

# A Note on Solid Coloring of Pure Simplicial Complexes

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## Abstract

We establish a simple generalization of a known result in the plane. The simplices in any pure simplicial complex in  $\mathbb{R}^d$  may be colored with  $d+1$  colors so that no two simplices that share a  $(d-1)$ -facet have the same color. In  $\mathbb{R}^2$  this says that any planar map all of whose faces are triangles may be 3-colored, and in  $\mathbb{R}^3$  it says that tetrahedra in a collection may be “solid 4-colored” so that no two glued face-to-face receive the same color.

## 1 Introduction

The famous 4-color theorem says that the regions of any planar map may be colored with four colors such that no two regions that share a positive-length border receive the same color. A lesser-known special case is that if all the regions are triangles, three colors suffice. For the purposes of generalization, this can be phrased as building a planar object by gluing triangles edge-to-edge, and then 3-coloring the triangles. Because the coloring constraint in this formulation only applies to triangles adjacent the dual graph—whose nodes are triangles and whose arcs join triangle nodes that share a whole edge—slightly more general objects can be 3-colored: *pure* (or *homogenous*)<sup>1</sup> *simplicial complexes* in  $\mathbb{R}^2$ , whose dual graph may have several components, with independent colorings. See Figure 1.

For simplicity, we will call such a complex a *triangle complex*, its analog in  $\mathbb{R}^3$  a *tetrahedron complex*, and the generalization a *d-simplex complex*. We permit these complexes to contain an infinite number of simplices; e.g., tilings of space by simplices are such complexes. The main result of this note is:

**Theorem 1** *A d-simplex complex may be (d+1)-colored in the sense that each simplex may be colored with one of d+1 colors so that any pair of simplices that share a (d-1)-facet receive different colors.*

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<sup>1</sup>Pure/homogenous means that there are no dangling edges or isolated vertices, and in general, no pieces of dimension less than  $d$  that are not part of a simplex of dimension  $d$ . So the complex is a collection of  $d$ -simplices glued facet-to-facet.

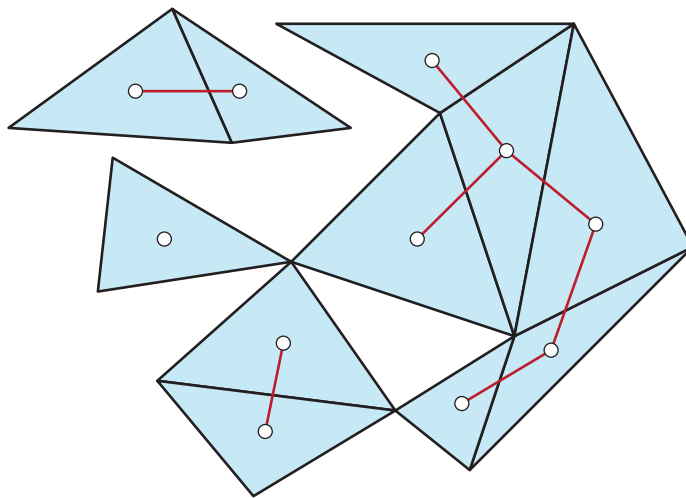


Figure 1: A triangle complex and its dual graph  $G$ .

One can think of the whole volume of each simplex being colored—so “solid coloring” of tetrahedra in  $\mathbb{R}^3$ . Although I have not found this result in the literature, it is likely known, as its proof is not difficult—essentially, remove one simplex and induct. Consequently, this note should be considered expository, and I will describe proofs in more detail than in a research announcement. Perhaps more interesting than the result itself are the many related questions in Section 5.

## 2 Triangle Complexes

Let  $G$  be the dual graph of a triangle complex, and let  $\Delta(G) = \Delta$  be the maximum degree of nodes of  $G$ . For triangle complexes,  $\Delta = 3$ . Let  $\chi(G) = \chi$  be the chromatic number of  $G$ . An early result of Brooks [Bro41] says that  $\chi \leq \Delta + 1$  for any graph  $G$ . For duals of triangle complexes, this theorem only yields  $\chi = 4$ , the 4-color theorem for triangle complexes. We now proceed to establish  $\chi = 3$  in three stages:

1. We first prove it for finite triangle complexes.
2. We then apply a powerful result of deBruijn and Erdős to extend the result to infinite complexes.
3. We formulate a second proof for infinite complexes that does not invoke deBruijn-Erdős.

The primary reason for offering two proofs is that related questions raised in Section 5 may benefit from more than one proof approach.

## 2.1 Finite Triangle Complexes

Let  $S$  be a triangle complex containing a finite number of triangles, and  $G$  its dual graph. Let  $C(S) = C$  be the convex hull of  $S$ , i.e., the boundary of the smallest convex polygon enclosing  $S$ . The proof is by induction on the number of triangles, with the base case of one triangle trivial.

**Case 1.** There is a triangle  $t$  with at least one edge  $e$  on  $C$ . Then  $e$  is *exposed* (i.e., not glued to another triangle of the complex), and  $t$  has at most degree 2 in  $G$ . Remove  $t$  to produce complex  $S'$ , 3-color  $S'$  by induction, put back  $t$ , and color it with a color distinct from the colors of its at most 2 neighbors in  $G$ .

**Case 2.** No triangle has an edge on  $C$ . Let  $v$  be any vertex of  $C$ , and let  $t$  be the most counterclockwise (ccw) triangle incident to  $v$ . See Figure 2. Then the ccw edge  $e$  of  $t$  incident to  $v$  is exposed. Then—just as in the

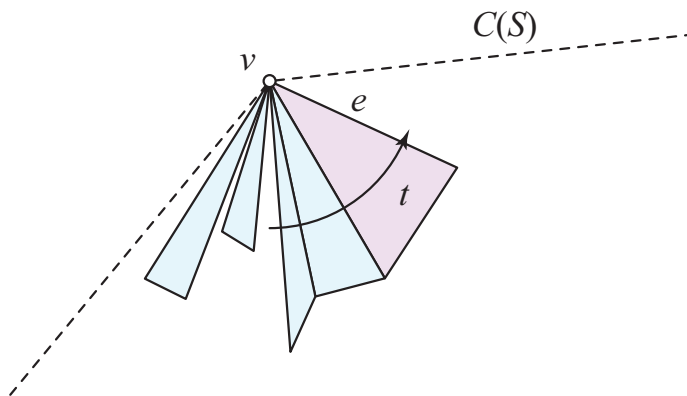


Figure 2: Triangle  $t$  has an exposed edge  $e$ .

previous case—remove  $t$ , 3-color by induction, put  $t$  back colored with a color not used by its at most two neighbors.

This simple induction argument establishes  $\chi = 3$  for finite triangle complexes.

## 2.2 deBruijn-Erdős

The result of deBruijn and Erdős is this [EdB51]:

**Theorem 2** *If a graph  $G$  has the property that any finite subgraph is  $k$ -colorable, then  $G$  is  $k$ -colorable itself.*

This immediately extends the result just proved to infinite triangle complexes. Note that the induction proof presented fails for infinite complexes, because it is possible that every triangle has degree 3 in  $G$  for infinite complexes, for example, in a triangular tiling.

### 2.3 Proof based on $K_r$

The alternative proof in some sense “explains” why a triangle complex is 3-colorable: because it does not contain  $K_4$  as a subgraph. Of course we could obtain this indirectly by using the above proof and conclude that  $K_4$  could not be a subgraph (because it needs 4 colors), but establishing it directly gives additional insight.

We rely here on this result, obtained independently by several researchers (Borodin and Kostochka, Catlin, and Lawrence, as reported in [Sta02]):

**Lemma 1** *If  $G$  does not contain any  $K_r$  as a subgraph,  $4 \leq r \leq \Delta + 1$ , then*

$$\chi \leq \frac{r-1}{r}(\Delta+2).$$

We will now show that  $K_4$  is not a subgraph of  $G$  for triangle complexes, which, because  $r = 4$  and  $\Delta = 3$ , then implies

$$\chi \leq \frac{3}{4}(3+2) = 3\frac{3}{4},$$

and so (because  $\chi$  is an integer),  $\chi \leq 3$ .

**Lemma 2**  $K_4 \not\subseteq G$ .

**Proof:** *Sketch.* We only sketch the argument, because in the Appendix we prove more formally the extension to  $\mathbb{R}^d$ , including  $d = 2$ .

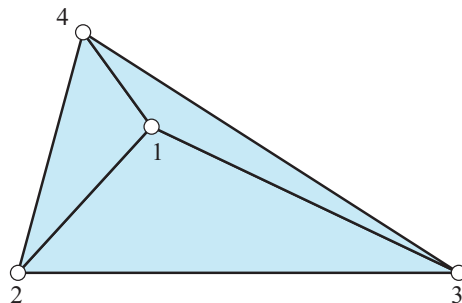


Figure 3: Triangles forming  $K_3$ .

If  $K_4$  is a subset of  $G$ , then  $K_3$  must be as well. The only configuration of triangles that realizes  $K_3$  is that shown in Figure 3: the three triangles share and surround a vertex (labeled 1 in the figure). Now consider attempting to extend this to  $K_4$  by gluing another triangle to the only uncovered edge of  $\Delta\{1, 2, 3\}$ , edge  $e = \{2, 3\}$ . Its apex, call it  $v_5$ , must lie below  $e$ , but because  $v_4$  lies above  $e$ , the new triangle  $\Delta\{2, 3, 5\}$  cannot share the edges  $\{2, 4\}$  and  $\{3, 4\}$ , which it

must to be adjacent to the other two triangles. Therefore,  $K_4$  cannot occur in  $G$ , and we have established the claim.  $\square$

And as we argued above, Lemmas 1 and 2 together imply that  $\chi(G) \leq 3$ : triangle complexes are 3-colorable.

### 3 Tetrahedron Complexes

Again we follow the same procedure as above, although we will defer consideration of  $K_5$  to general  $d$ -simplex complexes to the Appendix, Section 6. Now  $S$  is a finite tetrahedron complex,  $G$  its dual graph, and  $C$  the convex hull of  $S$ , the boundary of a convex polyhedron. Again the proof is by induction. Although we could repeat the structure of the proof for triangle complexes, we opt for an argument that more easily generalizes to  $d$  dimensions.

Let  $v$  be a vertex of the hull  $C = C(S)$ , and let  $S_v$  be the subset of  $S$  of tetrahedra incident to  $v$ . Let  $C_1 = C(S_v)$  be the convex hull of  $S_v$ . If there is a tetrahedron  $t \in S_v$  with at least one face  $f$  lying on  $C_1$ , then  $t$  has at most 3 neighbors in  $S$ . Remove  $t$ , 4-color the smaller complex  $S'$ , put  $t$  back, and color it with a color not used for its at most 3 neighbors. Note that it could well be that the face  $f$  lies on  $C(S)$  because  $C_1$  and  $C$  coincide at  $f$ . But having  $f$  on  $C$  is not the crucial fact; if it is on  $C_1$ , it is exposed, and induction then applies.

If no tetrahedron in  $S_v$  has a face on  $C_1$ , then there must be a tetrahedron  $t$  that has an edge  $e$  on  $C_1$  (in fact, there must be at least three such tetrahedra). See Figure 4.

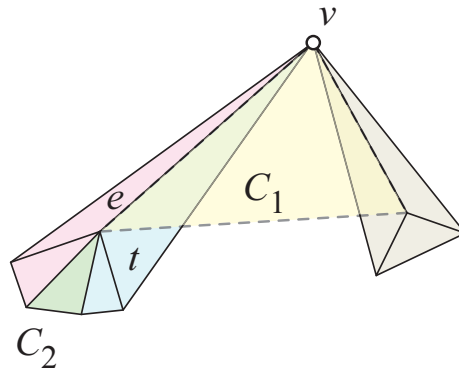


Figure 4: No tetrahedron has a face on  $C_1$ .

Let  $S_e$  be the subset of those tetrahedra in  $S_v$  that share  $e$ . Let  $C_2 = C(S_e)$  be the convex hull of these tetrahedra. It must be that at least one tetrahedron has a face on  $C_2$ . The tetrahedra sharing  $e$  are angularly sorted about  $e$ , and we can select the most ccw one (which might be the same as the most cw one if  $|S_e| = 1$ ). So we have identified a tetrahedron with an exposed face, and induction applies and establishes the result: finite tetrahedon complexes have

$\chi = 4$ . Infinite tetrahedon complexes follow from Theorem 2. And we could now work backward to conclude that  $G$  cannot contain  $K_5$  as a subgraph.

## 4 $d$ -Simplex Complexes

We repeat the outline just employed. The only difficult part is showing that in a finite  $d$ -simplex complex  $S$ , there must be a simplex with an exposed facet.<sup>2</sup> Then induction goes through just as before.

Say that a convex hull  $C$  of points in  $d$  dimensions is *full-dimensional* if  $C$  is not contained in a  $(d-i)$ -dimensional flat (hyperplane) for any  $i > 0$ .

Let  $v$  be a vertex of the hull  $C = C(S)$ , and let  $S_v$  be the subset of  $S$  of simplices incident to  $v$ . Let  $C_1 = C(S_v)$  be the convex hull of  $S_v$ ; this is a  $d$ -polytope that contains  $S_v$ . If there is a simplex  $\sigma \in S_v$  with at least one  $(d-1)$ -dimensional facet  $f$  contained in  $C_1$ , then  $\sigma$  has at most  $d$  neighbors in  $S$ , and induction establishes that  $S$  may be  $(d+1)$ -colored.

So suppose that no simplex in  $S_v$  has a  $(d-1)$ -dimensional facet on  $C_1$ . Let  $|S_v| = n$ . We must have  $n > 1$ , because otherwise  $C_1$  would bound a single simplex, and all of its facets would be on  $C_1$  and so exposed. We know  $C_1$  is full dimensional because it contains  $d$ -simplices. Let  $\sigma_1 \in S_v$  be a simplex that has a  $k$ -dimensional face  $f_1$  in  $C_1$ , such that  $k < d - 1$  is maximal among all simplices with faces in  $C_1$ . We claim that there must be another simplex  $\sigma' \in S_v$  that also has a face  $f'$  in  $C_1$ , where  $f' \neq f_1$ . For suppose otherwise, that is, suppose that all simplices in  $S_v$  share  $f_1$ . Then, because  $C_1$  is full-dimensional, one of these simplices  $\sigma''$  must have a vertex  $u$  not part of  $f_1$  on  $C_1$  (otherwise all simplices lie in the flat containing  $f_1$ ). But then  $\sigma''$  has a face (the hull of  $u$  and  $f_1$ ) on  $C_1$  of dimension larger than  $k$ , contradicting the choice of  $\sigma_1$ .

So  $\sigma'$  has a face on  $C_1$ , and  $\sigma'$  does not share  $f_1$ . Let  $S_{f_1}$  be all the simplices in  $S_v$  that share  $f_1$ , and let  $C_2$  be the convex hull of  $S_{f_1}$ . Because we know that  $\sigma' \notin S_{f_1}$ ,  $|S_{f_1}| < n$ .

Now the argument is repeated:  $C_2$  is full-dimensional because it includes at least one  $d$ -simplex  $\sigma_1$ . If some simplex in  $S_{f_1}$  has a  $(d-1)$ -dimensional facet on  $C_2$ , we have identified an exposed face. Otherwise, we select some simplex  $\sigma_2$  with a face  $f_2$  on  $C_2$ , and separate out into  $S_{f_2}$  all the simplices sharing  $f_2$ .  $S_{f_2}$  must have at least one fewer simplex than does  $S_{f_1}$ , following the same reasoning.

Continuing in this manner, we identify smaller and smaller subsets of  $S$ :

$$|S| \geq |S_v| > |S_{f_1}| > |S_{f_2}| > \dots$$

via repeated convex hulls  $C_1, C_2, \dots$ , and eventually either identify a simplex with a  $(d-1)$ -dimensional facet on the corresponding hull  $C_i$ , or reach a set of one simplex, which has all of its facets exposed. So there is always a simplex with an exposed facet:

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<sup>2</sup>We use *facet* for a  $(d-1)$ -dimensional face, and *face* for any smaller dimensional face.

**Lemma 3** *Any finite  $d$ -simplex complex contains a simplex with an exposed  $(d-1)$ -dimensional facet.*

Given the nearly obvious nature of this lemma, it seems likely there is a less labored proof that identifies an exposed simplex more directly.

This lemma then proves Theorem 1 for finite complexes, and deBruijn-Erdős establishes it for infinite complexes. Again we may now conclude that  $K_{d+2}$  cannot be a subgraph of  $G^{(d)}$ , where we use the notation  $G^{(d)}$  for the dual graph of a  $d$ -simplex complex. A geometric proof of this non-subgraph result is offered in the Appendix. With that, we obtain an alternative proof of Theorem 1, which we restate in slightly different notation:

**Theorem 3** *The dual graph  $G^{(d)}$  of a  $d$ -simplex complex in  $\mathbb{R}^d$  has chromatic number  $\chi \leq d + 1$ .*

**Proof:** Lemma 7 tells us that  $K_r$  is not a subgraph of  $G = G^{(d)}$ , with  $r = d + 2$ . We have that  $\Delta = d + 1$  because each  $d$ -simplex has  $d + 1$  facets. Therefore we have

$$4 \leq r = d + 2 \leq \Delta + 1 = d + 2$$

for  $d \geq 2$ . Therefore Lemma 1 applies, and yields

$$\chi \leq \frac{d+1}{d+2}(d+3).$$

Now we can see that

$$\frac{d+1}{d+2}(d+3) < d+2$$

by expanding  $(d+1)(d+3)$  and  $(d+1)^2$ :

$$d^2 + 4d + 3 < d^2 + 4d + 4.$$

Thus  $\chi$  is strictly less than  $d+2$ , which, because  $\chi$  is an integer, implies  $\chi \leq d+1$ .  $\square$

## 5 Beyond Simplices

One can ask for analogs of Theorem 1 for complexes composed of shapes beyond simplices. In the plane, a natural generalization is a complex built from convex quadrilaterals glued edge-to-edge. These complexes sometimes need four colors, as the example in Figure 5 shows. One does not need the 4-color theorem for this restricted class, even without the convexity assumption: there must exist a quadrilateral in a quadrilateral complex with an exposed edge, and 4-coloring follows by induction. Complexes built from pentagons can be proved 4-colorable by modifying the Kempe-chain argument;<sup>3</sup> so again the full 4-color theorem is not needed here.

<sup>3</sup>I owe this observation to Sergey Norin, <http://mathoverflow.net/questions/49743/4-coloring-maps-of-pentagons>.

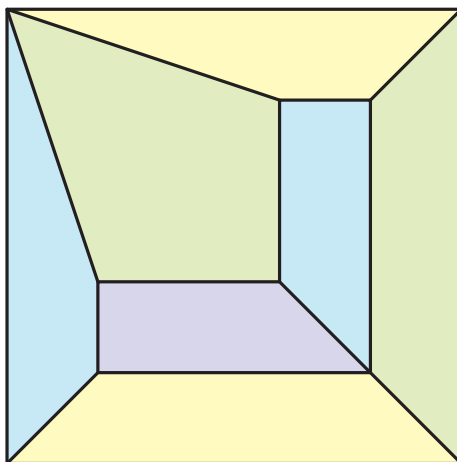


Figure 5: A convex quadrilateral complex that needs four colors [SW00, Fig.3a].

Sibley and Wagon proved in [SW00] the beautiful result: if the convex quadrilaterals are all parallelograms, then three colors suffice (essentially because there must be a parallelogram with two exposed edges). In particular, Penrose rhomb tilings (their original interest) are 3-colorable. Even more restrictive is requiring that the parallelograms be rectangles. Here with a student I proved in [GO03] that such *rectangular brick* complexes of genus 0 are 2-colorable. It is easily seen that complexes of genus 1 or greater might need three colors (surround a hole with an odd cycle).

We also explored generalizations to  $\mathbb{R}^3$  in [GO03]. Somewhat surprisingly, genus-0 complexes built from *orthogonal bricks* (rectangular boxes in 3D) are again 2-colorable. We also established that genus-1 orthogonal brick complexes are 3-colorable, and conjectured that the same result holds for arbitrary genus. I am aware of no substantive results on complexes built from parallelepipeds (aside from the observation in [GO03] that four colors are sometimes necessary), a natural generalization of the Sibley-Wagon result.<sup>4</sup> One could also generalize convex quadrilaterals to convex hexahedra (distorted cubes). All of these generalizations seem unexplored.

## 6 Appendix: $K_{d+2} \not\subseteq G^{(d)}$

Here we establish that  $K_{d+2} \not\subseteq G^{(d)}$  without appeal to deBruijn-Erdős. We partition the argument into four lemmas, the first three of which show that there is essentially only one configuration that achieves  $K_{d+1}$ , the analog of the configuration in Figure 3. The fourth lemma then shows that  $K_{d+2}$  cannot be achieved.

<sup>4</sup>Our attempted proof in [GO03] for zonohedra is flawed.



Let  $\sigma_1, \sigma_2$ , and  $\sigma_3$  be  $d$ -simplices. Suppose  $\sigma_1$  and  $\sigma_2$  share a  $(d-1)$ -facet. We will represent each simplex by the set of its vertex labels, with distinct labels representing distinct points in  $\mathbb{R}^d$ . When specifically referring to the point in space corresponding to label  $i$ , we'll use  $v_i$ . Let  $\sigma_1 = \{1, 2, \dots, d, (d+1)\}$   $\sigma_2 = \{1, 2, \dots, d, (d+2)\}$ , with  $\sigma_1 \cap \sigma_2 = f_{12} = \{1, 2, \dots, d\}$  their shared  $(d-1)$ -facet. Under these circumstances, the following lemma holds:

**Lemma 4** *If  $\sigma_3$  shares a  $(d-1)$ -facet with  $\sigma_1$  and a  $(d-1)$ -facet with  $\sigma_2$  (and so the three simplices form  $K_3$  in the dual), then the  $d+1$  vertices of  $\sigma_3$  are among the  $d+2$  vertices of  $\sigma_1 \cup \sigma_2 = \{1, 2, \dots, d, (d+1), (d+2)\}$ :  $\sigma_3$  cannot include a vertex that is not a vertex of either  $\sigma_1$  or  $\sigma_2$ .*

**Proof:** Suppose to the contrary that  $\sigma_3$  includes a new vertex labeled  $(d+3)$ . For  $\sigma_3$  to share a  $(d-1)$ -facet with  $\sigma_1$ , it needs to match  $d$  of the  $d+1$  vertices of  $\sigma_1$ . But it cannot match the facet  $f_{12} = \{1, 2, \dots, d\}$  because that is already covered by  $\sigma_2$ . Without loss of generality, let us assume that  $\sigma_3$  includes vertex  $(d+1)$  but excludes vertex  $k$  with  $1 \leq k \leq d$ . So the  $d+1$  vertices of  $\sigma_3$  are

$$\sigma_3 = \{(d+1), 1, 2, \dots, (k-1), (k+1), \dots, d, (d+3)\} .$$

Now comparison to  $\sigma_2$ ,

$$\sigma_2 = \{1, 2, \dots, d, (d+2)\}$$

shows that it is not possible for  $\sigma_3$  to match  $d$  of the  $d+1$  vertices of  $\sigma_2$  (as it must to share a  $(d-1)$ -facet): the two only share  $d-1$  labels:

$$\sigma_2 \cap \sigma_3 = \{1, 2, \dots, (k-1), (k+1), \dots, d\} .$$

This contradiction establishes the claim. □

**Lemma 5** *Suppose  $d+1$   $d$ -simplices are glued together so that their dual graph is  $K_{d+1}$ . Then all the simplices together include only  $d+2$  vertices.*

**Proof:** Let  $\sigma_1, \dots, \sigma_{d+1}$  be the simplices. By Lemma 4,  $\sigma_1, \sigma_2, \sigma_3$  together include only  $d+2$  vertices, the  $d+2$  vertices of  $\sigma_1 \cup \sigma_2$ . But then repeating the argument for  $\sigma_i$  for each  $i = 4, 5, \dots, d+1$  yields the same conclusion. □

We continue to study the  $K_{d+1}$  configuration in the above lemma. Let us specialize to  $d = 3$  to make the situation clear. We have four tetrahedra glued together to form  $K_4$ , and Lemma 5 says they have altogether 5 vertices. Because  $\binom{5}{4} = 5$ , only one of the possible combinations of the labels  $\{1, 2, 3, 4, 5\}$  is missing among the four tetrahedra. Without loss of generality, we can say that  $\{2, 3, 4, 5\}$  is missing, and that our four tetrahedra have these labels:

$$\{1, 2, 3, 4\}$$

$$\{1, 2, 3, 5\}$$

$$\{1, 2, 4, 5\}$$

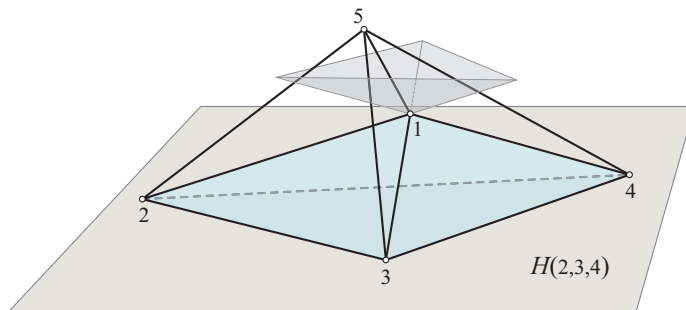


Figure 6: Four tetrahedra whose dual forms  $K_4$ .

$$\{1, 3, 4, 5\}$$

Our next claim is that  $v_5$  lies to the same side of the plane determined by the face  $\{2, 3, 4\}$  as does  $v_1$ . Refer to Figure 6.

Let  $H(i, j, k)$  be the plane containing the vertices with labels  $i, j$ , and  $k$ . Let  $H^+(i, j, k; m)$  be the open halfspace bound by  $H(i, j, k)$  and exterior to the tetrahedron  $\{i, j, k, m\}$ , and  $H^-(i, j, k; m)$  the analogous open halfspace including tetrahedron  $\{i, j, k, m\}$ . The claim is that  $v_5 \in H^-(2, 3, 4; 1)$ . The other three tetrahedra can each be viewed as the hull of  $v_5$  and one of the three faces of the  $\{1, 2, 3, 4\}$  tetrahedron above the base:  $\{1, 2, 3\}$ ,  $\{1, 2, 4\}$ , and  $\{1, 3, 4\}$ . Because a tetrahedron can only be formed by a point above each of these faces, we have that

$$v_5 \in H^+(1, 2, 3; 4)$$

$$v_5 \in H^+(1, 2, 4; 3)$$

$$v_5 \in H^+(1, 3, 4; 2)$$

So  $v_5$  must lie in the intersection of these three halfspaces, which is a cone apexed at  $v_1$  that is strictly above the base plane  $H(2, 3, 4)$ . See again Figure 6. And therefore  $v_5 \in H^-(2, 3, 4)$ , as claimed.

We now repeat this argument for  $d$ -simplices, where the logic is identical but is perhaps obscured by the notation.

The configuration of  $d + 1$   $d$ -simplices forming  $K_{d+1}$  in Lemma 5 uses only  $d + 2$  vertices. Because  $\binom{d+2}{d+1} = d + 2$ , only one of the combinations of  $d + 1$  labels is missing, which we take to be  $\{2, 3, \dots, (d+2)\}$  without loss of generality. So the labels of the  $d + 1$  simplices are:

$$\begin{aligned} &\{1, 2, \dots, d, (d+1)\} \\ &\{1, 2, \dots, d, (d+2)\} \\ &\{1, 2, \dots, (d+1), (d+2)\} \\ &\dots \\ &\{1, 3, \dots, d, (d+1), (d+2)\} \end{aligned}$$

**Lemma 6** *In the configuration of  $d + 1$  simplices forming  $K_{d+1}$  labeled as just detailed above,  $v_{d+2}$  lies in  $H^- = H^-(2, 3, \dots, (d+1); 1)$ , the same halfspace in which  $v_1$  lies.*

**Proof:**  $H(2, 3, \dots, (d+1))$  is the flat containing the “base” of the first simplex in the list above,  $\sigma_1 = \{1, 2, \dots, d, (d+1)\}$ . The remaining  $d$  simplices in the list share the facets of  $\sigma_1$  incident to  $v_1$ , each including  $v_{d+2}$ . Thus  $v_{d+2}$  is above each of those facets, i.e., it lies in the corresponding  $H^+$  halfspaces:

$$v_{d+2} \in H^+(1, 2, \dots, d; (d+1))$$

...

$$v_{d+2} \in H^+(1, 3, \dots, d, (d+1); 2)$$

And therefore  $v_{d+2}$  lies in the intersection of all these halfspaces, which is a cone apexed at  $v_1$  and lying strictly above  $H(2, 3, \dots, (d+1))$ . Therefore  $v_{d+2}$  is in  $H^-$ .  $\square$

Completing the argument is now straightforward.

**Lemma 7**  $K_{d+2} \not\subseteq G^{(d)}$

**Proof:** Assume to the contrary that  $K_{d+2}$  is a subgraph of  $G^{(d)}$ . Then  $K_{d+1}$  must be also. Using the notation of Lemma 6, that lemma establishes that in a configuration that realizes  $K_{d+1}$ , vertex  $v_{d+2}$  lies in  $H^- = H^-(2, 3, \dots, (d+1); 1)$ . Because  $\{2, 3, \dots, (d+1)\}$  is the only facet of the simplex  $\sigma_1 = \{1, 2, \dots, d, (d+1)\}$  not yet covered by another simplex, the last simplex  $\sigma_{d+2}$  must have labels  $\{2, \dots, d, (d+1), (d+2)\}$ . And therefore  $v_{d+2} \in H^+(2, 3, \dots, (d+1); 1)$ . But this is a contradiction, as it is saying that  $v_{d+2}$  must lie strictly to both sides of  $H(2, 3, \dots, (d+1))$ .  $\square$

## References

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