Pop-Up Card Design as a Vehicle to Teach Mathematics

Joseph O'Rourke* *and* Stephanie Jakus, Molly Miller, Duc Nuygen, Nell O'Rourke, Gail Parsloe, Ana Spasova, Faith Weller

August 2, 2005

Contents

1 Introduction **2**

2 Parallel Folds **3**
2.1 Via Cuts 3
2.1.1 Centered on Card Crease 3
Rhombus. 3
Question 2.1: 3
Question 2.2: 5
Parallelogram. 5
Question 2.3: 5
Question 2.4: 5
2.1.2 Centered on Any Crease 6
2.1.3 Pop-up Letters 6
2.1.4 Artistic Designs 8
2.2 Via Gluing 9
2.2.1 Pop-up Box 10
2.2.2 Pop-up Tents 11
Question 2.5: 11
Question 2.6: 12
2.2.3 Non-parallelogram Quadrilaterals 12
Question 2.7: 12
2.3 Connection to Pantographs 13

*Dept. Comput. Sci., Smith College, Northampton, MA 01063, USA. `orourke@cs.smith.edu`. Supported by NSF Distinguished Teaching Scholars award DUE-0123154.

3 Angle Folds and Rotary Motion **13**
3.1 Central V-Fold 13
Question 3.1: 14
Question 3.2: 14
Question 3.3: 15
3.1.1 Cone and Sphere 15
Question 3.4: 15
3.1.2 Medial Plane 16
Question 3.5: 16
Question 3.6: 17
Question 3.7: 17
3.1.3 Angle of V-Fold 19
Question 3.8: 19
3.2 Shifted V-Fold 19
3.2.1 Rotation of Non-Rib Segment 20
Question 3.9: 20
3.3 Planar Rotation 21

4 The Knight's Visor **21**
Question 4.1: 22
4.1 Flat Visor Curve 23
Question 4.2: 23
4.2 Visor Curve in 3D 26

Abstract

This is an unfinished draft. Please do not quote, and ignore the inconsistencies!

1 Introduction

A card is composed of two congruent rectangles, the *back* and the *front*, joined at the card *centerline*. We will call the card angle α.[1] We will view the back of the card as fixed to a table top, with the front of the card rotating from the closed position, when $\alpha = 0$, to the fully opened position, when $\alpha = 180°$. This notation is illustrated in Figure 2 below.

Creases in the card can be either *valley* or *mountain* folds. We will follow the origami convention of marking valley folds with a (green) dashed line, and mountain folds with (red) dash-dots.[2] Cuts in the card will be marked with solid (blue) lines.

[1] In geometry, the angle in 3D between two planes is known as a *dihedral angle*, from the Greek: di- (two) hedra (faces).

[2] Mnemonic: the dots are the mountain peaks.

2 Parallel Folds

We start with the simplest type of pop-up, based on "parallel folds." First we'll explore creating these pop-ups by cutting the card and bending out sections, and then by pasting in extra material. We will see that symmetric cuts produce pop-ups whose shape is that of a *rhombus*, a quadrilateral all of whose four sides have the same length, and asymmetric cuts produce *parallelograms*, quadrilaterals whose opposite sides have the same length, and are parallel.

2.1 Via Cuts

2.1.1 Centered on Card Crease

Rhombus. Make a single horizontal cut anywhere perpendicular to the card centerline, and symmetric with respect to the centerline, in that the cut extends the same distance to the left and to the right of the centerline. See Figure 1. This is easily accomplished by first, folding the card in half along the centerline, and then cutting with scissors perpendicular to the centerline. Reverse the

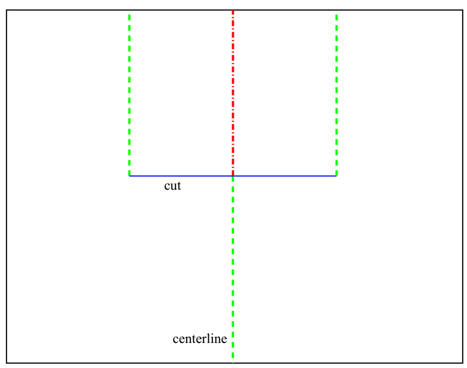

Figure 1: One symmetric cut parallel fold.

valley fold above the cut into a mountain fold, poping out the paper between. See Figure 2.

A minor variation is to start with two parallel horizontal cuts, which results in the pop-up standing out in the middle of the card, as shown in Figure 3.[3]

> **Question 2.1:** *[Grade $\geq$ 5.] Parallelogram Angles.* As the card flexes, what is the angle of the "apex" of the mountain crease (the angle at point y in Figure 2) in relation to the card angle (at point x in the figure)?
> ANSWER. α: the card angle and the apex angle are exactly the same.

[3] We thank Phillip Corker for a correction to this figure.

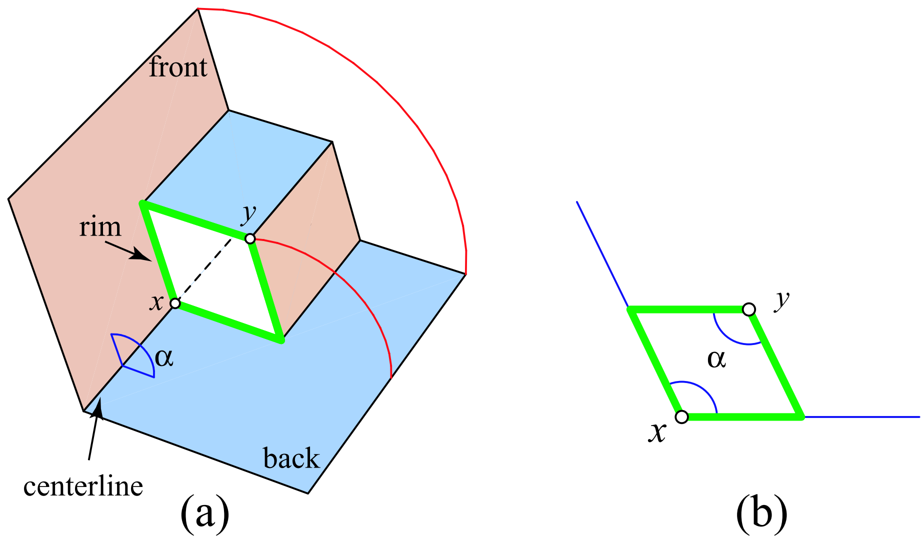

Figure 2: (a) Folding the template in Figure 1 produces pop-up whose rim is a rhombus. (b) View down the centerline.

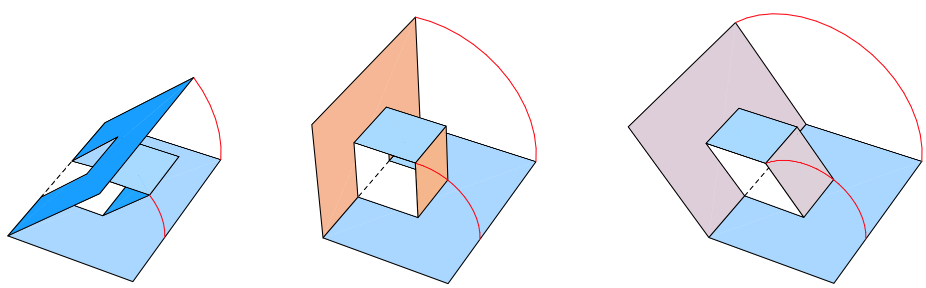

Figure 3: Parallel cuts producing a rhombus.

Question 2.2: *[Grade $\geq$ 4.] Shape of Pop-Up.* What is the shape of the "rim" of the pop-up, i.e., the shape composed of edges created by cuts?
ANSWER. A quadrilateral, more specifically a parallelogram, more specifically a rhombus.

When the card angle is 90°, the apex angle is also 90°, and the rim of the pop-up forms a square. For arbitrary card angles, the rim shape is a rhombus, that is, a parallelogram whose four edges are equal in length. The lengths are equal because they derive from a cut that extends the same length left and right of the centerline. This determines the bottom two sides of the rhombus. The top two sides are just the other side of the cut.

Parallelogram. A symmetric cut, i.e., one that extends the same distance to each side of the centerline, always produces a pop-up rhombus, as we've just seen.

Question 2.3: *Unequal Lengths.* Is it necessary for the cut to be symmetric with respect to the card centerline?
ANSWER. No, it "works" for any cut.

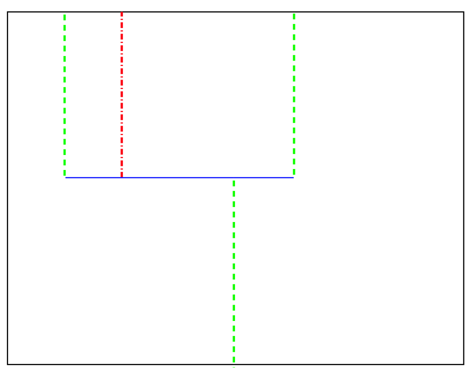

Figure 4: A cut assymetric with respect to the centerline.

If the two parallel cuts are asymmetric with respect to the centerline, as shown in the template in Figure 4, the pop-up rim shape is a parallelogram rather than a rhombus. See Figure 5. This is why the card angle and the apex angle are always the same, as opposite angles of a parallelogram are equal.

Question 2.4: *Path a Point in Space.* What is the path in 3D of the apex point (y in Figure 2) as the card opens?
ANSWER. As illustrated in several figures, this corner point follows a circular arc.

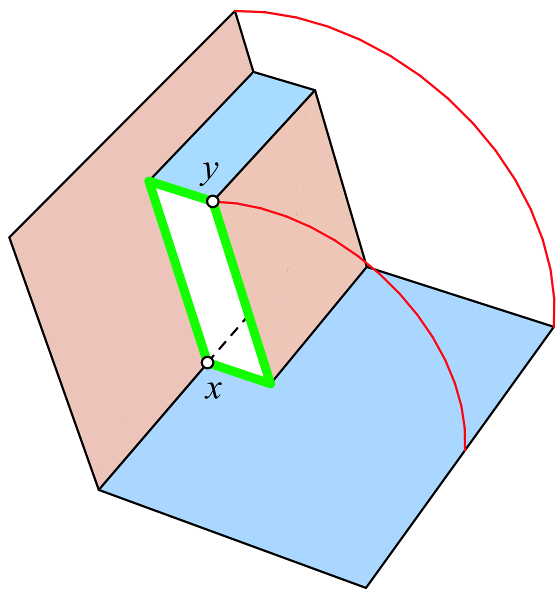

Figure 5: Parallelogram pop-up based on template in Figure 4.

The circle is centered on the crease lying on the back card face, and lies in a plane perpendicular to that face. Note that a corner of the card also follows a circular arc, centered on the centerline of the card, but an arc of a larger radius. These arcs will play a role in the connection to pantographs described in Section 2.3.

2.1.2 Centered on Any Crease

The basic parallel fold, although simple, already has the potential to make interesting and intricate structures. Returning to Figure 2, the pop-up was created by a cut perpendicular to the centerline. But a cut perpendicular to any valley crease would work as well. The basic parallel fold (Figures 1 and 2) creates two valley creases at the base of the pop-up. This leads to the idea of employing these valley creases in the same role, and building pop-ups on those. This process cna be iterated, as illustrated in Figure 6. and Figure 7.

2.1.3 Pop-up Letters

Returning to Figure 5, the large flat plane parallel to the front face created by the pop-up can be used as a base into which one can carve designs. An impressive effect can be achieved by carving an alphabet letter into this face, creating a pop-up 3D letter. See Figure 8 and Figure 9. Repeating this for several letters makes a very effective greeting card. One can find templates for the letters on the Web. For example, we used the templates at

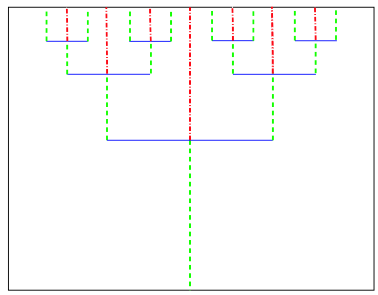

Figure 6: Template for multiple pop-ups.

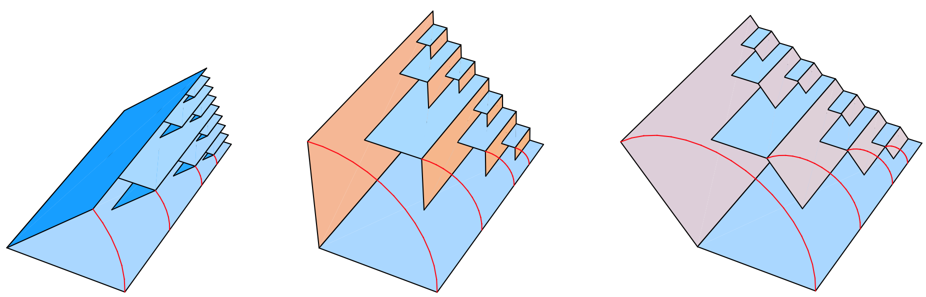

Figure 7: Each valley crease serves as a centerline for a “sub-pop-up.”

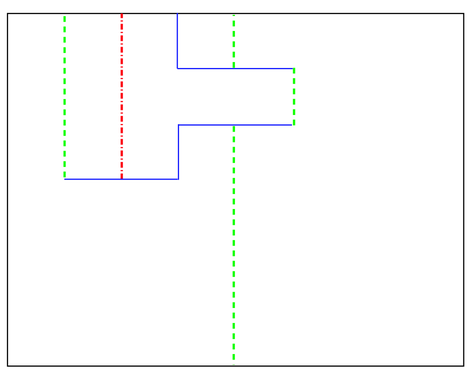

Figure 8: Template for pop-up `T`.

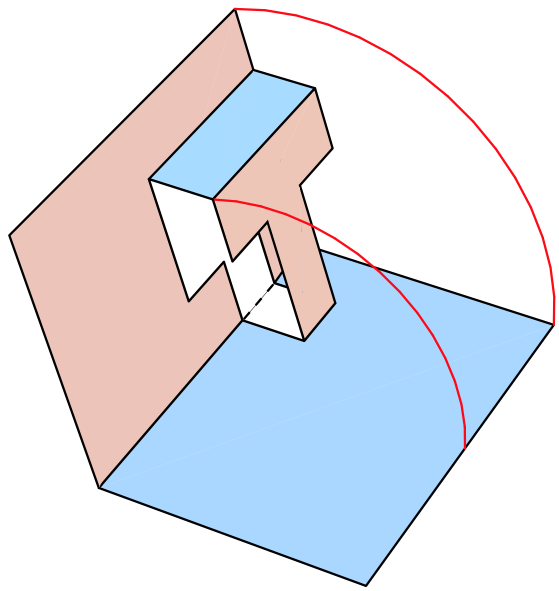

Figure 9: Pop-up letter `T`. Compare Figure 5.

http://niagaracalligraphyguild.netfirms.com/news0300/popup.htm to produce Figure 10.

Figure 10: 3D pop-up letters.

2.1.4 Artistic Designs

Even with just the simple parallel fold, artistic designs can be made. Figure 11 shows two intersecting "staircases," one vertical, and one horizontal. Each is constructed from parallel cuts staggered by the stair height. The design is based on a similar one in [Jac93, p. 62, Fig. 1], employing the `card3d` software at http://www.page.sannet.ne.jp/jun_m/card3d/index-eng.html to make

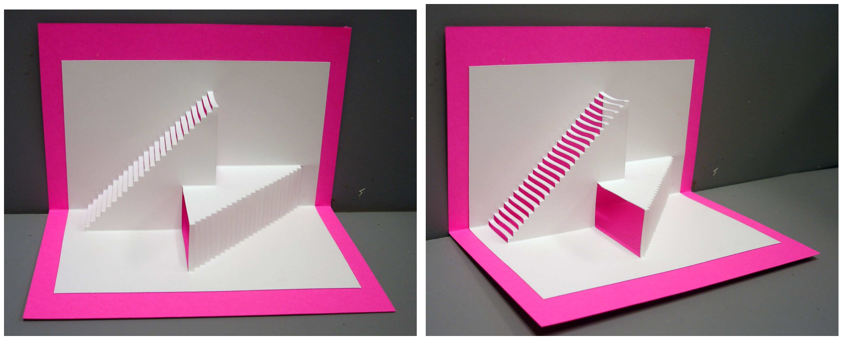

Figure 11: Two staircases. [Gail Parsloe]

the template. Figure 12 shows a beautiful design, again constructed solely of parallel cuts. This is the Logo of Evermore Enterprises. The pattern is available at their web site, `http://www.evermore.com/oa/index.php3`. One can view the central circular region as akin to the platform into which the `T` is cut in Figure 9.

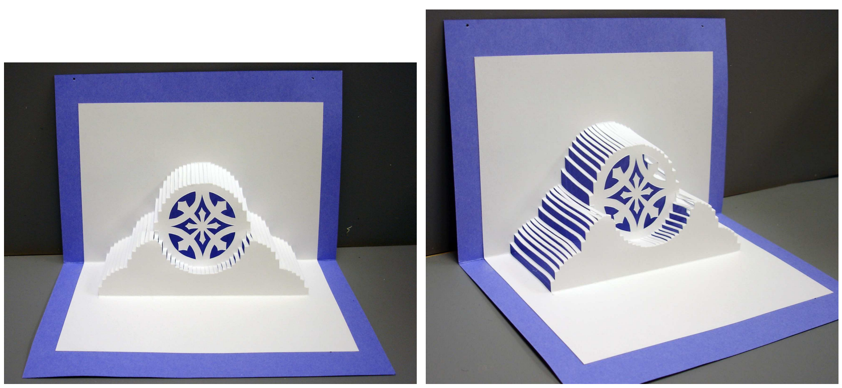

Figure 12: Evermore Enterprises Logo (Copyright 1999). [Gail Parsloe]

Parallel folds can also be used as a "platform." See Figure 13. [SAY MORE HERE]

2.2 Via Gluing

So far we have made our parallel-fold pop-ups by cutting into the card and bending out sections. Greater freedom is afforded by pasting in material, for then the pop-up is no longer restricted to be comprised only of the card material.

Figure 13: [Caption here.] `http://www.bobsliberace.com/christmas/xpop1.html`

2.2.1 Pop-up Box

An example is shown in Figure 14, which can be viewed as a pop-up box. The side view clearly shows that the construction produces two adjacent parallelograms. When the card is fully opened, the tops of the parallelograms become coplanar, forming a type of flat platform. Indeed, this is its main function in card design, to serve as a platform for display of characters, etc. [EXAMPLE

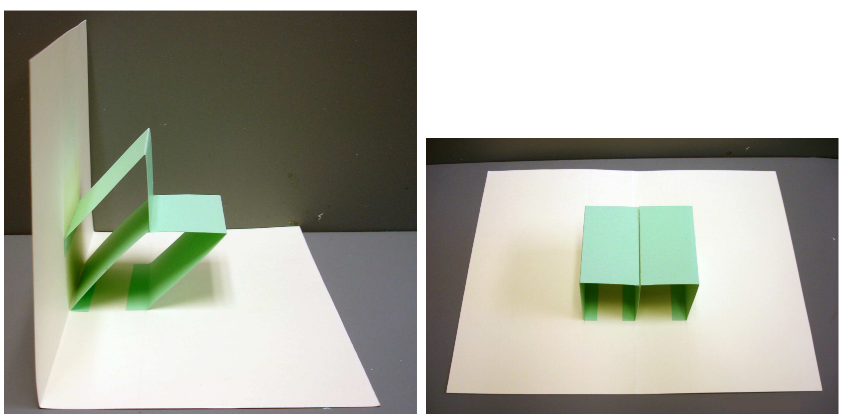

Figure 14: Pop-up box/platform.

HERE?]

2.2.2 Pop-up Tents

Having introduced the possibility of pasting in material, we now proceed to analyze quadrilateral pop-ups, generalizations of the parallelogram pop-ups produced by parallel folds. We start Figure 15.

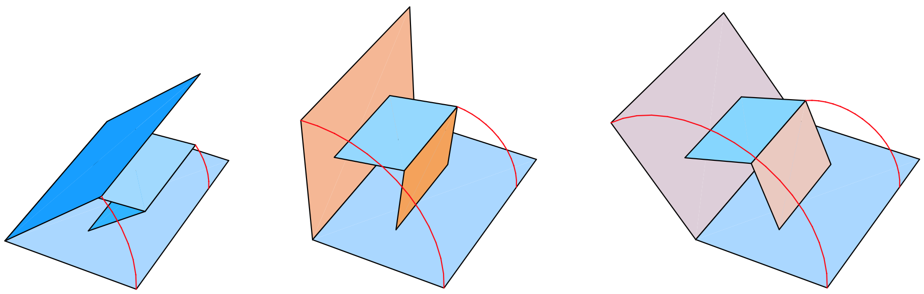

Figure 15: [Caption here.]

Question 2.5: *Apex Angle.* As the card flexes, what is the angle of the apex of the mountain crease in relation to the card angle?
Answer. It depends not only on α, but also on the two lengths that determine the tent: the base length b, and the side length a incident to the apex. It relationship can be computed by the Law of Sines. See below.

Call the apex angle β. Let $\alpha' = \alpha/2$ and $\beta' = \beta/2$. Then we can express β' in terms of α' using the Law of Sines (Figure 16):

$$\frac{a}{\sin\alpha'} = \frac{b}{\sin\beta'} \tag{1}$$

$$\sin\beta' = (b/a)\sin\alpha' \tag{2}$$

$$\beta = 2\sin^{-1}[(b/a)\sin(\alpha/2)] \tag{3}$$

For example, for $a = 2$ and $b = 1$, when the card is half-opened, i.e., $\alpha = 90°$, the apex angle is less than half the card angle: $\beta = 2\sin^{-1}(2\sin(\pi/4)) \approx 41°$. The example in Figure 15 uses $b = 2.5$ and $a = 3$, leading to $\beta \approx 72°$ when $\alpha = 90°$.

The diagonal bisecting α and β is a line of symmetry. Thus, the two triangles that are formed are congruent.

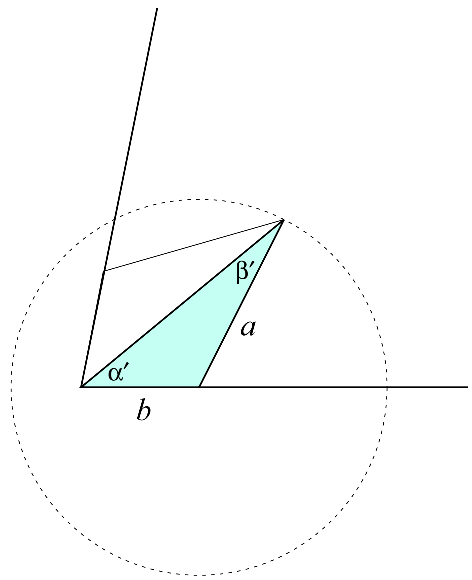

Figure 16: Applying the Law of Sines to the shaded triangle expresses β as a function of α.

Question 2.6: *[Grade $\geq x$.] Congruent Triangles.* What does this imply about the pairs of corresponding angles of the two triangles?
ANSWER. Because the triangles are congruent, all of the pairs of corresponding angles are equal.

2.2.3 Non-parallelogram Quadrilaterals

The tent described above is an example of a non-parallelogram quadrilateral.

(4)

Question 2.7: *[Grade $\geq x$.] Quadrilateral Edge Lengths.* Can a and b be arbitrary? What conditions must hold between a and b for the card to close flat ($\alpha = 0$) and open fully ($\alpha = 180°$)?
ANSWER. In order for a tent to close, a must be greater than or equal to b: $a \geq b$. Otherwise, the card cannot fully open, as shown in Figure 17.

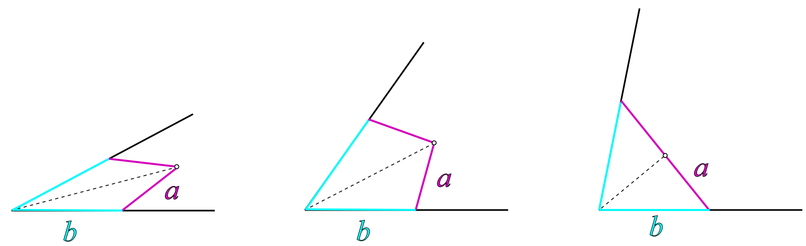

Figure 17: [Caption here.]

Now that we know one of the constraints for making a pop-up tent, let's next examine which quadrilaterals will also function as pop-ups. One question we want to answer is whether it is possible to make all the lengths of the sides of the quadrilateral different. This would create a nonsymmetrical quadrilateral.

see Figure 18.

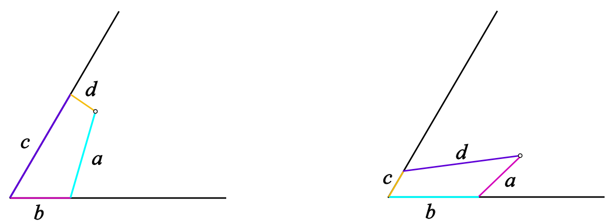

Figure 18: [Caption here.]

2.3 Connection to Pantographs

3 Angle Folds and Rotary Motion

3.1 Central V-Fold

We made the simplest parallel fold with a single cut symmetric on the centerline in Figure 1. With the exact same cut, but a different fold, we obtain the simplest *V-fold*; see Figure 19. We show two isosceles $45° - 45° - 90°$ triangles, one to each side of the centerline. The right one is labled (v, a, p). The central rib $R = vp$ (along which is a mountain fold) pops out toward the viewer as the card is opened, an effect that is useful for making, e.g., an opening mouth. see Figure 20.

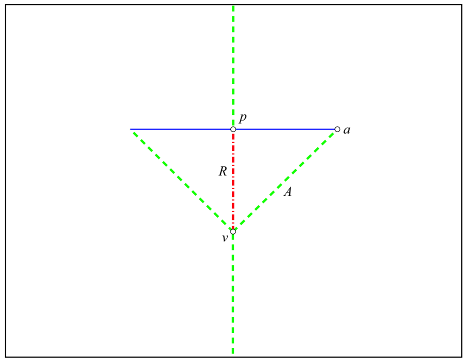

Figure 19: Template for centered V-fold.

Figures/birds.beak.photo.eps

Figure 20: Basic bird's beak.

Question 3.1: *[Grade ≥ X.] Motion in Medial Plane.* Let the *medial plane* be a plane through the centerline, midway between the front and back faces of the card. Let p be the tip of the central rib of the V-fold, i.e, the point where the cut crosses the centerline. See Figure 19. What is the path of p in the medial plane as the card opens?
ANSWER. It traces out a circle in the medial plane, centered on the V-fold apex v, with radius r, the length of the rib.

Although the motion of this tip point p in the medial plane is simple, its motion with respect to the back face of the card is perhaps less clear. It turns out that this motion will be the key to achieving rotary motion parallel to the back face, so we study it carefully.

Question 3.2: *[Grade ≥ X.] Motion of Rib.* Let the *rib* R of a V-fold be the mountain ridge that pops up from the cut centerline, the segment from v to p. What is the motion of R as the card opens when the back face is considered the base sy-plane?
ANSWER. R rides on a cone whose apex is at v, and whose axis A lies along the valley fold va of the V.

See Figure 21. Because the angle $\angle pva$ is 45°, $\triangle vap$ folds over so that vp is horizontal when the card is closed. And of course vp is vertical when the card is fully opened. So the full range of motion is 90°. This means that the angle at the apex of the cone is 90°, as is evident in Figure 22.

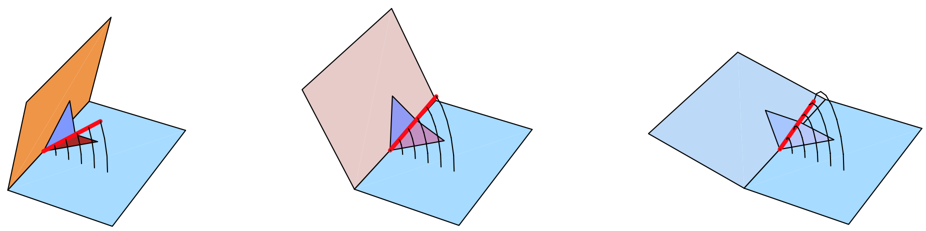

Figure 21: The central rib of a V-fold sweeps out a cone.

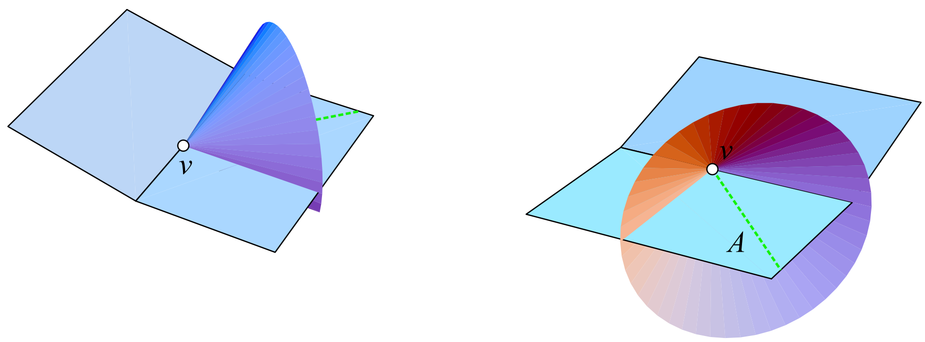

Figure 22: The cone intersects the card in a 90° angle at its apex.

> **Question 3.3:** *[Grade $\geq$ X.] Motion of Rib Tip.* What is the path of the rip tip p in space as the card opens?
> ANSWER. It follows a semicircle perpendicular to the back face.

In fact, any point on the rib follows a semicircle, as is evident in Figure 21.

3.1.1 Cone and Sphere

> **Question 3.4:** *[Grade $\geq$ X.] Cone/Sphere Intersection.* What is the intersection of a cone and sphere, when the cone apex is placed at the center of the sphere? I.e., what are the points on the surfaces that are shared by both the cone and the sphere?
> ANSWER. A circle. It lies in a plane orthogonal to the cone axis.

For any pop-up structure, if we identify a straight segment s on the structure with one endpoint fixed on the back face, then its other endpoint must lie on the surface of a sphere, just because s does not change in length. The rib $R = vp$

is such a segment, with v the fixed endpoint. Thus p is confined to the surface of a sphere centered on v, with radius $|R|$.

We can view the semicircle that p traces out as the top half of the circle that is the intersection of the cone on which R "rides" and the sphere to which p is confined. See Figure 23. The reason the semicircle is perpendicular to the back

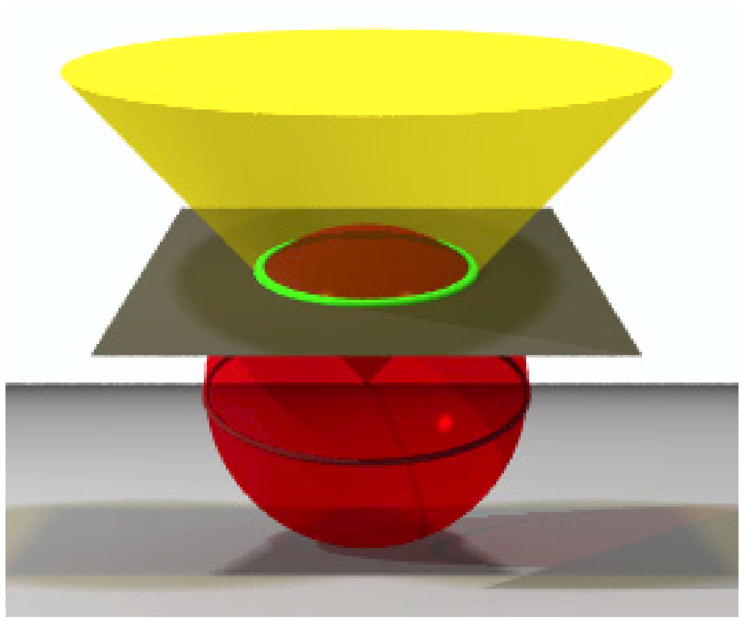

Figure 23: The intersection of a cone and a sphere is a circle lying in a plane.

face of the card is that the circle is perpendicular to the cone axis A, which lies in the back face.

3.1.2 Medial Plane

> **Question 3.5:** *[Grade $\geq$ X.] Different Viewpoints.* How can the circular motion of the rib tip p in the medial plane be reconciled with the circular motion of p with respect to the back plane? [REWORD]
> ANSWER. [This seems difficult to understand. Working on an answer...]

Let us take the view with the back face of the card fixed, in which case p rides on the rim of the cone shown in Figure 22, and traces out a semi-circle. Now think of the motion of the medial plane in this situation. Its angle with respect to the back face is half of the card angle α. So it opens from $0°$ to $90°$

as the card is fully opened. During this opening, it rotates through the cone, and the intersection of the cone rim with the medial plane traces out a quarter circle. See Figure 24 and 25.

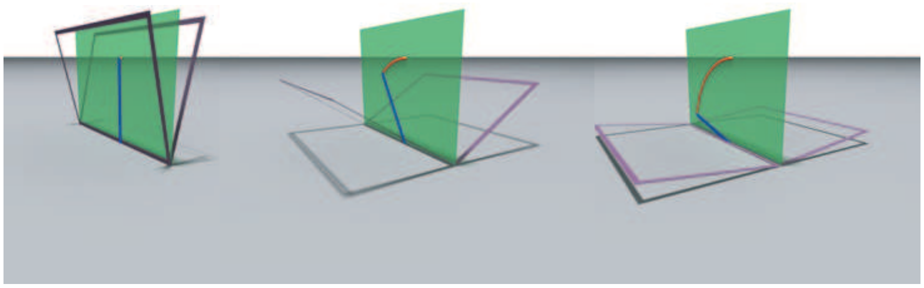

Figure 24: Medial plane fixed.

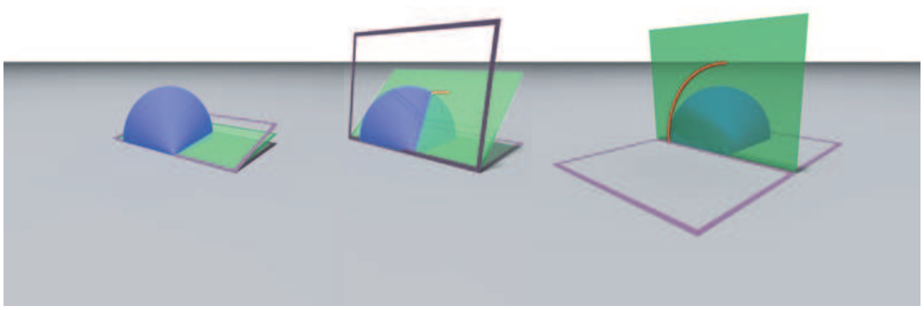

Figure 25: Medial plane (green) rotating through cone.

see Figure 26.

Question 3.6: *[Grade $\geq X$.] Medial Plane Card.*. Design a card that has a medial plane, i.e., a rectangle of paper that joins to the cardline and always lies halfway between the front and back card faces as the card opens.
ANSWER. One mechanism is to use a rhombus parallel fold and slots, as shown in Figures 27 and 28 below.

see Figure 28.

Question 3.7: *[Grade $\geq X$.] Medial Plane Rib.* Alter the medial plane card to have a thin rectangular "rib" that illustrates the motion of p tracing the quarter circle on the median plane. ANSWER. One design is illustrated in Figure 29.

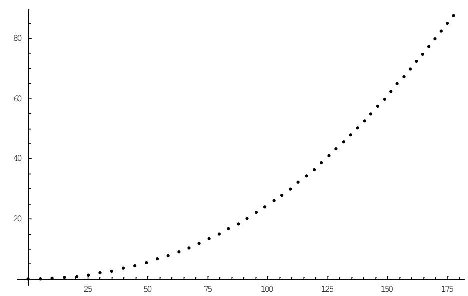

Figure 26: [Caption here.]

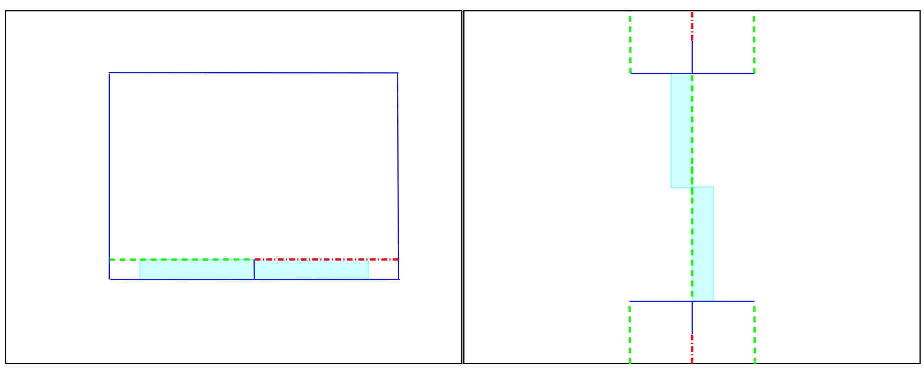

Figure 27: Left: Rectangle of paper with crease marks. Right: Paste regions.

Figures/plane.photo.eps

Figure 28: [Caption here.]

Figures/plane.rib.photo.eps

Figure 29: [Photo of Faith's medial-plane card.]

3.1.3 Angle of V-Fold

So far we have only explored V-folds that form two isosceles $45° - 45° - 90°$ triangles. Define the V-fold angle to be the angle of one of these triangles at the apex of the V, i.e., $\angle avp$ in Figure 19.

> **Question 3.8:** *[Grade $\geq$ X.]* *V-Fold Angles.* Suppose the V-fold angle is β. What is the angle between the start position of the V-fold rib when the card is closed, to the rib's final position when the card is fully opened?
> ANSWER. 2β.

Refer again to Figure 22. The cone along which the rib rotates intersects the back face of the card in two line segments, which are symmetric with respect to the axis A of the cone. When the card is fully opened, the V-fold triangle is flat, and the two angles of β separate the start and end positions of the rib. So the total rotation is 2β. [NOT WELL EXPLAINED]

3.2 Shifted V-Fold

In the same way that we used valley folds that were not the card centerline as the basis for creating additional parallel-fold pop-ups in Figure 7, one can use any valley fold as the basis for a V-fold. The structure most commonly used, and what we will describe here, is a parallel-fold pop-up to create a valley crease on the back face, with the V-fold based on this valley crease. Figure 30 shows an example.

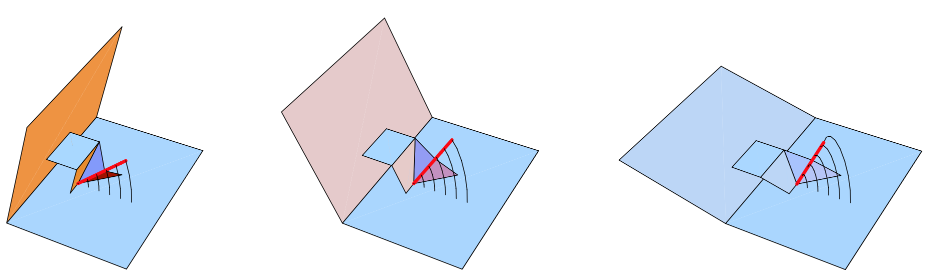

Figure 30: [Caption here.]

This can be constructed by gluing a single rectangle of paper to the card, appropriately creased, as shown in Figure 31. The cone is now displaced toward

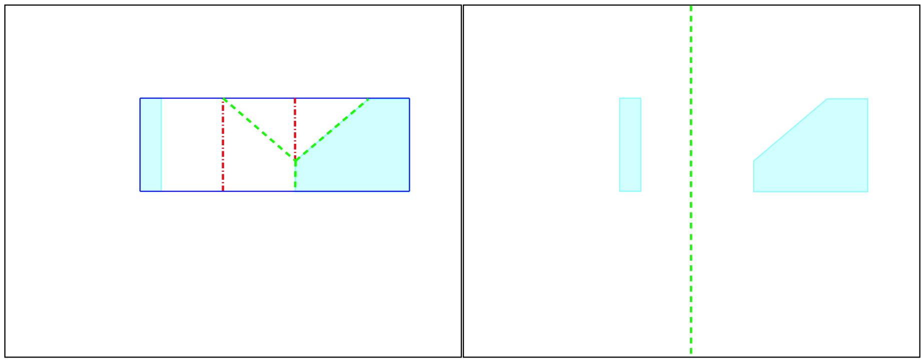

Figure 31: Left: Rectangle of paper with crease marks. Right: Paste regions.

the center of the back card face, which is often required for the rotation effects we study below.

see Figure 32.

Figures/shifted.V.photo.eps

Figure 32: [Caption here.]

3.2.1 Rotation of Non-Rib Segment

So far we have only tracked the motion of points along the rib segment of a V-fold, but in order to achieve pure planar rotary motion, we will need to examine the motion of a segment parallel to the rib, but with one endpoint at point a' in Figures 30 and 31. Call this segment $R' = a'b'$. It remains parallel to R, the V-fold rib, throughout its movment. When the card is fully opened, it is aligned with the (then flattened) mountain crease of the rectangle parallel fold, qa' in the figures.

> **Question 3.9:** *[Grade $\geq X$.] Swept surface.* As the card is opened, segment R' sweeps out a surface. Describe this surface qualitatively.
> ANSWER. The point a' traces out a semicircle centered on uv. The point b' does not trace out a semicircle, but moves in a more complicated manner. See Figure 33.

We have not computed the shape of the curve traced out by b', but it does not appear to lie in a plane. The surface is what is known as a *ruled surface* in mathematics, for it is a surface swept out by a line segment moving in space.

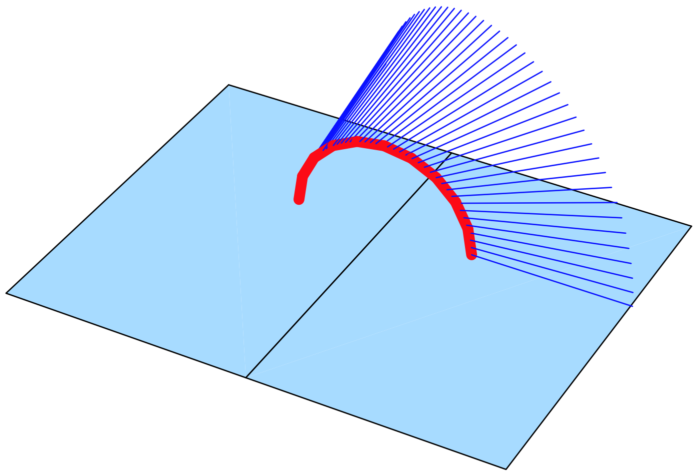

Figure 33: Surface swept by segment R'.

3.3 Planar Rotation

in Figures 34

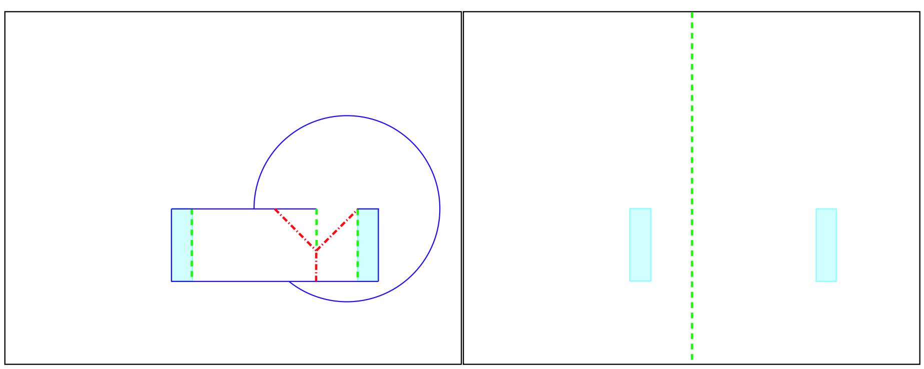

Figure 34: Left: Rectangle of paper with crease marks. Right: Paste regions.

see Figure 35.

4 The Knight's Visor

Although Figures 11 and 12 give some idea of the power of parallel cuts, there is an elegant and surprising construction that goes beyond all that we have shown so far, but still based on parallel cuts. (This construction is described as a "Multi-slit variations" in [Jac93, p. 62, Fig. III].) Draw a circle in the center of a card, centered on the centerline, an cut equally-spaced, parallel, symmetric cuts terminating on the circle boundary. See Figure 36. Note that the valley

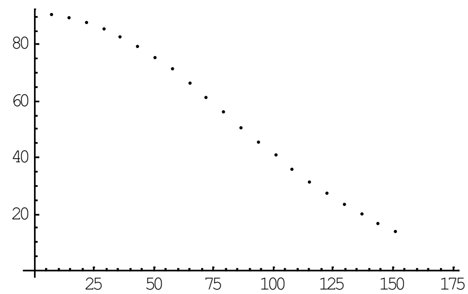

Figure 35: [Caption here.]

creases at the ends of each strip should follow the circle, i.e., these creases are not parallel to the card centerline.

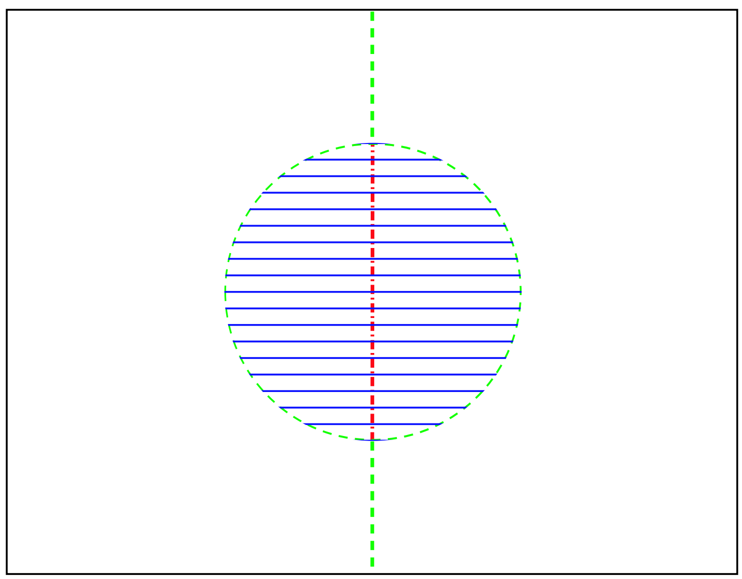

Figure 36: Template for the Knight's Visor.

> **Question 4.1:** *[Grade ≥ X.] Shape of Knight's Visor.* Before bending the card, can you predict the shape it will take when opened roughly midway?
> ANSWER. See Figure 37!

We have studied just one angled crease, i.e., one crease angled to the centerline, and the motion is quite complex. Now each half-strip attached to the

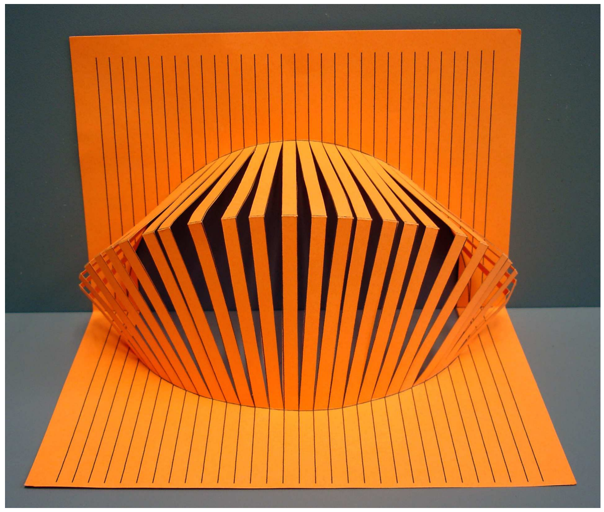

Figure 37: The Knight's Visor pop-up.

back face of the card rides on a cone whose axis is along its angled crease, and the half-strip attached to the front face of the card rides on another cone, but this time attached to the front face. And all the cone axes make different angles with the centerline. The result is a bowing that is small for long strips and gradually increases for the short strips. We call this shape the Knight's Visor; see Figure 38!

4.1 Flat Visor Curve

Question 4.2: *Flat Visor Curve.* What is an equation for the curve formed by the tips of the ribs when the card is closed, i.e., $\alpha = 0$?
Answer. It is an apparently unstudied curve, which we christen the *Knight's Visor curve.* Derivation and properties below.

Let the circle C at which the cuts terminate be a unit circle centered on the origin, and consider the rib R at $x = a$, i.e., the segment from $(a, 0)$ to the circle at $C(a) = (a, \sqrt{1-a^2})$. See Figure 39.
Differentiating the function $y = \sqrt{1-x^2}$, we find the slope of the tangent line to C at $C(a)$ to be $\frac{a}{\sqrt{1-a^2}}$. The line goes through $C(a) = (a, \sqrt{1-a^2})$ so it has the equation

$$y = (x-a)\frac{a}{\sqrt{1-a^2}} + \sqrt{1-a^2} \tag{5}$$

R reflects over this tangent, and we seek to compute its reflected tip $p(a)$, the reflection of $(a, 0)$. The line connecting $(a, 0)$ to $p(a)$ is perpendicular to the

Figure 38: Close helmets with visor, circa 1550. Left: `http://www.by-the-sword.com/acatalog` Right: `http://www.gemstone.play.net/etimes`

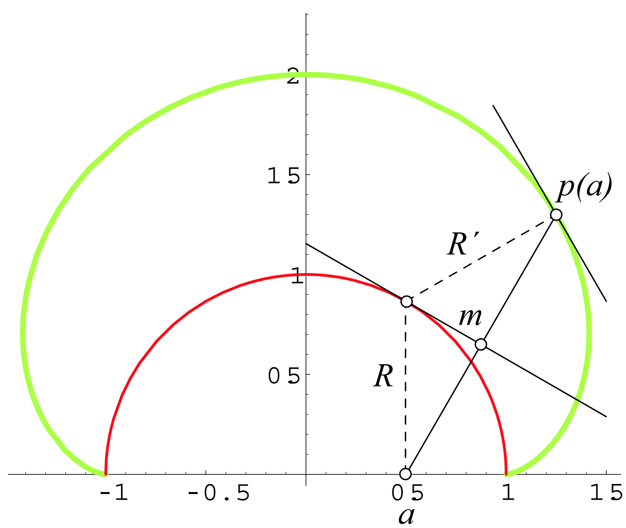

Figure 39: Geometry of the visor curve. $p(a)$ is a point on the curve, deriving from $(a, 0)$.

tangent line to $C(a)$ so it has slope $\frac{-\sqrt{1-a^2}}{a}$ and equation

$$y = (x-a)\frac{-\sqrt{1-a^2}}{a} \tag{6}$$

Intersecting this line with the tangent line (Eq. (5)) yields coordinates for the midpoint m of the isosceles triangle in the figure. Then we can find $p(a) = (x(a), y(a))$ as $(a, 0)$ plus twice $m - (a, 0)$. This yields

$$\begin{aligned} x(a) &= a(3-2a^2) && (7) \\ y(a) &= 2(1-a^2)^{3/2} && (8) \end{aligned}$$

This is a parameterization of the flat Knight's visor curve with a ranging in $[-1, 1]$. When $a = 0$, $p(0) = (0, 2)$; when $a = \pm 1$, $p(a) = 0$.

With this curve parametrization in hand, we can develop several properties of the curve.

1. The curve is tangent to the x-axis at $x = \pm 1$. The slope of the curve at a is $y'(a)/x'(a)$, which, evaluated at $a = \pm 1$, yields 0.

2. The tangent to the curve at $p(a)$ is the reflection of the x-axis over the tangent line at $C(a)$. In other words, the reflected rib R is orthogonal to the tangent at $p(a)$. We can establish this by showing the dot product is zero. The tangent direction is $T = (x'(a), y'(a))$. The reflected rib R' is

   $$(\ x(a) - a,\ y(a) - \sqrt{1-a^2}\ )$$

   Taking the dot product $T \cdot R'$ (and simplifying) indeed yields 0.

We have expressed the visor curve in polar coordinates, but the expression is complex. The curve does have an interesting implicit form, however. We find the implicit form from the parametric form using the equation of the circle. Recall the parametric form

$$\begin{aligned} x &= a(3-2a^2) \ = \ 2a(1-a^2) + a && (9) \\ y &= 2(1-a^2)^{3/2} && (10) \end{aligned}$$

Solve the second equation for $1 - a^2$ and then substitute the result $1 - a^2 = g(y)$ to the first equation. We get $x(a)$ as a function of a and y. Solve this new equation for a and we get $a = f(x, y)$. We know that $a^2 + (1 - a^2) = 1$. Substitute f and g to this equation, we get a new equation:

$$x^2 = 2\left(\left[\frac{y}{2}\right]^{2/3} + 1\right)^2 \left(1 - \left[\frac{y}{2}\right]^{2/3}\right) \tag{11}$$

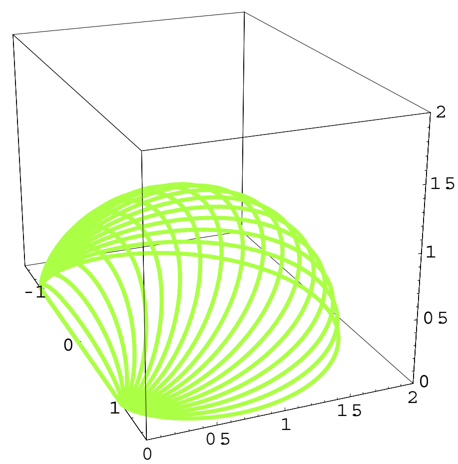

Figure 40: Progression of visor curves from $\alpha = 0$ to $\alpha = 180°$.

4.2 Visor Curve in 3D

We now compute the visor curve for all card angles α. It starts with the flat visor curve just described, and ends with the diameter of the circle along the centerline of the card. Figure 40 shows all the curves for α from 0 to 180°.

Let M be the medial plane through the card centerline, halfway between the front and back faces of the card, i.e., at angle $\alpha/2$. The visor curve lies in M. We employ the same geometric ideas that we used to find the flat curve. Again we focus on the rib R connecting $(a, 0, 0)$ to the circle at $(a, \sqrt{1-a^2}, 0)$. It reflects to rib R', whose tip is the point $p(a)$ on the curve, just as before. But now $p(a)$ is a point in 3D. Because R' is the same length as R, its tip $p(a)$ lies on the sphere S centered at $(a, \sqrt{1-a^2}, 0)$. So we now know that $p(a)$ lies in M and on S. The intersection of a plane and a sphere is a circle. We need one more constraint to pin it down. Let Q be the vertical plane that contains both a and $p(a)$. The point we seek lies at the intersection of S and M and Q. See Figure 41, and Figure 42, the view from underneath. The intersection of M and Q is a line in space, which penetrates the sphere at two spots, one $p(a)$, and the other beneath the card. So we can compute $p(a)$ by constructing the line $L = M \cap Q$, computing $S \cap L$, and choosing as $p(a)$ the solution with positive z-coordinate. It was through this computation that we computed the curves displayed in Figure 40.

References

[Jac93] Paul Jackson. *The Pop-Up Book*. Henry Holt and Co., 1993.

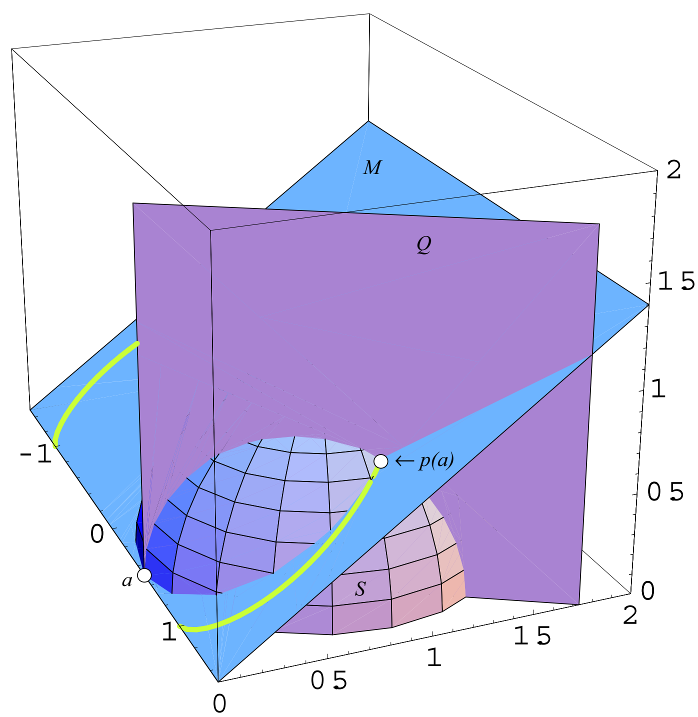

Figure 41: Construction of the visor curve (green) in 3D.

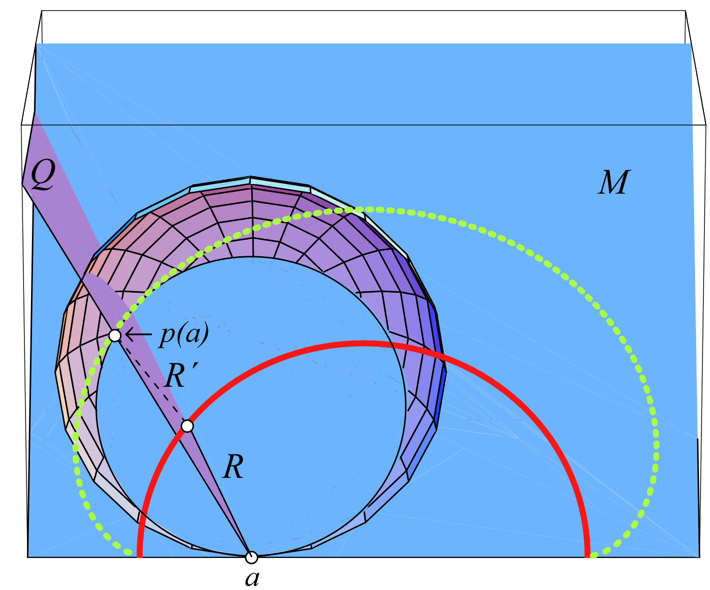

Figure 42: View of Figure 41 from underneath.