Polygons needing many flipturns

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Dedicated to Godfried Toussaint on the occasion of his 60th birthday

Abstract

A *flipturn* on a polygon consists of reversing the order of edges inside a pocket of the polygon, without changing lengths or slopes. Any polygon with n edges must be convexified after at most $(n-1)!$ flipturns. A recent paper showed that in fact it will be convex after at most $n(n-3)/2$ flipturns. We give here lower bounds. We construct a polygon such that if pockets are chosen in a bad way, at least $(n-2)^2/4$ flipturns are needed to convexify the polygon. In another construction, $(n-1)^2/8$ flipturns are needed, regardless of the order in which pockets are chosen. All our bounds are adaptive to a pre-specified number of distinct slopes of the edges.

1 Background

A *polygon* is a set of distinct points $v_0, \ldots, v_{n-1}$ such that its *edges* (the open line segments (v_i, v_{i+1}) for $0 \leq i < n$, addition is modulo n) are disjoint. A polygon is called *convex* if the interior angle at each v_i is not more than π. If a polygon P is not convex, then it can be made convex with the help of some transformation. One simple way to do so is to compute the *convex hull*, i.e., the smallest polygon that contains P and is convex. Another approach to convexify a polygon is to move the location of its points, without changing edge lengths. That this is always possible is not at all trivial; see [CDR03].

Another approach is to change the polygon only in the vicinity of the convex hull. In particular, two operations are *flips* and *flipturns*, which are defined as follows. A *pocket* of P is a set of contiguous edges such that none of them intersects the convex hull, but the two endpoints are on the convex hull. A *flip* of a pocket consists of reflecting the pocket along the line through the two endpoints of the pocket (also called the *lid* of the pocket). A *flipturn* instead rotates the pocket by 180°, and thus reverses the order of edges in the pocket. See Figure 1 for an example.

Flips have a long and intricate history; see the overview paper by Toussaint [Tou99]. We study here flipturns, which were mentioned for the first time in a paper by Grünbaum [Grü95]. The question studied is: how many flipturns are needed until the polygon is convexified? This number may depend on the order in which pockets are chosen; we thus consider both the *shortest flipturn sequence* (which is the shortest sequence of flipturns needed until a given polygon is convexified), and the *longest flipturn sequence*.

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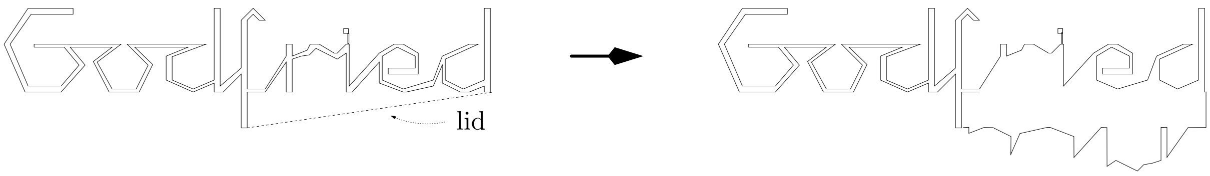

Figure 1: An example of a flipturn.

Joss and Shannon (unpublished, but reported in [Grü95]) observed that any flipturn sequence has length at most $(n-1)!$. This holds because every flipturn creates a different permutation of the edges, and no permutation can repeat. Also, Joss and Shannon conjectured that the shortest flipturn sequence has length at most $n^2/4$ flipturns for any polygon.

While this conjecture is still open, Ahn et al. showed at least that a quadratic upper bound is correct: any polygon can be convexified with $n(n-3)/2$ flipturns [ABC+00]. In fact, their bound holds for any flipturn sequence, and thus the longest flipturn sequence has length at most $n(n-3)/2$. The bound by Ahn et al. depends on using "modified flipturns" in case of degeneracies. Aichholzer et al. [ACD+02] studied the case when such flipturns are not allowed, and give larger (but still quadratic) upper bounds. The special case of orthogonal polygons (where all edges are horizontal or vertical) was studied even earlier; see [DORS88].

In this paper, we give lower bounds on the length of the longest and the shortest flipturn sequence, and show that both may be quadratic in the number of vertices of the polygon. More precisely, we first construct a polygon that – with a bad sequence of flipturns – needs $(n-2)^2/4$ flipturns to convexify. Note that this lower bound is very close to the (conjectured) upper bound by Joss and Shannon.

The upper bounds of [ABC+00] and [ACD+02] are adaptive in the number of slopes s of edges of the polygon. Motivated by this, we also make our lower bounds adaptive in s, and establish that the length of the longest flipturn sequence is $\theta(ns)$. In fact, our lower bound is quite close (for small s) to the upper bound of Ahn et al.

Then we turn to lower bounds on the shortest flipturn sequence. We give a polygon that requires $(n-s)(s-1)/2$ (or $(n-1)^2/8$ for $s = n-2$) flipturns to convexify, and that only has one pocket at any given time. Table 1 summarizes our results.

	In terms of n		In terms of n and s	
Lower bounds	Shortest	Longest	Shortest	Longest
	$(n-1)^2/8$	$(n-2)^2/4$	$(n-s)(s-1)/2$	$(n-s)(s-1)/2 + (s-2)^2/4$
Upper bounds	$n(n-3)/2$ [ABC+00]		$n(s-1)/2 - s$ [ABC+00]	
	n^2-4n+1 [ACD+02]		$ns-\lfloor (n+5s)/2 \rfloor -1$ [ACD+02]	

Table 1: Bounds on the length of the shortest and longest flipturn sequence to convexify any polygon with n edges and s slopes. Upper bounds are for the longest flipturn sequence and hence also hold for the shortest flipturn sequence. Bounds from [ABC+00] use modified flipturns whereas the ones from [ACD+02] do not.

For ease of understanding, we only give the ideas in the main part of the paper; the precise coordinates and argumentation why they suffice are left to the appendix.

2 Lower bounds for the longest flipturn sequence

In this section, we construct polygons for which the longest flipturn sequence has quadratic length. The crucial idea is as follows: Assume k consecutive edges $e_1, \ldots, e_k$ are sorted by slope, and followed by an edge e^* whose slope is smaller than all other slopes. Then e^* belongs to a pocket that also contains e_k. If e^* is short enough (relative to the length of e_k and the slopes of e^*, e_k and e_{k-1}), then this pocket will consist of e^* and e_k only.

Now flipturn the pocket; then e^* belongs to a pocket which (if e^* is short enough) consists of e_{k-1} and e^* only. Flipturn this pocket; if e^* is short enough, we now have a pocket with e_{k-2} and e^*. This continues, until after k flipturns we exchange e^* and e_1, and the edges are sorted by slope. See Figure 2 for an illustration.

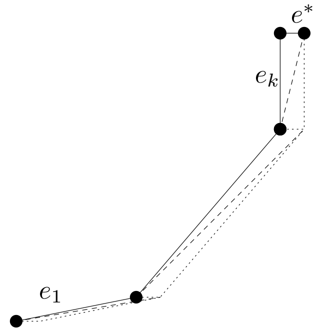

Figure 2: Edge e^* is sliding past $e_1, \ldots, e_k$ with k flipturns.

Now we apply this idea repeatedly. Assume that we are given some slopes $c_1 < c_2 < \ldots$. In what follows, e_i is an edge of slope c_i; there will be only one such edge per slope. For ℓ even, we will construct a polygonal chain C_ℓ with ℓ edges such that some flipturn sequence (applied to a polygon that contains C_ℓ as part of it) has length $\ell^2/4$ before C_ℓ is convexified. For $\ell = 2$, C_1 consists of two edges e_2 and e_1 (in this order), which need $1 = \ell^2/4$ flipturns.

For $\ell > 2$, we construct the polygonal chain C_ℓ by taking the polygonal chain $C_{\ell-2}$ (applied to slopes $c_2, \ldots, c_{\ell-1}$), and attaching to it two edges e_ℓ and e_1. See Figure 3(a). By induction, there are $(\ell-2)^2/4$ flipturns before $C_{\ell-2}$ is convexified. If e_1 is short enough, then the pocket formed by e_ℓ and e_1 will not interfere with the flipturns for $C_{\ell-2}$. Once $C_{\ell-2}$ is convexified, edge e_1 slides down with $\ell-1$ flipturns. This yields a total of $(\ell-2)^2/4+(\ell-1) = \ell^2/4$ flipturns for C_ℓ. See Figure 3(b) and (c).

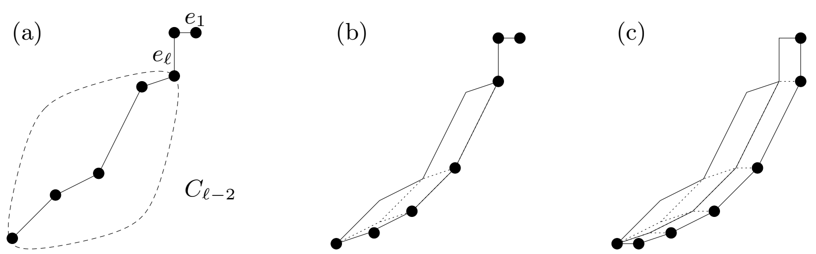

Figure 3: (a) A polygonal chain that may need $\ell^2/4$ flipturns. (b) Chain $C_{\ell-2}$ is convexified. (c) The last pocket is removed with $\ell - 1$ flipturns.

Since C_{n-2} can be completed to a polygon after adding two more edges, this yields the desired lower bound of $(n-2)^2/4$ flipturns for a polygon. However, we will now slightly change the construction to accommodate a limited number s of slopes. We assume for now that the number of slopes s is even and $s \geq 6$.

Let C_{s-2} be the polygonal chain with $s-2$ edges (applied to slopes $c_2, \ldots, c_{s-1}$) that needs $(s-2)^2/4$ flipturns to convexify. Add k pairs of edges, alternatingly of slope c_s and of slope c_1. All edges of slope c_1 have the same length while the edges of slope c_s become increasingly longer. Thus the k pairs form k pockets. See Figure 4(a).

We first convexify C_{s-2} with $(s-2)^2/4$ flipturns. Then we slide down the edges of the k pockets, starting with the lowest pocket. Each such edge needs $s-1$ flipturns, for a total of $(s-2)^2/4 + k(s-1)$ flipturns. See Figure 4(b) and (c).

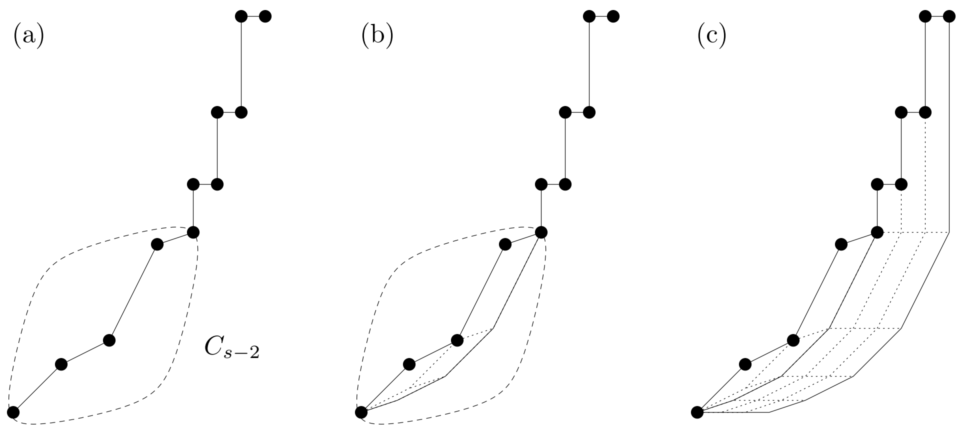

Figure 4: Lower bound construction for a limited number of slopes.

We close the polygon with two more (potentially very long) edges; one of slope c_s and the other of slope c_{s-1} (but in the other direction.)[1] Chain C_{s-2} uses $s-2$ edges, so the number k of pairs is $k = (n-2-(s-2))/2 = (n-s)/2$, and the total number of flipturns is $(s-2)^2/4 + (n-s)(s-1)/2$.

Theorem 1 *There exists a polygon with n edges and s distinct slopes ($s \geq 6$, n and s even) for which the longest flipturn sequence has length at least* $(s-2)^2/4 + (n-s)(s-1)/2$.

Note that our lower bound equals $\frac{n(s-1)}{2} - \frac{s^2}{4} - \frac{s}{2} + 1$, which is close (if s is small) to the upper bound of $\frac{n(s-1)}{2} - s$ given by Ahn et al. [ABC+00]. Also, it is maximized for $s = n-2$, and then evaluates to $(n-2)^2/4$.

We briefly touch on the other cases of s and n: For s odd and $s \geq 5$, the base case for the construction of C_ℓ changes to using just one edge, which needs $0 = (\ell^2-1)/4$ flipturns. The lower bound hence reduces by $\frac{1}{4}$. For $s = 3, 4$, we need to add one more edge to complete the polygon, and for $s = 2$ we need to add two more edges. To cover the other parity of n, we can add one more edge. The lower bounds in these cases can be obtained by replacing n with $n + a$, where a is the number of added edges.

[1]Here we use $s \geq 6$; for $s = 2, 4$ the first edge of the chain has slope c_{s-1} and would overlap the new edge.

2.1 Order matters

The lower bound for our polygon critically depends on the order in which we choose pockets for executing flipturns. In our construction we always chose the lowest pocket to flip; if instead we choose the topmost pocket, then the polygon is convexified after $k+s = (n+s)/2$ flipturns. Similar as done in [ACD$^+$02] for orthogonal polygons, we can therefore observe the size of the gap between the longest and the shortest flipturn sequence.

Theorem 2 *For infinitely many n, and any $s \geq 6$ and even, there is a polygon with n vertices and s slopes where the shortest and longest flipturn sequence differ in length by $ns/2 - n - s - s^2/4 + 1$.*

3 Lower bounds for the shortest flipturn sequence

In this section, we construct another polygon which needs quadratically many flipturns and has only one pocket at any given time; the lower bounds polygon hence hold for the shortest sequence, which settles an open question in [ACD$^+$02].

The idea is the same as before: Repeatedly some edge is "sliding down" a chain of edges that are sorted by slope. The difference is that the next edge to slide down will be hidden inside the edge that is sliding down, and so on recursively. Thus, we thus first study how to slide down a parallelogram, where the parallelogram shape hides all later edges.

Assume that one edge e of the polygon has been replaced by a parallelogram R with two opposite corners at the endpoints of e. Assume furthermore that at most one point of R is on the convex hull of P. Then we can define pockets just as before, and apply flipturns just as before; this rotates the parallelogram (and may bring it to the convex hull.) See Figure 5.

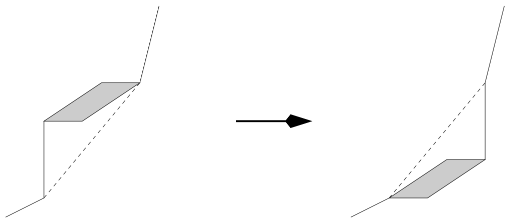

Figure 5: Flipping a pocket with a parallelogram.

The parallelogram can "slide down" similar as an edge. Let $c_1 < c_2 < \ldots < c_s$ be a set of slopes; for ease of description we assume $c_1 = 0$ and $c_s = \infty$. Let $e_2, \ldots, e_s$ be edges of slope $c_2, \ldots, c_s$, followed by a parallelogram R (Figure 6(a)). We may have additional horizontal edges before e_2 and additional vertical edges after R. If the width of R is small enough, then R forms a pocket with e_s. After a flipturn, it will form a pocket with e_{s-1}. This continues for $s-1$ flipturns; then R is before e_2. See Figure 6(b).

At this point R is on the convex hull and the concept of a flipturn is not well-defined. Now R reveals itself to consist of two horizontal edges before and after a (smaller) parallelogram R'. Parallelogram R' now slides back up. More precisely, R' forms a pocket with one of the horizontal edges of R. After a flipturn, R' is hence before e_2. Then it forms a pocket with

e_2, and after another flipturn, it is before e_3. This continues for $s-1$ flipturns, until R' is before e_s. At this point, R' is on the convex hull. Now R' reveals itself to consist of two vertical edges plus an (even smaller) parallelogram R'' where one pair of edges is horizontal. See Figure 6(c).

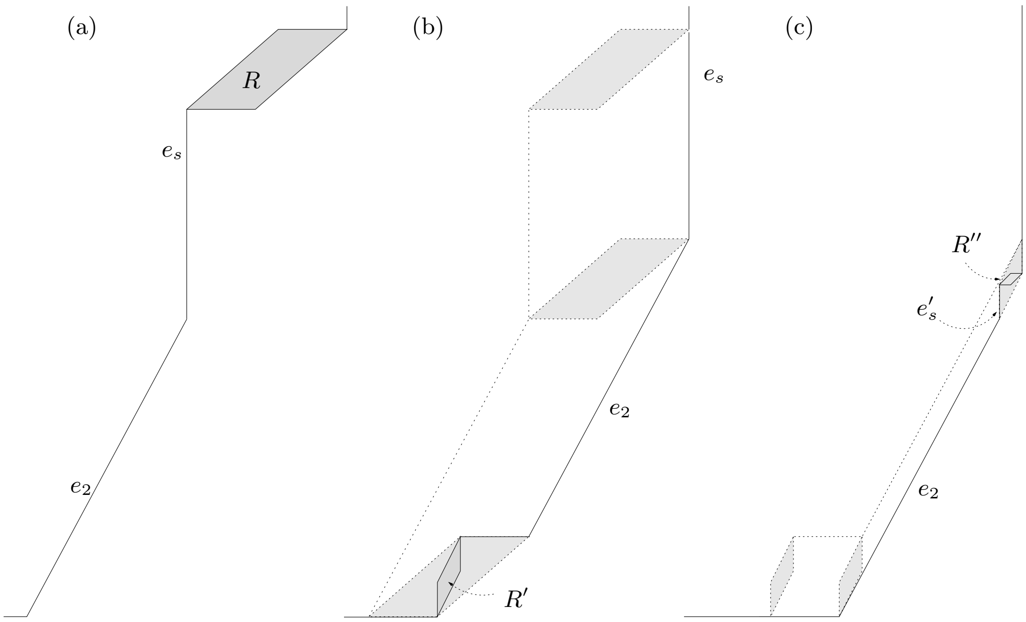

Figure 6: Sliding a parallelogram down and up.

Now we go into recursion. We are exactly in the same situation as when sliding down R, except that edge e_s has been replaced by a new (shorter) vertical edge e'_s. The process thus iterates, until R is reduced to a single horizontal edge, which slides to its place with $s-1$ flipturns. Note that there always is only one pocket, which is incident to the parallelogram.

We compute the number of flipturns as follows. In the outermost recursion, we have $s-1$ edges $e_2, \ldots, e_s$, the parallelogram R, and two more edges (of slope c_s and c_{s-1}) to close off the polygon.[2] For each recursion, we remove 4 edges from the parallelogram and do $2(s-1)$ flipturns. In the final recursion, we have one edge and $s-1$ flipturns. If k is the number of recursions, then the number of edges is $(s-1)+2+4k+1$, so $k=(n-s-2)/4$. The total number of flipturns hence is $k \cdot 2(s-1)+(s-1) = (n-s-2)(s-1)/2+(s-1) = (n-s)(s-1)/2$.

Theorem 3 *There exists a polygon with n edges and s distinct slopes ($s \geq 4$, $n-s=2 \pmod 4$) with a unique convexifying flipturn sequence that has length $(n-s)(s-1)/2$.*

Note that this bound is maximized for $s=(n+1)/2$, and then yields $(n-1)^2/8$ flipturns. Very similar lower bounds can be derived for $s \leq 4$ and all other values of $n-s \pmod 4$, by adding one more edge to close the polygon if needed, and/or by stopping the recursion by letting one vertical edge slide up.

[2]The first edge has slope c_2, hence this construction works for $s \geq 4$.

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A Suitable edge-lengths

In this appendix, we fill in details of how to choose integer coordinates such that all edges are sufficiently short to create pockets (and flipturns) as desired.

A.1 The construction for Theorem 1

We first study the construction for the longest flipturn sequence, which consists of a polygonal chain C_{s-2} which needs $(s-2)^2/4$ flipturns, followed by k pairs of edges with alternatingly the largest and the smallest slope. We know $k = (n-s)/2$, but will write k for simplicity.

We describe coordinates via the x-*extent* of each edge, which is the length of the projection of the edge onto the x-direction. We choose slopes and x-extents as follows:

- We set $c_i = i$ for $i = 1, \dots, s-1$. [3]
- An edge of slope c_i, $1 \leq i \leq s-1$ has x-extend 2^i.
- The x-extent of the k edges of slope c_s are 2^s, 2^s+1, 2^s+2, ... In particular, these edges are increasingly longer, as required.
- To close the polygon, draw a line of slope c_{s-1} above the polygonal chain. Connect to this line by extending the first edge of C_{s-2} (which has slope $c_{s/2+1}$), and by adding an edge of slope c_s at the other end of the polygonal chain. Neither of these new edges adds any pockets; we will not give precise coordinates for the vertices, since choosing the edges long enough will clearly suffice to make the polygon simple.

See Section A.4 for the full construction. Now we verify that every edge slides down correctly. Say edge e_j is sliding down and forms a pocket with e_i, where $2 \leq j < i \leq s$. The lid of the pocket of e_j and e_i must have slope greater than the one of the edge before, which is c_{i-1}. If x_j and x_j are the x-extents of e_j and e_i, then this becomes (see also Figure 7)

$$\frac{x_i c_i + x_j c_j}{x_i + x_j} > c_{i-1}, \quad \text{or} \quad \frac{x_i}{x_j} > \frac{c_{i-1} - c_j}{c_i - c_{i-1}} \tag{1}$$

For our slopes and x-extents, this evaluates to $2^{i-j} > i - 1 - j$, which holds for all $i > j$.

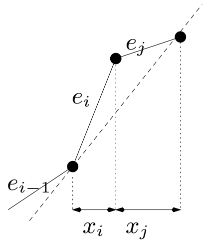

Figure 7: Slope considerations.

[3]We preferred simplicity of description over smallest possible coordinates. Smaller (but still exponential) coordinates could be achieve by setting $c_1 = 0, c_s = \infty$ and $c_i = i - 1$ for all other i.

A.2 A polynomial construction

Note that our coordinates are exponential in the number of vertices of the polygon, and we have not been able to find x-extends and slopes that avoid this and satisfy Equation (1). However, at a slight loss in the lower bound, we can obtain polynomial coordinates. Replace C_s by a polygonal chain that contains $e_2, \ldots, e_{s-1}$ in order sorted by slope, and then keep k pairs of edges of slope e_s and e_1. Now only edges of slope c_1 slide down, so Equation (1) only needs to hold for $j = 1$ and simplifies to $x_i > i - 2$. With $x_i = i - 1$, the coordinates are then polynomial in n. The length of the longest flipturn sequence of this polygon is $k(s-1) = (n-s)(s-1)/2$.

A.3 The construction of Theorem 3

Now we give the construction for the shortest flipturn sequence, which consists of $s-2$ edges with slopes $c_2, \ldots, c_{s-1}$, followed by a parallelogram R_k that hides k recursions. (We have $k = (n-s-2)/4$, but will use k for simplicity.) More precisely, R_k consists of two horizontal edges and a parallelogram R'_k, whereas R'_k consists of two vertical edges and a parallelogram R_{k-1}, and so on recursively.

The parallelograms are defined as follows (see also Figure 8):

- R_0 consists of one horizontal edge of length 2.
- R'_1 consists of two vertical edges of length 4 around R_0.
- R_i, $i \geq 1$ consists of two horizontal edges of length $7 \cdot 8^{i-1}$ around R'_i.
- R'_i, $i \geq 2$, consists of two vertical edges of length $28 \cdot 8^{i-2}$ around R_{i-1}.

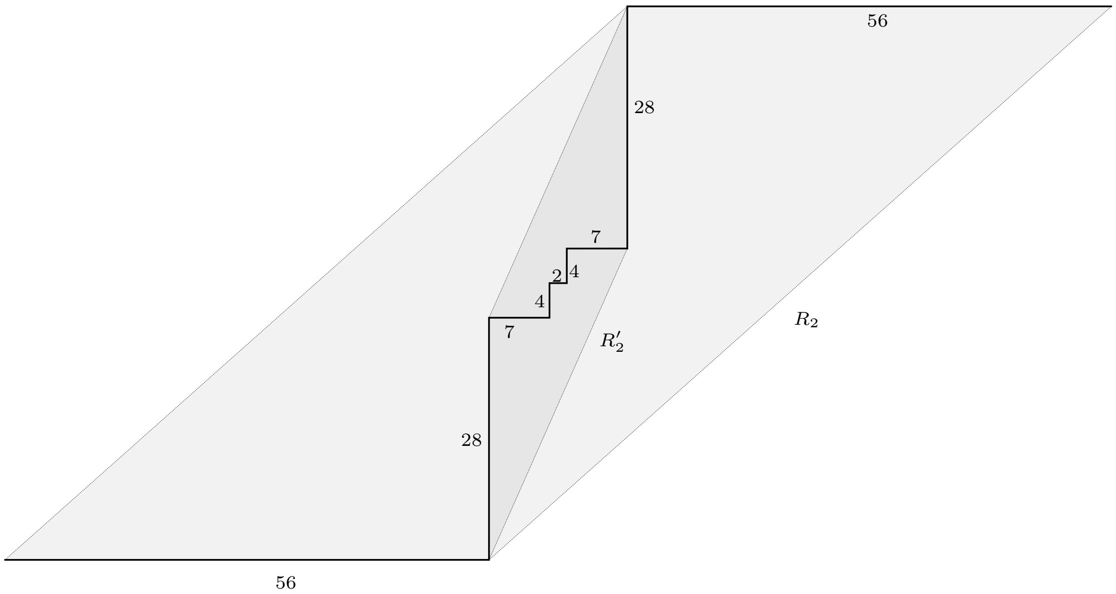

Figure 8: The construction of R_i.

Some easy math shows that R_i has width $2 \cdot 8^i$, height 8^i, and its non-horizontal edges of R_i have slope less than 1. On the other hand, R'_i has width $2 \cdot 8^{i-1} = \frac{1}{4}8^i$, height 8^i, and its non-vertical edges have slope greater than 2.

The remaining slopes can be chosen quite arbitrarily, as long as they are between 1 and 2; we set $c_i = 1 + 1/s$ for $i = 2, \ldots, s$. The edge lengths of edges $e_2, \ldots, e_s$ have to be chosen such that parallelogram R_k can slide down one edge at a time and R'_k can slide up one edge at a time. All other parallelograms R_i and R'_i are even shorter than R_k and R'_k, and so will also slide correctly if R_k and R'_k do. As we will show now, it suffices to set the x-extent of e_i to $3s \cdot 8^k$ (for $2 \leq i < s$), and the y-extend of e_s, which is vertical, to $3 \cdot 8^k$.

Let $Y_k = 8^k$ and $X_k = 2Y_k$ (these are the height and width of R_k), and denote the x-extent of e_i by x_i; we have $x_i = 3sY_k$. For $3 \leq i \leq s-1$, the slope of the lid formed by e_i and R_k is then (see also Figure 9(a))

$$\frac{c_i x_i + Y_k}{x_i + X_k} = \frac{(c_{i-1} + \frac{1}{s})3sY_k + Y_k}{3sY_k + 2Y_k} = \frac{3sc_{i-1} + 4}{3s + 2} > c_{i-1},$$

since $c_{i-1} < 2$. So R_k forms a pocket with e_i, but not e_{i-1}, for $3 \leq i \leq s-1$. Similarly one shows that this holds for $i = s$ (the math changes slightly since $c_s = \infty$.) Also, by $c_{i-1} > 1$ the lid of this pocket has slope greater than 1, so R_k (for which the non-horizontal edges have slope less than 1) is truly inside the pocket.

Similar math shows that R'_k also slides up correctly. Let $Y_k = 8^k$ and $X_k = \frac{1}{4}Y_k$; these are the height and width of R'_k. For $1 \leq i \leq s-2$, the slope of the lid formed by R'_k and e_i is then (see also Figure 9(b))

$$\frac{Y_k + c_i x_i}{X'_k + x_i} = \frac{Y_k + (c_{i+1} - \frac{1}{s})3sY_k}{\frac{1}{4}Y_k + 3sY_k} = \frac{3sc_{i+1} - 2}{3s + \frac{1}{4}} = \frac{12sc_{i+1} - 8}{12s - 1} < c_{i+1},$$

so R'_k forms a pocket with e_i, but not e_{i+1}, for $1 \leq i \leq s-2$. Similarly one shows that this holds for $i = s-1$ (the math changes slightly since $c_s = \infty$.) Also, the lid of this pocket has slope less than 2, so R'_k (for which the non-vertical edges have slope greater than 2) is truly inside the pocket.

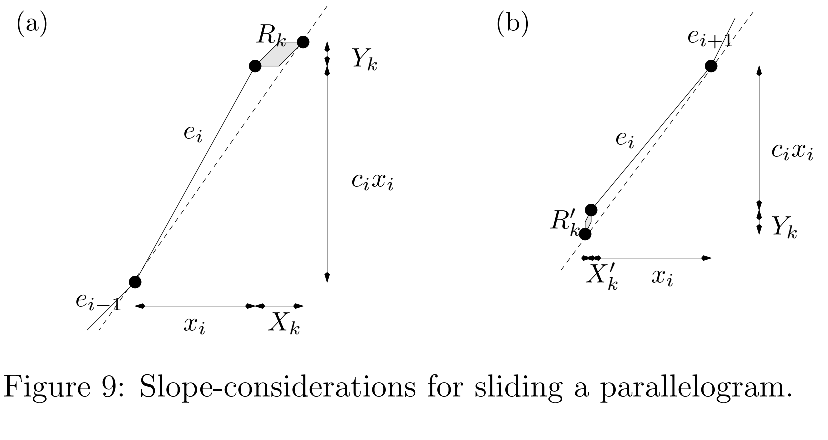

Figure 9: Slope-considerations for sliding a parallelogram.

A.4 Full constructions

For convenience of the reader, and to give an idea of the size differences involved, we have included here some full constructions to scale.

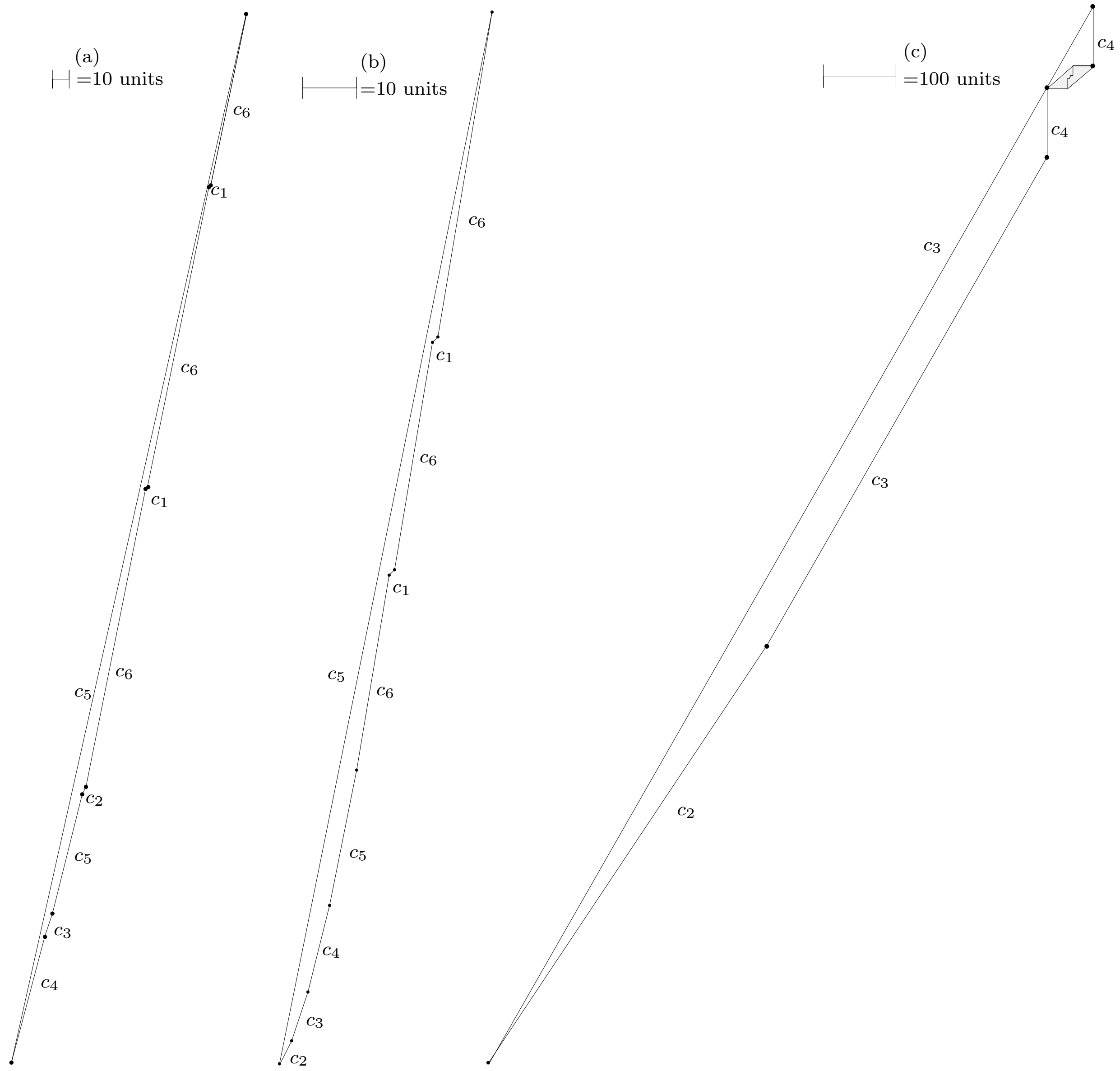

Figure 10: The full constructions to scale. (a) The construction of Section 2 for $s = 6$ and $n = 10$. (b) The construction of Section A.2 with polynomial coordinates for $s = 6$ and $n = 10$. (c) The construction of Section 3 for $s = 4$ and $n = 14$.