

Zonohedra get their name from the zone of faces forming a "belts" around their "middles," each composed of faces with two edges equal in length and parallel. Each face belongs to two zones that cross through it, and cross again at an antipodal position on the surface. The zones are evident in Fig. 2.2 below.

We find the answer to the question we posed above in [Cox73, p. 258]:

Lemma 2.2.1 *Every zonohedron generated by n segments "may be dissected (in various ways) into $\binom{n}{3}$ parallelopipeds, one for every three of the n segments."*

Thus the zonohedron in Fig. 2.1 is built from $\binom{4}{3} = 4$ bricks. It is also worth noting that the brick graph of the rhombic dodecahedron is K_4, thereby establishing that some zonohedra require four colors.

A more complex example of a zonohedron is shown in Fig. 2.2. This was generated by code written by David Eppstein [Epp95] from 30 random vectors. Thus, it may be dissected into $\binom{30}{3} = 4060$ bricks.

Although we have not be able to find a precise proof of Lemma 2.2.1 in the literature, the following was provided by Eppstein[1]:

Proof: "By induction on number of zones and dimension. Choose one zone, and view the zonohedron as being formed by the product of a smaller zonohedron and a single line segment. The facets of the smaller zonohedron are partitioned into two subsets by whether they point towards one end or the other of the segment; you can form the larger zonohedron by gluing the smaller zonohedron to a shell formed by the product of the lower facets and the segment. The lower facets look like a zonohedron one dimension down, so can be divided into parallelograms, and when you take the product of them with a line segment you get a zonohedron." □

Our goal is to prove that any of the dissections of a zonohedron into bricks is 4-colorable. For that we need to connect zonohedra to arrangements.

2.3 Zonohedra and Arrangments

The combinatorics of the faces of a zonohedron are equivalent to those of a simple arrangement of planes in 3-space [Zie94, p. 207]. Each zone of faces corresponds to one plane of the arrangement, perpendicular to the parallel edges defining the zone. Each point of the arrangement where three planes meet corresponds to a brick. The brick shape is determined by the three edges corresponding to the

[1] Personal communication, Oct. 2002

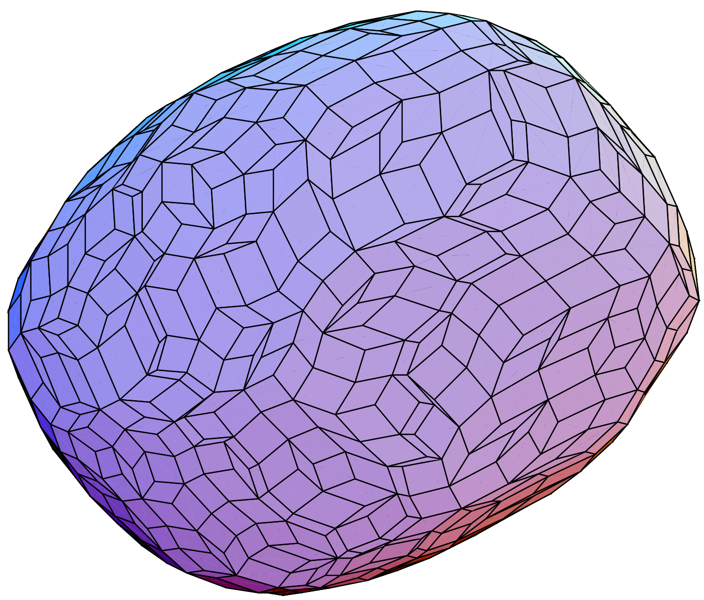

Figure 2.2: Zonohedron generated by 30 random vectors.

three planes: it is the product of the three segments. The arrangement specifies how the bricks are glued together. This is best illustrated in two dimensions.

Fig. 2.3 shows a 2D zonogon based on a 4-star, partitioned into 6 parallelogram bricks. The six bricks are numbered, and correspond to the intersection

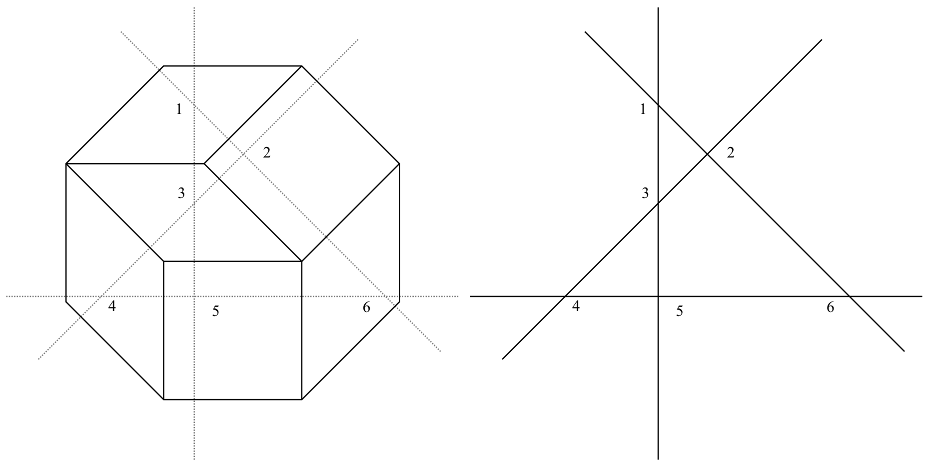

Figure 2.3: An octagon zonogon and its dual arrangement.

points of pairs of lines orthogonal to the zones. Each face of the arrangement corresponds to a vertex of the dissection. Thus the degree-4 vertex shared by bricks $\{2, 3, 5, 6\}$ corresponds to a quadrilateral face of the arrangement.[2] The reason there are "various ways" to dissect a zonohedron into bricks is that there are several combinatorially distinct arrangement constructible from the same set of parallel lines.

Define the *skeleton* of an arrangement of planes as the collection of vertices and finite-length edges connecting those vertices (excluding the infinite rays incident to one vertex). Note the skeleton is a geometric object. It is merely the arrangement with all infinite rays "clipped" off. The way in which the combinatorics of a zonohedron are equivalent to those of the arrangement may be stated as follows:

Lemma 2.3.1 *The skeleton of an arrangement, viewed as a graph, is isomorphic to the brick graph of the associated gluing of bricks.*

[2]That it is actually geometrically inside the face is an accident of the example.

Recall that a *corner* brick is one with a vertex all three of whose incident faces are exposed. A corner brick has degree ≤ 3. (As we mentioned, the similar concept in 2D is called an *elbow* in [SW00].)

Theorem 2.3.2 *A zonohedron built from n bricks has at least four corner bricks if $n \geq 4$. If $n < 4$, all of its bricks are corners.*

Proof: Take the convex hull of the skeleton of the arrangement. If the arrangement is formed from at least four planes, this hull is at least a tetrahedron. Let v be a vertex of the hull. Reorient so that v is the unique leftmost point of the hull. Because the arrangement is simple, every vertex of the arrangement has degree 6, formed by the intersection of three lines at the vertex. But for v in particular, three of edges incident to v are infinite rays. We pause to justify this claim.

Let π be the vertical plane supporting the hull at v; it does not including any face of the hull, because v is strictly leftmost. Let A, B, and C be the three lines of the arrangement that meet at v. None of those lines lie in π, because π includes no hull faces. Therefore, each penetrates π and v, and π partitions the three lines into six rays. Three of those rays must be infinite, because there are no vertices of the arrangement left of v. The claim is established.

As these three rays are clipped off in the skeleton, v has only degree 3 in the skeleton. By Lemma 2.3.1, v corresponds to a brick b, which also therefore has degree 3 in the brick graph. Finally, it is clear that the vertex of the brick that dually corresponds to the infinite cone with those three infinite rays as ribs is "exposed," and therefore satisfies the definition of a corner. Thus b is a corner, and the same holds true for the other vertices of the hull of the skeleton. $\square$

Corollary 2.3.3 *A zonohedron built from bricks is 4-colorable.*

Proof: Remove a corner, apply induction, and reglue the corner brick, using the color not used among its three neighbors. $\square$

Because all the properties we used generalize to arbitrary dimensions, we have also established this:

Theorem 2.3.4 *A zonotope in d dimensions built from n bricks has at least d corner bricks if $n \geq d + 1$. If $n < d + 1$, all of its bricks are corners. Such a zonotope is $(d + 1)$-colorable.*

We have not found this theorem in the literature, but have no doubt it is known to zonotope experts.

Because we feel that zonohedra are in some sense the worst case for coloring, we believe this:

Conjecture 2.3.1 *All objects built from bricks are 4-colorable.*

Chapter 3

Orthogonal Bricks

In this section we will look at objects built from orthogonal bricks, parallelopipeds whose faces are all rectangles. When working with orthogonal bricks, we can assume without loss of generality that all faces are parallel to the coordinate planes. This assumption simplifies the reasoning. In particular, it makes our original question about corner bricks almost trivial.

Theorem 3.0.5 *Every object P built from orthogonal bricks has at least one corner brick.*

Proof: Let B be the bounding box for the object P. The top view of B is a rectangle R. All four sides of R must touch some brick. Let b be the rightmost brick amongst those that touch the front edge of R. Brick b must be exposed above (because it is visible in the top view), exposed to the right (because it is rightmost), and exposed to the front (because it touches the front face of B). Therefore b is a corner brick.

$\square$

Corollary 3.0.6 *Every object built from orthogonal bricks is* 4*-colorable.*

Proof: The set of all objects built from orthogonal bricks forms a strongly shellable class, because removal of any brick leaves an object built from orthogonal bricks. Theorem 3.0.5 shows they have a corner brick, and a corner brick is by definition a (≤ 3)-brick. Lemma 1.1.1 then applies and establishes the claim. $\square$

Chapter 4

Crack Lemma

In looking at objects built from orthogonal bricks, it is often useful to look at those bricks with the same x, y, or z extent, bricks where the cracks align. Call an arc of the brick graph an *x-arc* if the two adjacent bricks share a face parallel to yz; and similarly for y- and z-arcs.

Let $\pi = (b_1, b_2, \ldots, b_k)$ be a path of bricks, i.e., a collection of bricks which induce a path in the brick graph G. Call the path an xy-path if all the arcs of the path are either x- or y-arcs; and similarly define yz- and xz-paths. Call the *z-extent* of a brick to be the interval of z-values it covers; and similarly for x- and y-extents.

Lemma 4.0.7 *In an xy-path of bricks in G, the z-extent of every brick in the path is identical.*

Proof: The proof is by induction on the number of bricks in the path. Let $\pi = (b_1, b_2, \ldots, b_k)$ be the xy-path of bricks.

1. Base case, $k = 2$. Let the arc between b_1 and b_2 be an x-arc wlog. Then, by the definition of an object built from bricks, b_1 and b_2 must share the whole of a yz-face between them. This means both their y- and z-extents must be identical.

2. General case. Suppose the lemma holds up to $k - 1$. Thus the z-extent of b_1 is the same as that of b_{k-1}. There are two possibilities: Either b_k

connects to b_{k-1} via an x-arc or via a y-arc. In the former case, they must share a yz-face and therefore have identical y- and z-extents; in the latter case, they must share a xz-face and therefore have identical x- and z-extents. In either case, they have the same z-extents.

□

A *coordinate plane* is one whose normal is parallel to either the x, y, or z axes; we will identify them by their normals. Say that a plane P *cuts* a brick b if P includes a point strictly interior to b.

Lemma 4.0.8 *If a coordinate z-plane P cuts every brick in a set of bricks B, and if the open (geometric) set $Q = \mathrm{int}(P \cap B)$ is connected, then there is an xy-path in G between any two bricks $b_1 \in B$ and $b_k \in B$.*

Proof: We first observe that any two bricks "adjacent" in Q are connected by an x- or y-arc in G. For if $\mathrm{int}(P \cap (b_1 \cup b_2))$ is a connected set, then the two bricks must share a face, either an xz- or a yz-face, corresponding to an x- or y-arc in G

It now follows that the connected set Q induces a connected subgraph in G whose nodes are the bricks in B and whose arcs are all x- or y-arcs. A path in this subgraph between b_1 and b_k then is an xy-path in G connecting them. (In particular, a shortest path (fewest arcs) is a simple path.) See Fig. 4.1. □

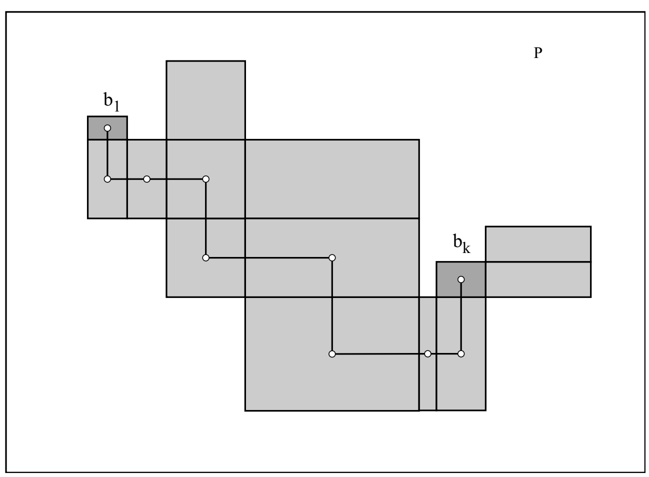

Figure 4.1: Coordinate plane P intersects a set of bricks B.

Chapter 5

Orthogonal Balls are 2-Colorable

In this section we prove this theorem:

> Any ball O (i.e., any genus-zero object) built from orthogonal bricks is 2-colorable.

5.1 Genus Definition

Before we discuss the 2-colorability of balls, it is first useful to say a few things about how we are defining the genus of any particular object built from bricks.

Let O be an object built by bricks, and let $\epsilon > 0$ be a number much smaller than any brick side length. Define O_ϵ to be the object reduced in size by ϵ, meaning that all faces move inwards ϵ. We define the *genus of object* O to be the genus of the surface of O_ϵ, where the latter genus is the familar concept from topology. Thus, vertex-to-vertex and edge-to-edge contacts do not affect the genus of an object. See Fig. 5.1. Note that this shrinking is necessary to obtain a surface, for a *surface* requires the neighborhood of each point to be homeomorphic to a disk. The genus of the surface may be computed via Euler's formula (on either O or O_ϵ—the results will clearly be the same).

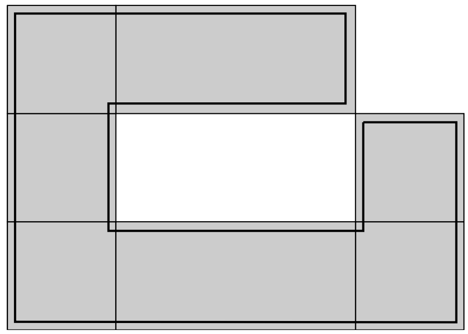

Figure 5.1: A genus-0 object built from bricks. The central rectangle is not a hole when the object's boundaries are displaced inward by ϵ.

It should be noted that most of the theorems we will prove about genus-zero objects would still apply, with modified proofs, if we allowed edge-to-edge and vertex-to-vertex connections to alter the genus.

5.2 2-Colorings

Lemma 5.2.1 *Any genus-zero 2D collection of rectangular bricks is 2-colorable.*

Proof: The proof is by induction. Let R be the collection of rectangles. Let b be the x-most, y-most brick, i.e., the highest brick touching the right side of the bounding rectangle. Remove b, producing $R' = R \setminus b$. Removal of a brick on the boundry of a genus zero object cannot increase the genus, though it may cut the object into disconnected pieces. Therefore, R' has genus zero as well, and the induction hypothesis applies.

Consider all of the ways that the at most two neighbors of b, b_1 and b_2 can be colored. If b has just one neighbor or if b_1 and b_2 have the same color, b can be colored with the opposite color. If the two neighbors have opposite colors, then since the induction hypothesis gives us a proper 2-coloring, then we know b_1 and b_2 cannot share a neighbor, and so the space diagonally to the left and beneath b, call it c, has no brick. See Figure 5.2.

There are now two possible cases. The first is that there is no path in the brick graph of R' between b_1 and b_2, meaning that the brick graph has been divided into two disconnected pieces. Then all of the colors in the piece containing b_2 can be reversed without affecting the color of b_1. Thus, b_1 and b_2 can be made the same color, and b can be colored the opposite as above. The second case is that there is a path in R' between b_1 and b_2. However, that would mean that when b is included, there is a cylce in R which surrounds c, making the genus of R greater than zero, contradicting our original hypothesis. □

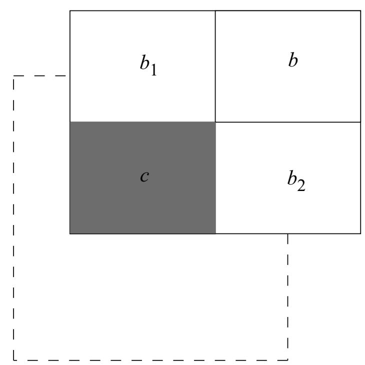

Figure 5.2: A situation where a clash might occur. Note that if there exists a path from b_1 to b_2 other than through b, such as the one along the dotted line, then the grey area becomes a hole in the object, and the object is no longer genus-zero.

Say that a collection of bricks form an *xy-layer* (or just a *layer*) if a z-plane cuts each brick. A *connected layer* is a layer for which the brick graph is connected. By Lemmas 4.0.8 and 4.0.7, the top of all bricks in a connected layer lie at the same z-value. (Note this does not necessarily hold for layers with several connected components.)

We now turn attention to layers that are not genus zero. Let γ be a cycle in a layer. Two bricks in γ are called *opposing* if they are both cut by either a x- or a y-coordinate plane, i.e., they include points at the same height z, and with either the same y- or x-coordinate respectively.

Lemma 5.2.2 *Let O be a genus-zero object containing a connected layer A that has genus greater than zero. Then any pair of opposing bricks in a cy-*

cle surrounding a hole of A have the same extent orthogonal to their plane of opposition, i.e., their "cracks" align.

Proof: Let b_1 and b_k be two bricks of γ that are cut by a y-coordinate plane P, i.e., one parallel to the xz-plane. We now argue that there is a path in $P \cap O$ connecting b_1 to b_k, either above A or below A.

Suppose otherwise; suppose there is no path connecting the bricks above A in $P \cap O$, and none below A. Then we could thread a rope through the top of A, staying entirely inside P, through the interior of the cycle γ, and out the bottom of A. See Figure 5.3. Closing the loop exterior to the object then shows that O has genus at least 1, a contradiction.

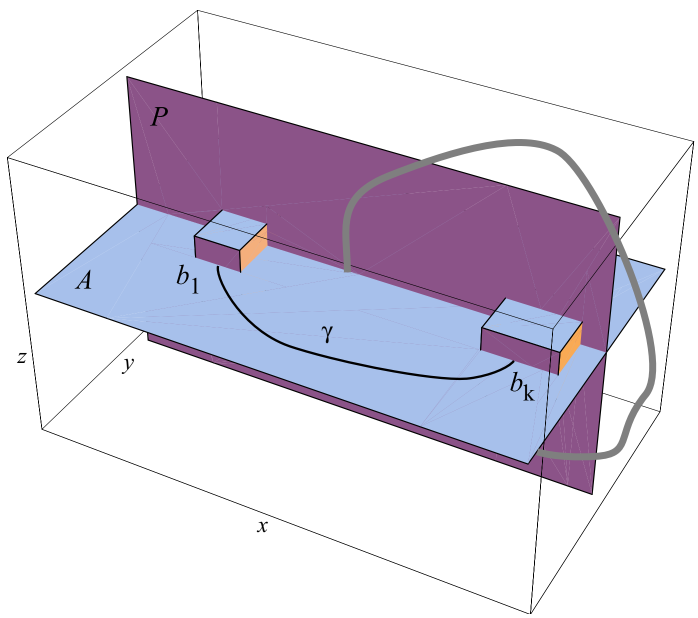

Figure 5.3: A rope showing that the object has genus at least 1.

Now Lemma 4.0.7 applies and shows that all the bricks in the path in P have the same y-extent, and in particular, b_1 and b_k do. □

This lemma will permit us to "fill in" holes in a layer.

Define the intersection of two adjacent layers A and B to be those bricks in layer A which share a face with a brick in layer B, and those bricks in layer B which share a face with a brick in layer A. We will need one more lemma, related to the intersection of two layers, to reach our theorem:

Lemma 5.2.3 *The intersection of two genus-zero layers in a genus-zero object is connected.*

Proof: First note that by Lemma 5.2.2 we can assume that all layers in a genus-zero object are genus-zero, since we can fill in the holes of those that are not.

Now, suppose that that there exist two adjacent layers A and B such that the intersection of A and B is not connected. Then there exists two bricks b_1 and b_k in layer B (and their corresponding adjacent bricks a_1 and a_k in layer A) such that there exists no path between them in the intersection. However, since by definition layer A and layer B are connected, then there does exist a path π_A in layer A from a_1 to a_k and a path π_B in layer B from b_1 to b_k. Without loss of generality, we may assume that none of the bricks in either path are in the intersection, since otherwise we could move b_1 and b_k to the last bricks in the paths which are in the intersection.

Now, assume that layer A is below layer B and lay a rope across the brick in π_A adjacent to a_1 and thread it on top of the A layer until you reach a place where the rope can be threaded down to beneath the B layer. See Figure 5.4. This rope will be external to the object, since because layer B is genus zero, it cannot wrap around and trap the rope, and since the entire object is genus zero, there cannot be a cycle of layers which might trap the rope either. The same applies to layer A. However, this rope has now been threaded through the object, showing that the genus is greater than zero, a contradiction. Therefore, the intersection of any two adjacent layers must be connected. □

Theorem 5.2.4 *Any ball O (i.e., any genus-zero object) built from orthogonal bricks is 2-colorable.*

Proof: Let P be a z-plane that cuts the lowest brick of O. In general $P \cap O$ will contain several connected components. We imagine sweep P upwards over O. Each connected component in one layer becomes a node in a layer graph G_L. Two nodes of this graph are connected by an arc if the corresponding connected layers have a non-empty intersection. See Fig. 5.5.

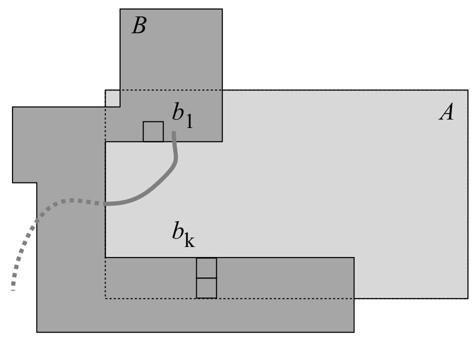

Figure 5.4: Threading a rope through two layers with a disconnected intersection.

(Note that each object has three layer graphs, one for each possible sweeping coordinate plane.)

Because O is genus zero, G_L cannot contain any cycle. Therefore it is bipartite. Therefore it may be 2-colored. Let the colors be a and b. Thus each connected layer is assigned a “super” color by this assignment.

Let A be one connected layer. We now argue that its brick graph is 2-colorable. If it is genus zero, it is 2-colorable by Lemma 5.2.1. If it is not genus zero, then opposing bricks around a cycle have the same extent by Lemma 5.2.2. This permit us to “fill in” holes in A with bricks that match that extent. This renders A genus zero, and it is again 2-colorable.

Finally, we use the super 2-coloring by $\{a, b\}$ to create a 2-coloring of the brick graph. Let A and B be two connected layers adjacent in G_L. Color each brick $b_2 \in B$ the opposite the color of its adjacent brick $b_1 \in A$ (if there is such a brick). We will never get any clashes, since if two bricks in B are both adjacent to bricks in A, then there will be a path between them such that every brick in the path is adjacent to a brick in A by Lemma 5.2.3. Thus, the 2-colorings of the layers will mesh smoothly together, and the entire graph is 2-colorable. □

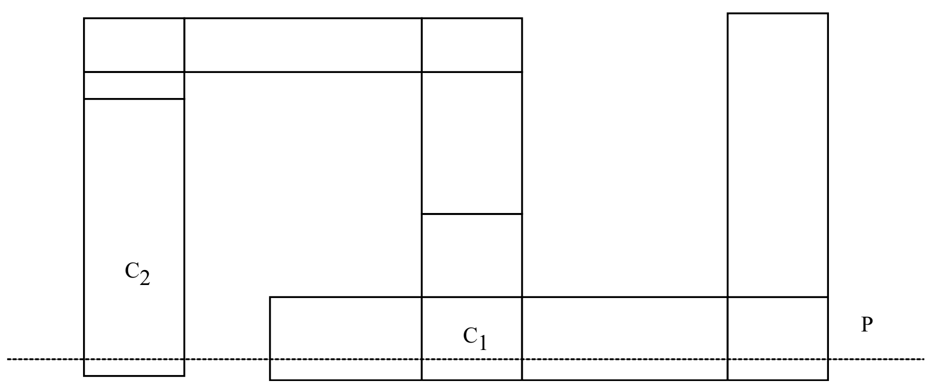

Figure 5.5: Sweeping P up through the layers.

Chapter 6

Objects of Higher Genus

Let O, an object built from orthogonal bricks, be an object of arbitrary genus. Call a hole in O a *dividing hole* if a plane parallel to one of the coordinate planes passes through the hole in such a way that the plane is cut into two disconnected pieces by the hole.

Theorem 6.0.5 *If an object built from orthogonal bricks has no dividing holes, then it is* 2*-colorable.*

Proof: In any given layer, look at a cycle around a hole, and let b_1 and b_k be bricks which are cut by a plane P parallel to a coordinate plane. Since the hole is not a dividing hole, the hole does not cut P into disconnected pieces, so there must be a path between b_1 and b_k. Thus, Lemma 4.0.7 applies, allowing us to "fill in" the hole, giving a genus-zero object. □

Objects built from orthogonal bricks are not 2-colorable in general; in fact, one can find an example in 2D of a genus-one object built from orthogonal bricks which is 3-chromatic because of the existence of an odd cycle. See Figure 6.1.

By Sibley and Wagon's theorem, every 2D object built from orthogonal bricks is 3-colorable, but in 3D, it becomes more difficult to see if that is still the case.

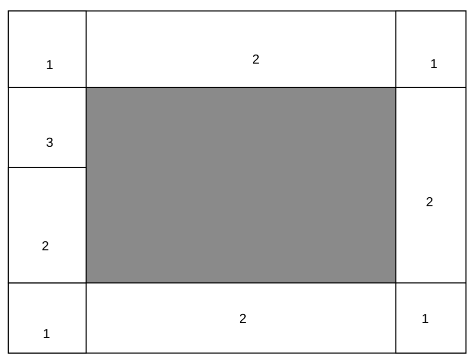

Figure 6.1: An example of a 3-chromatic graph built from orthogonal bricks.

Chapter 7

Shine-Through Holes

In this section we will look at some special cases of objects built from orthogonal bricks which are 3-colorable. Define a hole to be *shine-through* if it is such that a light oriented parallel to one of the three coordinate axis will be able to shine through to the other side. Note that shine-through holes are dividing holes.

Theorem 7.0.6 *If an object built from orthogonal bricks has only shrine-through holes which are all oriented in the same direction, then it is* 3*-colorable.*

Proof: Assume, without loss of generality, that the holes are oriented parallel to the z-axis. Then, starting with the left-most hole, cut the object from the lower left corner to the front of the object. Do this for every hole. If while cutting the hole, you run into another hole, simply finish the cut there. See Figure 7.1. When all of these holes have been cut, you will be left with a genus 0 object, which is 2-colorable. Two-color the object.

In order to repair the cuts, there are two cases. The first is that the coloring on both sides of the cut matches up, so that the two halves can be simply merged. The second is that the coloring schemes clash, so a change has to be made. Merge and erase all the cuts which can be merged, the assign A or B to the remaining cuts in an alternating fashion, starting with the left-most. Then, on the A cuts, on the row of cells immidiately to the left of the cut, change all of those colored 1 to color 3, and on the right side of the cut, change all of those colored 2 to color 3. On the B cuts, do the reverse, changing 2 to 3 on left side and 1 to 3 on the left side. Each of these changes will locally eliminate the clash, and the fact that they are done in an alternating way will insure that they never cause a clash with each other. See Figure 7.2. □

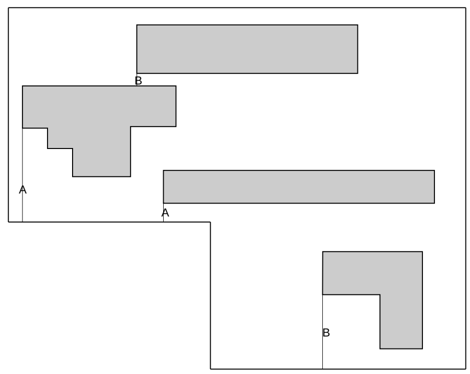

Figure 7.1: Cutting the cycles to get a genus 0 object.

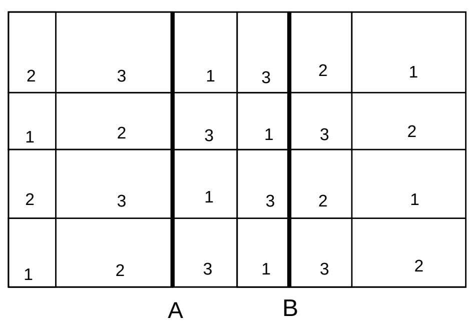

Figure 7.2: Repairing along the cuts. Even when the cuts are very close, there are no clashes.

A similar proof can be worked out for objects with shine-through holes which are oriented in two directions:

Theorem 7.0.7 *If an object built from orthogonal bricks has only shrine-through holes which are all oriented in one of two directions, then it is* 3*-colorable.*

Proof: Assume, without loss of generality, that all holes are parallel to either the z-axis or the y-axis. Then, consider the holes in different orientations separately, and follow the same scheme as above, making all cutting planes parallel to the yz plane. The only difficulty that may arise here is if holes in two different directions intersect. The crack lemma will not hold here, since there is no path in a yz plane which connects bricks on opposing sides of the hole. Thus, the crack which we are cutting along on the top of the hole may fail to exist at the bottom of the hole. If this happens, then simply treat four "quadrants" of the hole as separate holes and cut them using different cuts, as illustrated in Figure 7.3. Now, since all of the cutting planes are parallel, we can assign them A or B in an alternating fashion, as described in the theorem above, and then use the repair scheme described above to repair the cuts. □

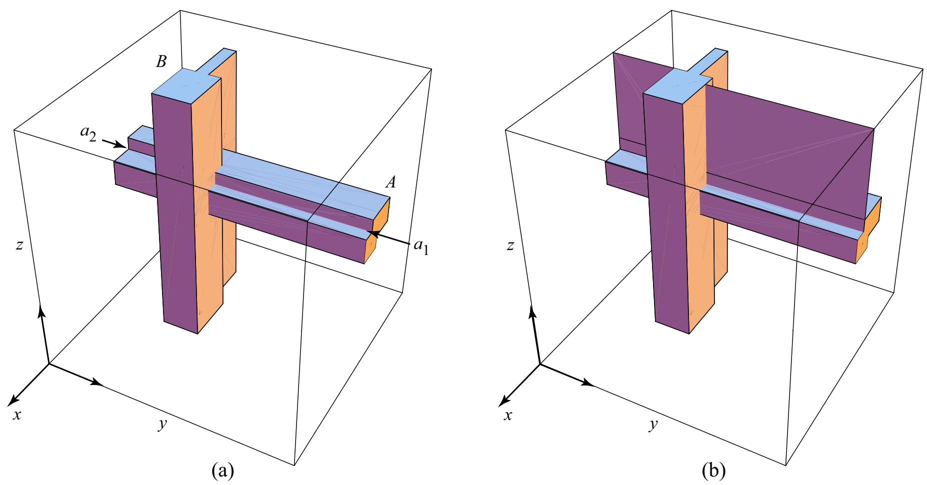

Figure 7.3: An orthogonal object object built from bricks constructed by drilling two holes through a cube. (a) Hole A, parallel to y, and hole B, parallel to z, intersect. Cracks a_1 and a_2 of A do not align. (Not all cracks are shown in the figure.) (b) The yz quarter-planes incident to a_1 and a_2 are therefore not coplanar.

Another way to 3-color an object of orthogonal bricks with shine-through holes is to cut the shape up into larger, two-colorable shapes we call superblocks. Define two holes to be *well-separated* if a cut can be made from the right side of each to the boundry without intersecting the other.

Lemma 7.0.8 *If an object made of orthogonal bricks can be cut into an* 2*-colorable arrangement of* 2*-colorable superblocks, then the object itself is* 3*-*

colorable.

Proof: Two-color bricks in the individual superblocks using colors 1 and 2 and then color the arrangement of superblocks using colors A and B. Then, merge the superblocks. If the colors do not clash, simply merge the cuts. If they do, change the coloring in the following fashion:If the block is colored A, change 1 to 3; if the block is colored B, change 2 to 3. □

Theorem 7.0.9 *If all the holes of an object built from orthogonal bricks are shine-through and well-separated, the object is* 3*-colorable.*

Proof: Cut each hole from the right side to the boundry, making a grid-like arrangement of superblocks. This arrangement will be 2-colorable, since it is simply a grid, and thus the 3-colorability follows from lemma 7.0.8. □

In fact, it is not actually neccessary for the holes to be well-separated in every direction. It is enough for one direction to have well-separated hole to guarentee that the superblock arrangement will be 2-colorable.

Theorem 7.0.10 *If all of the holes of an object built from orthogonal bricks are shine-through and well-separated in at least one of the three axis directions the object is* 3*-colorable.*

Proof: Assume that the holes in the z direction are well-seperated. In the z direction, cut the holes into superblocks as described above. In the x and y directions, cut the holes to both boundries using cutting planes parallel to the xy plane. When two holes intersect in such a way that the cut to the boundry from one of them cannot be continued to the other side, because the crack no longer exists, simply shift the cut to the right until a crack is found where it can continue. This will lead to a jagged, but still two-colorable arrangment of superblocks in each layer, every layer on top of it will be identical, making the entire arrangment of superblocks two colorable, leading to a 3-coloring for the object. □

Chapter 8

The General Case

While this focus on shrine-through holes may seem arbitrary, shine-through holes are not as specific a case as they may at first appear. By Theorem 6.0.5 and the application of the crack lemma, non-dividing holes can be "filled in," so the only holes we have to worry about are dividing holes. Shine-through holes are the easiest examples of dividing holes, since they divide planes in two directions rather than just one. However, many of our theorems for shine-through holes could be modified to be used with dividing holes, as with the theorem below:

Theorem 8.0.11 *If all of the dividing holes of an object divide parallel planes, the object is* 3*-colorable.*

Proof: Identical to the proofs of Theorem 7.0.6 and Theorem 7.0.7. For each hole, use a plane divided by that hole as a cutting plane. □

Corollary 8.0.12 *Every genus-one object built from orthogonal bricks is* 3*-colorable.*

So if there exists a 4-chromatic object built from orthogonal bricks, what might it look like? Our theorems do not cover obejcts with shine-through holes which are "crowded" (not well-seperated) in all three directions or "crooked" (not shine-through) holes which do not divide parallel planes. We do believe that these objects are 3-colorable, however, and that perhaps modified versions of our proofs might cover more of them.

Although we cannot offer a general proof which works for arbitrary genus, given the high number of specific cases for which we can prove 3-colorability, we offer the following conjecture:

Conjecture 8.0.1 *All objects built from orthogonal bricks are 3-colorable.*

In some ways, it would be surprising if this conjecture is true. In two dimensions, all genus-zero objects built from orthogonal bricks are 2-colorable, while objects of arbitrary genus are 3-colorable. It would be interesting if these theorems transfered directly to three dimentions without any need to add additional colors, and it leads to some interesting speculation about what the case might be in higher dimentions.

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