Romantic Mathematical Art: Part I

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The Mathematical Intelligencer

ISSN 0343-6993
Volume 42
Number 2

Math Intelligencer (2020) 42:70-77
DOI 10.1007/s00283-020-09968-y

Romantic Mathematical Art: Part I

JIM HENLE

To see a World in a Grain of Sand
And a Heaven in a Wild Flower
Hold Infinity in the palm of your hand
And Eternity in an hour
—William Blake, *Auguries of Innocence*

This is a column about the mathematical structures that give us pleasure. Usefulness is irrelevant. Significance, depth, even truth are optional. If something appears in this column, it's because it's intriguing, or lovely, or just fun. Moreover, it is so intended.

There is Romantic music. There is Romantic literature. There is Romantic painting. The Romantic era in art lasted a century or more. Is there such a thing as Romantic mathematics?

It would seem unlikely. Consider these phrases frequently used to characterize Romanticism: "senses over intellect," "emotion over reason," "freedom from rules." But there are words connected to Romanticism that have affinities to mathematics: "mysterious," "unbounded," "imaginative," "remote," "unattainable," and "paradoxical."

I can think of two quite specific areas of mathematics and mathematical art that are relevant here: infinity ("unbounded," "remote") and impossibility ("unattainable," "paradoxical"). The importance of these concepts in art is witnessed by the fact that virtually no website on Romanticism is complete without an image of Caspar David Friedrich's 1818 painting *Wanderer Above the Sea of Fog*,

a work that captures both impossibility and infinity as intense feelings.

This Part I of "Romantic Mathematical Art" (Part II will appear in a subsequent issue of this journal) is devoted to the first of these notions, infinity, though as the *Wanderer* illustrates, infinity and impossibility are cousins. That may be why for thousands of years, people have been both attracted and frightened by infinity. In our own era, fear of

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https://doi.org/10.1007/s00283-020-09968-y

infinity, apeirophobia, has become a recognized psychological condition.

Of course, this article can't cover all the beautiful bits. Infinity is, after all, big.

Infinite Geometric Art

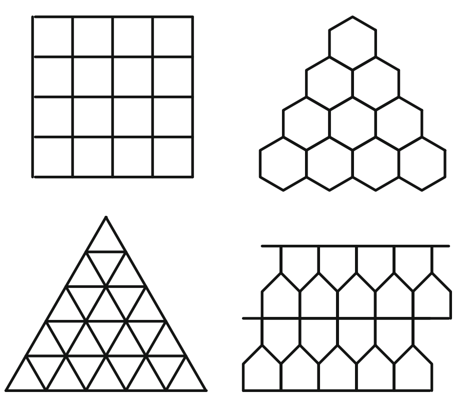
When I was in college in the 1960s, the undergraduate math majors were treated to a lecture from Emeritus Professor Bancroft Brown. He was a geometer. He reminisced about the old days. He led us to believe that geometry was a backwater, a field that had simply played itself out. His message, in essence, was that back in the old days, "we had fun."

What Brown didn't know was that the revival of geometry was already underway, led by Donald Coxeter,[1] although it had not yet gathered strength. Most of what I include here was discovered since Brown's lecture. Fun is now quite general.

The visual appeal of geometric mathematical structures has made geometry a prime area for mathematical art. I can't do it justice here; I can only summarize and touch on a few of the highlights, emphasizing the wide public interest.

Tiling the Plane

Filling the plane with shapes is an ancient pastime practiced by almost every culture. The discovery of single figures that tessellate the plane—that is, fill it entirely—is also ancient.

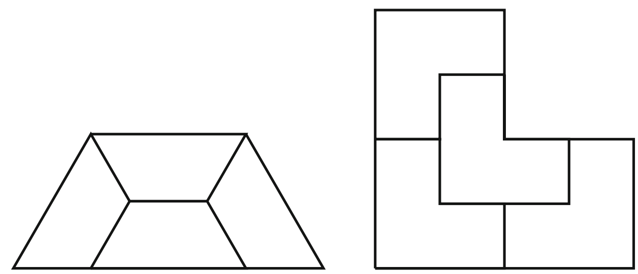


Any three- or four-sided polygon can tile the plane. With five sides, though, things get exceptionally tricky. Kepler may have been the first to explore tiling with pentagons.[2] The complete picture wasn't filled in until Michaël Rao's 2017 article "Exhaustive search of convex pentagons which tile the plane."[3] You have to look no further than last year's *Intelligencer* to see the attraction of pentagonal tiling in both ancient and contemporary minds.[4]

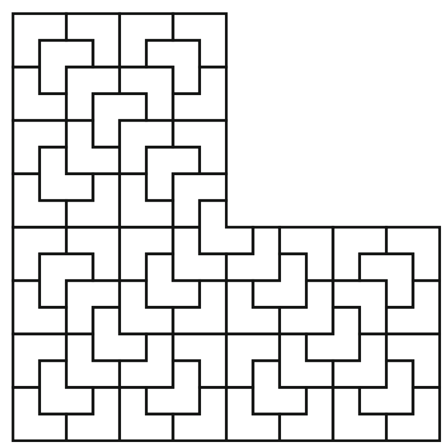
The status of tiling is witnessed by the number of non-mathematicians who have participated in it. A wonderful example is Marjorie Rice. Excited by Martin Gardner's column in *Scientific American*, she joined the search for tessellating pentagons. I strongly recommend Doris Schattschneider's essay on this subject, "In praise of amateurs."[5]

Rep-Tiles

A rep-tile is a figure that can be carved up into smaller versions of itself.



Rep-tiles lead to interesting plane tilings because the dissecting and composing can be replicated forever.



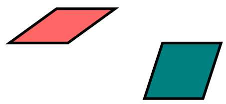
This is another area for amateurs. Lee Sallows,[6] for example, has pushed rep-tiles into self-tiling tile sets, which he calls "setisets."[7]

Penrose Tiles

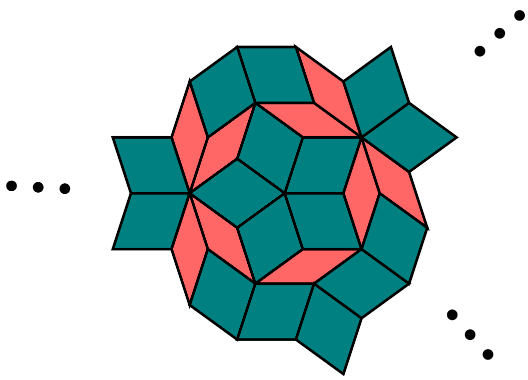
What makes for truly romantic art is a structure whose unique characteristics are not apparent in any finite part.

[1]See Siobhan Roberts, *The King of Infinite Space: Donald Coxeter, the Man Who Saved Geometry*, Walker & Company, 2006.
[2]See Kepler's 1619 book *Harmonices Mundi*.
[3]Available online at arXiv:1708.00274.
[4]See Frank Morgan, ''My undercover mission to find Cairo tilings,'' *Mathematical Intelligencer* 41:3 (2019), 19–27.
[5]In *The Mathematical Gardner*, edited by David A. Klarner, pp. 140–166, Prindle, Weber & Schmidt, 1981.
[6]An *Intelligencer* author (1990, 1992, 1995, 1997).
[7]See the Wikipedia article https://en.wikipedia.org/wiki/Self-tiling_tile_set.

The tiles invented by Roger Penrose are certainly an example of that. These two marvelous shapes



together can tile the plane.

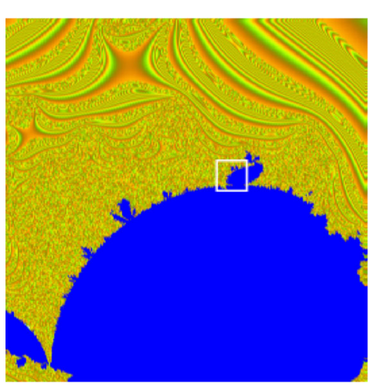


And a few composition rules ensure that no matter how they are combined, the tiling completely lacks global symmetry.

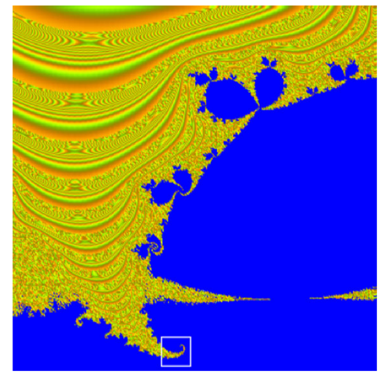
As evidence of the attraction of these particular tiles as mathematical art, I note that its features are so attractive that Penrose thought it advisable to patent them.[8] There is an important amateur here too, Robert Ammann.[9]

Fractals

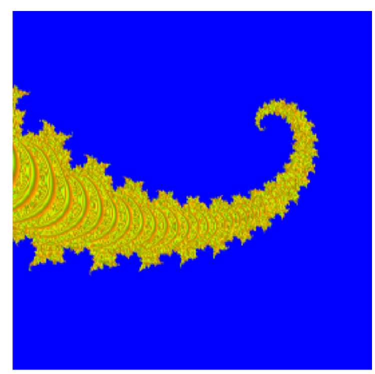
Fractals appeared before Bancroft Brown's lecture, but they took off as mathematical art only with the application of computers and their exploration by Benoit Mandelbrot. In Mandelbrot's hands and the hands of others, fractals were revealed as fantastic infinite toys.



If you pick one region of a fractal and magnify it,



you reveal new details, new structure, new mysteries. And if you then select a detail of the magnified region, you reveal another level of structure and mystery. And so on, world without end. Every fractal opens a door to infinity.



The language of fractals and the pleasure that they give have become part of the general culture. Googling "fractals" gets me roughly the same number of hits as "Beethoven," "quantum computing," and "The New York Yankees." It gets many more hits than "Frank Sinatra," "Downton Abbey," and "Fifty Shades of Grey."[10]

Dragons

Another infinite treasure is the dragon curve invented in 1966 by John Heighway.[11]

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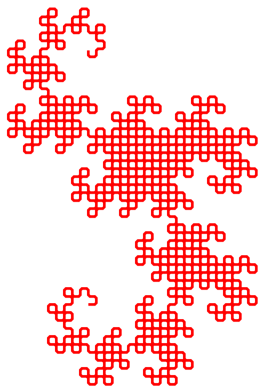
[8]And it was advisable. The patent was subsequently violated by the Kimberly-Clark company, and Roger Penrose sued.

[9]See Marjorie Senechal, ''The mysterious Mr. Ammann,'' *Mathematical Intelligencer* 26:4 (2004), 10–21.

[10]I checked out all 138,000,000 hits and decided that the Wikipedia page was the best for an overview and introduction. By the way, have you, dear reader, ever donated to Wikipedia? They appreciate even the *tiniest* contributions.

[11]The first paper on the dragon curve was ''Number representations and dragon curves,'' by Chandler Davis (editor emeritus of the *Intelligencer*) and Donald Knuth, *J. Recreational Math.* 3 (1970), 133–149. The dragon curve also appeared in a previous column of mine, ''Numeralogy,'' *Mathematical Intelligencer* 41:4 (2019), 22–27.



The dragon has attributes of fractals and attributes of rep-tiles. Is it art? I have to tell you that the dragon has its origin in strips of folded paper.[12] Anything that involves folding paper is art.

Sandpiles

Like all of the above examples, sandpiles are based on a specifically finite idea, yet they produce structures of infinite detail and interest. Sandpiles are all about grains of sand spilling, four grains at a time. Sandpiles sound like kids' stuff, so clearly, it's art (and watch the "Rampage" episode of the TV series Numb3rs).

I recommend (and others do too) the paper "The amazing, autotuning sandpile," by Jordan Ellenberg.[13]

M. C. Escher

The most singular romantic mathematical artist must be Maurits Cornelis Escher (1898–1972). His work openly incorporates the infinite, the impossible,[14] and often both.[15]

For Escher's place in the pantheon of mathematical artists, you can read Martin Gardner or Doris Schattschneider.[16]

Infinite Combinatorial Art

Combinatorial and numerical structures, missing the visual impact of geometry, may be less immediately attractive. But they can take you and they can hold you forever.

Hypergame

Definition 1. A two-player game is *finite* if the game always ends after a finite number of moves.

Definition 2. *Hypergame* is a two-player game in which the player going first chooses a finite game, then the second player makes the first move of that game, and the players continue to play the chosen game to its end (which ends the play of hypergame).[17]

Theorem 1. *Hypergame is a finite game.*

Proof. The proof is simple. A finite game is chosen. The chosen game, being finite, must end after a finite number of moves, say n. So this instance of hypergame will have ended after $n + 1$ moves, a finite number. □

Corollary 1. *Hypergame is not a finite game.*

Proof. The proof is simple. All we have to do is give an example of an instance of hypergame that never ends. Here it is: The first player chooses hypergame. That is permitted, since by Theorem 1, hypergame is a finite game. Then the second player makes the first move of hypergame (the chosen game) and chooses hypergame. This can continue forever. □

There's a paradox here. How are we to extract ourselves? The answer given by most mathematicians is that hypergame doesn't exist in any consistent mathematical universe. The axiom system that underlies mathematics (for most mathematicians), Zermelo–Fraenkel set theory, does not allow sets to be too universal (such as, for example, the set of all sets). Hypergame, if it existed, would arguably contain everything, since every mathematical object could be imagined as part of a finite game.

In this way, the hypergame paradox is very much like Russell's paradox, another wonderful creation of mathematical art.[18]

There are other approaches to logical paradoxes like these (coming up in Part II).

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[12]Wikipedia is as good a place as any for an introduction and the story.

[13]It appeared in *Nautilus*, April 2, 2015. Available online at http://nautil.us/issue/23/dominoes/the-amazing-autotuning-sandpile.

[14]A quick cruise of www.mcescher.com/gallery/ should convince you of this.

[15]For "both" see the cover of the *Mathematical Intelligencer* 18:2 (1996), which featured H. M. S. Coxeter's article "The trigonometry of Escher's woodcut 'Circle Limit III,'" pp. 42–46.

[16]"The eerie mathematical art of Maurits C. Escher," in *Mathematical Carnival*, Martin Gardner, Mathematical Association of America, 1989, or "The mathematical side of M. C. Escher," *Notices of the AMS* 57:6 (2010), 706–718, and *Visions of Symmetry*, Abrams Books 2004, both by Doris Schattschneider.

[17]Hypergame was invented by Bill Zwicker. The theorems and proofs that follow are his as well.

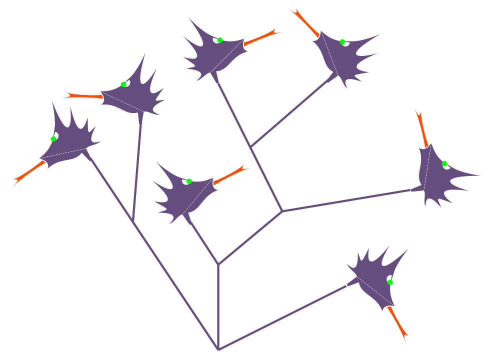
[18]A straightforward account of this can be found at https://brilliant.org/wiki/russells-paradox/. Wikipedia and the *Stanford Encyclopedia of Philosophy* add many nuances. I will also say more at the column website: www.math.smith.edu/~jhenle/pleasingmath/.

The Hydra Game

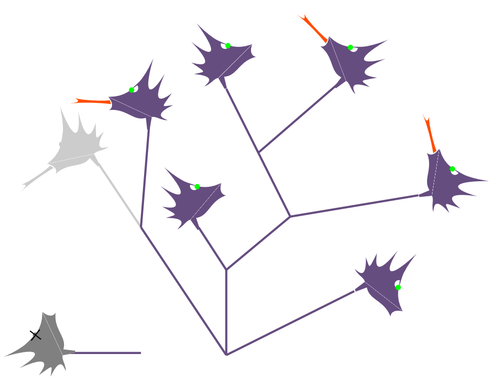
In most works of infinite art, the infinity is hidden. Hypergame is an example of this—infinity surfaces only in the corollary. Another example is the two-envelope paradox, which will appear in Part II. Yet another example is the hydra game.

Jeff Paris, Leo Harrington, and Lawrence Kirby, inspired by a theorem of R. L. Goodstein, invented the hydra game. It is all about cutting the heads off a finite dragon.

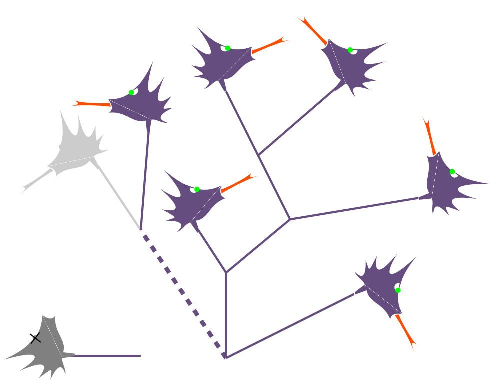
Imagine a dragon in the form of a mathematical tree with multiple heads.



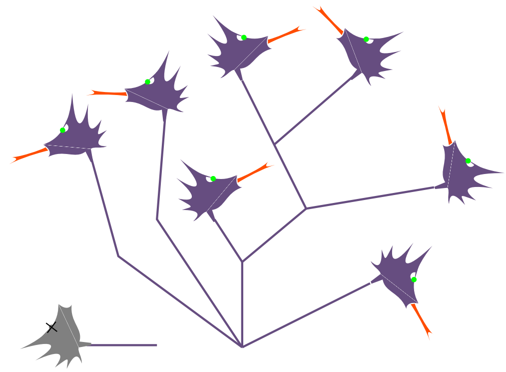
The challenge is to slay the dragon by cutting off its heads, one by one. The problem is that the heads grow back. When you cut off your first head,



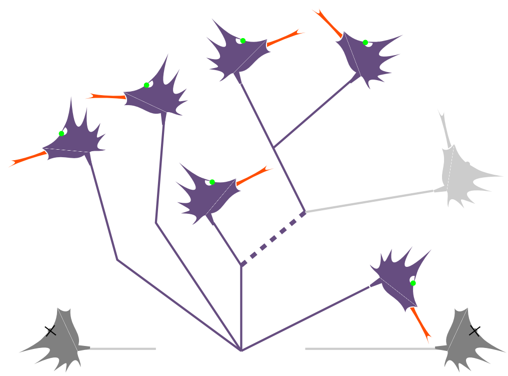
you go back one segment of the branch from the cut,



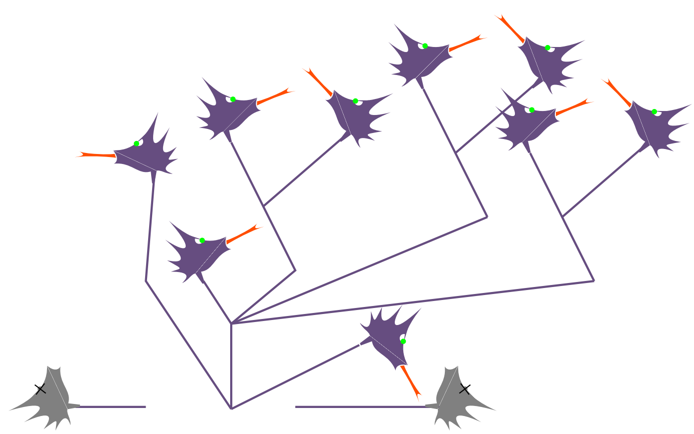
and a new branch, identical to the branch above, grows in.[19]



When you make your *second* cut,



the same thing happens except that now *two* copies of the branch above grow back.



After the third cut, three branches grow back. After the fourth cut, four branches. Looks like trouble!

Paris, Harrington, and Kirby prove that no matter how you choose which head to cut next, eventually you *will* slay the dragon. What is especially cool is that they show that even though this is a theorem about finite objects, the

[19] If you cut off a head attached to the root of the tree, then there can be no moving downward, and so no new heads grow.

existence of infinity is essential to the proof. In other words, although this is a theorem about arithmetic (finite numbers), the theorem is not provable using the axioms of arithmetic. It is provable, though, in higher systems such as analysis (infinite decimals) and set theory.

$3n+1$

The hydra is not well known, but it's good background for the $3n+1$ problem, which is well known indeed. The problem originated with Lothar Collatz and is also known as the Collatz problem, but it has many names attached to it:[20]

Choose a positive integer n. Now repeatedly do:

- if n is even, replace n with $n/2$;
- if n is odd, replace n with $3n+1$.

Collatz's conjecture is that no matter what number you start with, this process will eventually reach 1. Countless programming students have written programs to test this. It has been tested for all numbers below a fantabulously large number. Is it true for *all* numbers?

Some n reach 1 pretty quickly. For example, $n = 26$ takes 11 steps:

$$26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.$$

But some numbers take absurdly many steps. My personal favorite is $n = 27$, which takes 42 steps, climbing all the way up to 9232 before finally collapsing to 1.

Mathematician (and outstanding mathematical artist) J. H. Conway has speculated that the conjecture may be true but not provable.[21]

But not provable in what system? The fact that every hydra game will end is not provable in arithmetic. But it is provable in analysis (which includes an axiom for infinity). That might be possible here. Paul Erdős's conclusion was, "Mathematics is not ready for such problems."

The Collatz conjecture is high mathematical art. It can be understood by any third-grader. Probably millions have spent happy hours playing with it.

I'm an example. I have no special background in the various fields that touch on the problem, but I can't help playing with it. As I write this, I've just thought of something that might ....

Really Big Numbers

At the end of the nineteenth century, Georg Cantor proved that there are infinite sets of different sizes. It soon became clear that there are infinitely many infinite sizes.[22] In the twentieth century, this area of mathematics became an art that shares attributes of *Star Wars* films, Stephen King novels, and extreme sports.

It started with mathematicians inventing axioms of infinity, axioms stating that large cardinal numbers exist with special properties. More often than not, the properties are so special that it is impossible to prove that such cardinals exist. And usually, no one can prove that they don't exist.

The first such axiom was invented by Felix Hausdorff in 1908. Another was invented by Stanisław Ulam in 1930. In the late twentieth century, a flood of new axioms came into the world.[23]

I claim that these large cardinal axioms are art, although they're rather esoteric. I justify their status as art by the following:

1. Axioms are worthless if they are inconsistent.
2. But each new cardinal axiom increases the danger of inconsistency.
3. Despite the danger, researchers of almost every philosophical persuasion are unwilling to give up large cardinals.

The danger of inconsistency is that an inconsistent axiom system is worthless. You can prove anything if your system is inconsistent. Every statement is trivially both true and false.

All this makes the lure of the infinite multifaceted. Large cardinals are powerful, enabling one to prove fantastic theorems (see the section on astrology below). Large cardinals open up galaxies of mathematical objects far, far away. But there is danger.

Research in this area is like Olympic-level downhill skiing. It's thrilling. But you could wipe out. A skiing accident can sideline you for a year, even end your career. In the same way, the discovery of an inconsistency can erase years or even a life of painstaking research.

For a personal story, see the column website.

Astrology

Astrologers make predictions about our lives on Earth based on facts about the heavens—the positions of the stars and planets. For many, this is fun but absurd. Now consider the mathematical equivalent, that facts about finite natural numbers might hinge on the existence of certain ridiculously large cardinals. Absurd! But in fact, this can actually happen. Ordinary astrology is complete nonsense. But there are true examples of mathematical astrology. For (some) details, see the column website.

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[20] See Wikipedia, and see also *The Ultimate Challenge: The $3x+1$ Problem*, edited by Jeffrey Lagarias, American Mathematical Society, 2010.

[21] J. H. Conway, On unsettleable arithmetical problems,'' *American Math. Monthly* 120:3 (2013), 192–198.

[22] See the column website for a proof of this.

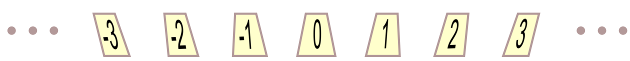
[23] I will give some examples at the column website.

Magical Mathematical Art

Magic is a significant genre of mathematical art. It's worth a column by itself, except that two exceptional books of mathematical magic have appeared in the last few years, *Magical Mathematics: The Mathematical Ideas That Animate Great Magic Tricks*, by Persi Diaconis and Ron Graham, and *Mathematical Card Magic: Fifty-Two New Effects*, by Colm Mulcahy. These present the genre better than I could possibly do.[24]

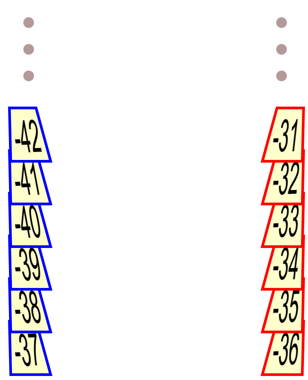
For some reason, neither book contains any tricks that use an infinite deck of cards. That makes the idea of an infinite card trick a challenge for some unprincipled mathematical artist. I accepted the challenge. Here's my trick:

Standing before a large audience, possibly infinite, I pull out an infinite deck of cards imprinted with the set of integers.



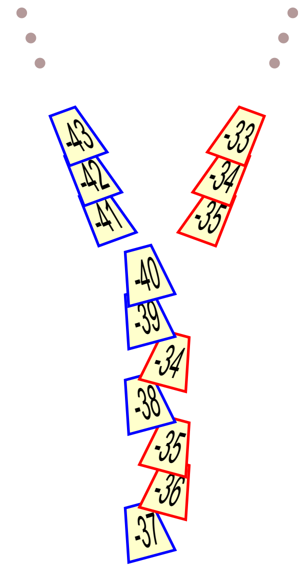
I invite the audience to cut the deck wherever they like. For example, they might choose to divide it between −37 and −36.

That gives us two infinite decks, each one with a bottom card but no top card.



(I colored the edges to help you tell them apart.)

Next, I let the audience riffle shuffle the two decks together. That means they decide, for each card in the shuffled deck, whether it comes from the left (blue) deck or the right (red) deck. The process might look like this.



I say to them, "I need volunteers for a game! The game will have winners and losers. I'm going to give each winner a $10 prize! There could be a lot of winners, so I'm asking for a penny from each volunteer. With luck (or magic) the pennies will pay for the prizes! For added protection, though, my accountant has asked that I take only a finite number of volunteers. So I'll take any number of volunteers—a million, a billion, a trillion, whatever—so long as it's just a finite number."

Now I deal out cards to all the volunteers. How many cards per volunteer? Once again, I leave that up to the audience. (And here is the only time I do something sneaky: if the number of volunteers is even and the number of cards per volunteer is odd, I deal myself in, saying, "I'm dealing myself in because I want to be a winner too!")

I add, "Pardon me for dealing from the bottom of the deck! It's a habit I picked up in grad school."

Now I tell everyone to add up the numbers on the cards they hold. "That's your score," I say, "but it's going to change." I tell them to even up their scores as much as possible. If, for example, one person's score is 134 and another's is 99, the 134 should be lowered by 17 to 117, and the 99 raised by 17 to 116. The volunteers should keep making exchanges until no pair of volunteer scores differ by more than 1. The winners will be those with the higher of the two scores.

"Here's an example," I say. "Suppose there are three people, A with score 2, B with score 5, and C with score 7. If A evens up first with B and then with C, then A and C will be winners (try it). But if instead C evens up with B, and then C evens up with A, and then C evens up with B again,

[24]*Magical Mathematics* is published by Princeton University Press, 2012, and *Mathematical Card Magic* is published by CRC Press, 2013.

then B and C will be winners. And if C evens up with B, then with A, and then B evens up with A, then A and B will be the winners—three completely different results!"

"Hey!" I say. "This doesn't look good. In all these cases—three people and two winners—I'm out \$19.97. Luckily, I've already bought my ticket out of town."

This evening-up process might take a long time depending on the number of volunteers, but at this stage everything is finite, so it won't take *infinitely* long. When the exchanges are over, we can expect there to be two numbers, one apart, such that everybody's score is one of the two numbers.

Now I announce in a deep, impressive voice: "You're a winner if your score is higher than somebody else's score. As promised, I will now give \$10 to every winner!"

But amazingly, I pay nothing. There are no winners. When all the exchanges have been made, everyone has the exact same score.

Is this a good trick? I don't know. I've never tried it. An infinite deck on Ebay was cheaper than the ones on Amazon, but the cost was still out of sight.

Why does the trick work? The key is the Gilbreath principle, a lovely magical mathematical trick invented some sixty years ago. I'll explain how it works in the next column and on the column website: www.math.smith.edu/~jhenle/pleasingmath/.

And if you have any comments or questions, shoot me an email at pleasingmath@gmail.com.

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