

The Entertainer

Jim Henle

The Mathematical Intelligencer

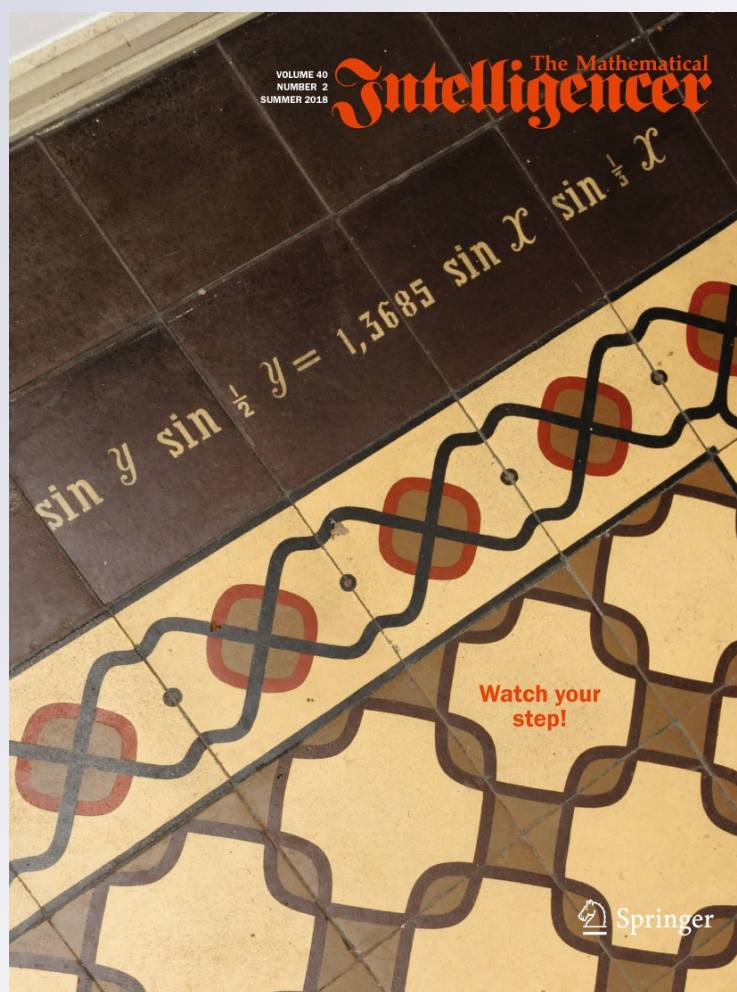
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The Entertainer

JIM HENLE 

This is a column about the mathematical structures that give us pleasure. Usefulness is irrelevant. Significance, depth, even truth are optional. If something appears in this column, it's because it's intriguing, or lovely, or just fun. Moreover, it is so intended.

Raymond Smullyan—logician, magician, mathematician, puzzlist, and Taoist philosopher—passed away last year at the age of 97. In each of his vocations, he made serious contributions. But in his heart, and from his head to his toes, he was an entertainer.

In gatherings large and small, among friends and among strangers, Ray was “on.” If there was a piano, there would be music. If there was a deck of cards, there would be tricks. And if there was conversation, there would be stories, jokes, and paralyzing paradoxes.

Smullyan had important things to say about logic, about knowledge, about mathematics, and about the meaning of life. To bring his ideas to the public, he created libraries of fantasies, puzzles, and conundrums. It's my belief that these were more than means to an end. They were really his greatest joy. The professor professed—so the entertainer could entertain.

Smullyan came to Smith College a number of times at my invitation, but I can't claim I really knew him. I base my conclusions on his choice of careers, his books (especially the memoirs), and on the recollections of others. In *Four Lives: A Celebration of Raymond Smullyan*,¹ violinist Claudia Schaer wrote:

I soon learned that he brings his deck of cards everywhere and does magic tricks not just in restaurants, but at almost any opportunity, bringing a smile to strangers who find themselves in his proximity. He just as gladly played at the piano.

One of his thesis students, Robert Cowen, wrote:

Whatever Ray does, be it in mathematics, puzzles, music, or magic, is characterized by beauty and elegance. Also, it is always very entertaining. In fact, Ray is the ultimate entertainer; he will always have something to delight you!

The Logician

There were logic puzzles before Raymond Smullyan,² but he raised the genre to the level of art. His first puzzle book, *What Is the Name of This Book?*³ is pure pleasure. The puzzles are populated by knights—who always tell the truth, and by knaves—who always lie.⁴

Here is one of Smullyan's elegant puzzles:

I saw a pair on the island of knights and knaves. I asked, “Is either of you a knight?” One fellow replied and from his answer I knew what sort person he was

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¹Jason Rosenhouse and Raymond Smullyan, Dover Books, 2014.

²The history of the logic puzzle goes back at least 150 years to Lewis Carroll. Arguably, it is thousands of years old, and begins, perhaps, with the liar paradox of Epimenides.

³Prentice-Hall, 1978.

⁴This sort of logic puzzle appears in Maurice Kraitichik's *Mathematical Recreations* (W.W. Norton, 1942). Kraitichik may have originated the form.

and also what sort the other person was. What did I find out?

The answer is at the end of this column.

After ringing changes on knights and knaves, he finds that some of his characters are sane and see the world as it is and some are insane and see it as it is not. When the puzzle possibilities of this have been exhausted, we find knights and knaves speaking in unknown tongues. Here is one of Smullyan's most baroque puzzles:

I was in Transylvania, where humans tell the truth and vampires lie. Some are sane, some are insane. Either the words "bal" and "da" mean "yes" and "no" or they mean "no" and "yes." To escape Dracula, I must find a sentence S that unlocks all secrets, that is, for any statement P , if you ask anyone (human or vampire, sane or insane)

Is P true if and only if S is true?

They will answer "Bal" if P is true and "Da" if P is false (no matter what "bal" actually means).

I found such a sentence. Can you find one?

Typically, a Smullyan puzzle book has an agenda. More than a few of these books conclude with Gödel's incompleteness theorems and their famously difficult proofs.⁵ Smullyan's many mathematical contributions center on the work of Gödel and Tarski.⁶ My personal feeling is that the puzzles are not especially helpful in understanding or explicating Gödel's work. But that's a pedagogical judgment. If I look at these as mathematical structures in and of themselves, they are vastly entertaining.

Here is one of the best, in a simplified form: For his proof, Gödel produces a sentence that is true but cannot be proved by creating a sentence that says, in effect,

"You can't prove me!"

There are many amazing ideas in Gödel's proof, but the most amazing (to me) is that somehow the sentence succeeds in referring to itself. Smullyan creates a marvelously simple language that does that.

*Consider all words (that is, all strings of symbols) composed of just the three letters **N**, **P**, and **R**. Here is how to interpret sentences in this language:*

P xyz | means that xyz can be proved
N xyz | means that it is not true that xyz
R xyz | is an abbreviation of xyz repeated.

Now suppose that provability and truth were the same. Consider the sentence:

NPRNPR.

*The **NP** at the front means that the sentence is saying: "**RNPR** can't be proved."*

*But **RNPR** is the repeat of **NPR**, that is, **NPRNPR**. In other words, the sentence says*

"You can't prove me!"

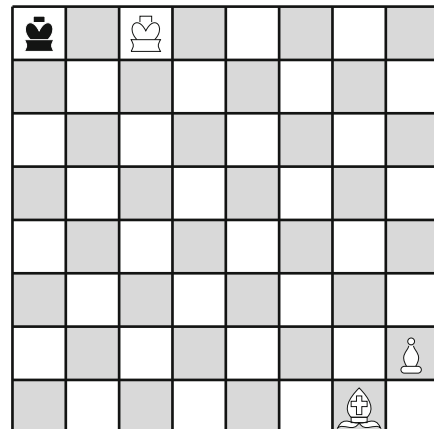
*It follows that proof and truth don't agree in this language, because that would mean that **NPRNPR** is true if and only if it can't be proved.*

In no sense is this Gödel's theorem. But it's great fun.

The Puzzlist

Smullyan's wealth of puzzle books establishes him as a master puzzlist.⁷ His genius emerged at an early age. As a high-school student, Smullyan invented a new puzzle form. Only much later did he discover that the form already existed: retrograde analysis. A problem in retrograde analysis presents pieces on a chess board and asks for information about the game that left the pieces where you see them. No chess strategy is involved. The problems are entirely about the logic of the rules of chess. It's a most attractive puzzle form, and Smullyan's contributions are wonderfully subtle, imaginative, and surprising.

Here is my favorite, which appeared on the cover of his first book of retrograde problems, *The Chess Mysteries of Sherlock Holmes*.⁸



"Black moved last, Watson. What was his last move —and White's last move?"

Black's last move must have been for the king to go one square up, but how could that be? He would have been in check from the White bishop and there doesn't seem to be any way White could have put the Black king in check. The bishop couldn't have moved to where it is unless the king were already in check, an impossibility. The only other way a piece can be put in check is a revealed check, that is,

⁵The Lady or the Tiger? Alfred A. Knopf, 1982; Forever Undecided, Random House, 1987; The Gödelian Puzzle Book, Dover, 2013.

⁶Formal systems, computability, provability, and recursion.

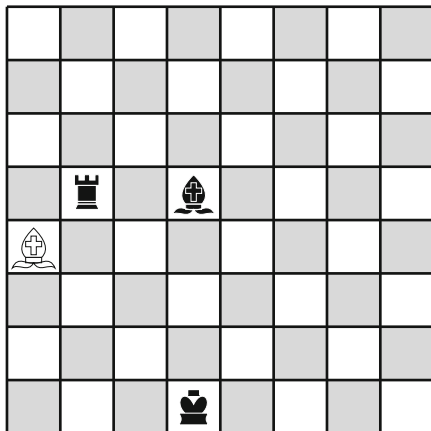
⁷Martin Gardner described Smullyan's first puzzle book as "The most original, most profound and most humorous collection of recreational logic and mathematics problems ever written."

⁸Alfred A. Knopf, 1979.

some piece that was between the king and the bishop moved out of the way, revealing the check. But there is no white piece on the board that could have revealed the check!

The solution is at the end of this column!

Here's one more puzzle, reportedly the favorite of Smullyan himself. It appeared on the cover of his second retrograde book, *The Chess Mysteries of the Arabian Knights*:⁹



"The White king, Haroun Al Rasbid, has made himself invisible. Where is he?"

The Mathematician

One of Smullyan's puzzle books, *Satan, Cantor, and Infinity*,¹⁰ has as its climax Cantor's diagonalization proof that the set of real numbers is uncountable, that is, that there is no mapping from the natural numbers onto the set of decimals. Here, I think, Smullyan makes a significant contribution to pedagogy.

I, like many mathematicians, remember the moment I heard Cantor's proof as the moment I knew I wanted to be a mathematician. Consequently, I have revealed it, one way or another, to untold numbers of students. But each time (until I read *Satan, Cantor, and Infinity*) there would be at least one student hung up on a particular point.

I usually presented the proof as it was presented to me by my eleventh-grade teacher Miss Orndorff. Suppose there is a mapping from the natural numbers onto the decimals. Miss Orndorff wrote the start of such a mapping, just using decimals between 0 and 1.

0 \rightarrow .8274864345...
 1 \rightarrow .7878333463...
 2 \rightarrow .3734653920...
 3 \rightarrow .0103389425...
 \vdots

Then she constructed a decimal that was not in the range of the mapping by choosing a number whose digits differed from the digits on the diagonal,

0 \rightarrow .8274864345...
 1 \rightarrow .7478333463...
 2 \rightarrow .3734653920...
 3 \rightarrow .0100389425...
 \vdots

She explained that such a number, say, .2759..., can't be the first number in the range because it has a different first digit. And it can't be the second number in the range because it has a different second digit, and so on. Thus the number is not in the range, and so there is no mapping between the set of natural numbers and the set of decimals.

Whenever I presented the proof, someone would complain, "But you could add this number to the list! You could create a mapping that would include it!"¹¹

Smullyan's tremendously entertaining version avoids this problem. He tells a story about Satan. He imagines Satan teasing the souls who arrive in Hell. He offers one soul a deal: He, Satan, will write a natural number on a piece of paper. Each day in Hell, the damned soul may try once to guess the number. If the soul names it, it is released and goes to heaven. If not, it can guess the next day. Satan won't change the number.

It's easy to see that there's a strategy for escaping Hell. You simply guess: 0 on the first day, 1 on the second day, then 2, and so on.

Satan loses souls this way, so he changes the game. Instead of a natural number, he writes a positive or negative integer for the condemned to guess. But once again there's a strategy for escape. A soul can guess integers in this order: 0, 1, -1, 2, -2, 3, -3,

Satan changes the game a second time. Now it's fractions that he writes. But as with the natural numbers and the integers there is a strategy. A soul playing this game can first run through all fractions with numerator and denominator less than 2. Then the soul can name the additional fractions with numerator and denominator less than 3, and so on.

In each of these cases, there is a strategy that guarantees escape in a finite number of days. Furthermore (important point here), *even if the devil is told the strategy before he chooses his number, the soul will still escape in a finite number of days.*

Finally, Satan changes the game one last time. Now the numbers he writes are infinite decimals. And now there is no strategy that will work. Unlike the game with the natural numbers, the game with the integers, and the game with fractions, if the devil is told the strategy, if he is told the order in which decimals will be guessed, he can (using Cantor's trick) write down a decimal that the soul will never guess.

⁹Alfred A. Knopf, 1981.

¹⁰Alfred A. Knopf, 1992.

¹¹This is not a valid objection, of course, because Cantor's proof shows that the altered mapping would still be missing decimals.

I use this approach now when I teach this proof. It's a lot of fun. It's



the same proof, but the presentation has a way of sorting out the quantifiers.

The Magician, the Philosopher, and the Raconteur

Smullyan's first job was doing magic tricks. This was before he graduated from high school. He was a "close-up" magician, not a stage performer. Ultimately, magic was a sideline for him. But informally, magic was a unifying theme to his interests. It was the magical aspects of mathematics, for example, that attracted him to that subject.

In addition to mathematics, magic, in the form of paradox, was at the center of Smullyan's logic and, arguably, of his philosophy. He loved paradoxes. When he found one, he would nurse it, tease it, and bless it. I suspect it was paradox that drew him to Taoism.

A rich paradox is also a good joke. Smullyan loved a good joke. A Smullyan lecture, especially in his later years, was rather like stand-up comedy.

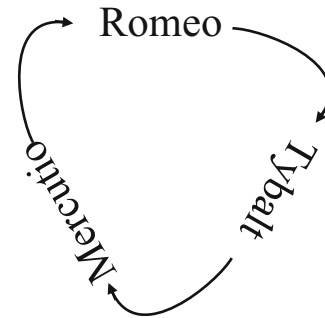
Still the Entertainer

Raymond Smullyan's art can be enjoyed on at least two levels. At one level is the pleasure of wrestling with his puzzles. That pleasure can be limited. But there's a deeper level. Smullyan's puzzles inspire some of us to create our own puzzles. *Full disclosure: I and my coauthors sprinkled knights and knaves all over our logic book.*¹² *Satan is there too.*

I am an appreciator at both levels. I submit this as evidence:¹³

In one district of the city of Verona there are only knights and knaves. All of these are either Montagues or Capulets. Three residents, Romeo, Mercutio, and Tybalt, stood in a circle talking to each other. These gentlemen were not the ones in Romeo and Juliet, so their houses are not necessarily those of Shakespeare's

characters. Each pointed to the man at his left and stated his house (Montague or Capulet) and type (knight or knave).



Each man gave the same description of his neighbor. Tybalt disputed what Romeo said about Mercutio. What can you infer from this?

Of the three, at least two must be of the same house. If I add that Mercutio and Romeo are from the same house, what more can you say?

One of my students, Gabrielle (Gabby) Manna, also appreciated Smullyan's work at this high level. She wrote a series of puzzles involving a dysfunctional family that was truthful or dishonest on a weekly schedule:

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Mom	Truth	Lie	Lie	Truth	Lie	Truth	Truth
Dad	Lie	Truth	Truth	Truth	Lie	Lie	Truth
Son	Lie	Lie	Truth	Lie	Truth	Truth	Lie

The son lives with the mother on weekdays and the father on weekends. The following phone conversation took place on a weekday. Gabby asks you to determine what day it is.

Dad: Johnny will stay with you this weekend because I will be on a three-day business trip. I told him to tell you this yesterday.

Mom: No, you didn't tell him anything, and you're not going on a business trip. To prove it, I'm going to ask him. [She calls out to her son.] Johnny, did your father tell you anything about this coming weekend yesterday?

Son: [entering] Yes he did, yesterday. He said he'd be picking me up an hour late this Saturday.

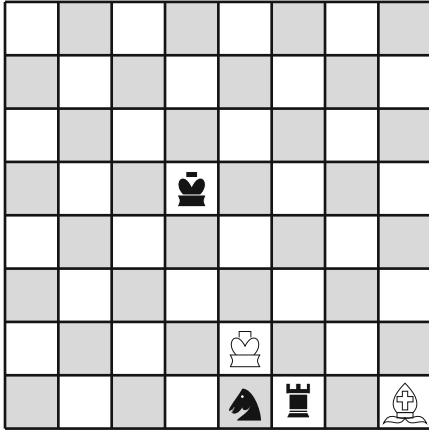
Mom: You're lying. I know your father didn't mention anything to you about the weekend.

Dad: Yes I did! I told him about it yesterday!

I was much taken with Smullyan's retrograde chess problems. Here is something I put together recently:


¹²*Sweet Reason: A Field Guide to Modern Logic*, second edition, with Jay Garfield and Tom Tymoczko, Wiley-Blackwell, 2011.

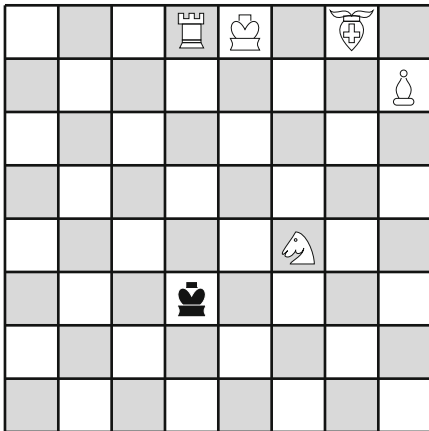
¹³I invented this puzzle, but it would be a Herculean task to check whether Smullyan invented it earlier!



After many moves, the chessboard appears as above. What were the last two moves?

I'm not much of a chess player. Perhaps as a consequence, I'm drawn to chess's eccentric relations. Years ago I bought a copy of *A Guide to Fairy Chess*.¹⁴ It describes a menagerie of odd and creative chess pieces. I can imagine tantalizing retrograde analysis problems involving the *reflecting bishop*, the *grasshopper*, the *edgehog*, the *nightrider*, or the *magnetic queen*.

Many retrograde problems involve situations in which a king is simultaneously in check from two different pieces. With a fairy piece, this number could be increased, for example: The  is a reflecting bishop. It moves like a bishop but can bounce off the sides of the board. Three pieces simultaneously checking the king was the best I could do with a reflecting bishop. The problem seems to be that 8 is an even number. I don't think I can do anything about that. On a 7×7 chess board, however, I can manage four simultaneous checks.



Smullyan's Satan stories also stimulated my puzzlistic tendencies. Here is a problem I put on a final exam in a first-semester logic course:

The devil has a new deal for condemned souls. He gives them an essay test. There's only one question. Satan writes the answer on a piece of paper and puts it in his pocket. He won't change it. But he doesn't tell you the question! You have to write the answer to a question you don't know! And the answer has to be exactly what he wrote! He won't even tell you how many pages it is (don't you hate professors who do that?!)

If you succeed, you go to heaven. If not, then you try again the next day. By the way, you don't know the language that the devil wrote his answer in. It does use the Roman alphabet.

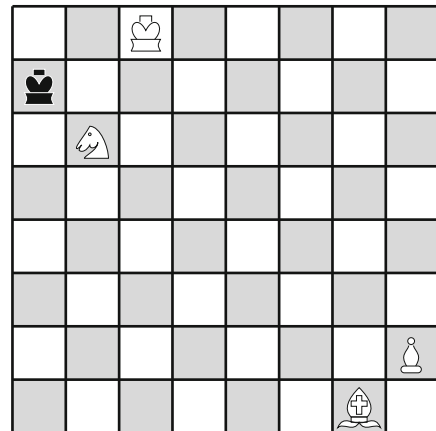
Two Answers

For the first knights and knaves problem:

The answer is "No." This tells you that the speaker is a knave, for a knight would answer "Yes." Further, it tells you that the other gentleman is a knight, for if he were a knave, the answering knave would have said "Yes."

For the first chess problem:

It's true that there is no white piece on the board that might have revealed the check. But *there could have been*. The black king could have captured such a piece, a knight. Indeed, this is what the board must have looked like two moves previously:



Answers to all the other puzzles can be found at www.math.smith.edu/~jhenle/Pleasingmath/.

And readers can send comments and new puzzles to me at pleasingmath@gmail.com.

¹⁴Anthony Dickens, Q Press, 1969.