Numeralogy

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This is a column about the mathematical structures that give us pleasure. Usefulness is irrelevant. Significance, depth, even truth are optional. If something appears in this column, it's because it's intriguing, or lovely, or just fun. Moreover, it is so intended.

I never could make out what those damned dots meant.

—Lord Randolph Churchill

That's right, *numeralogy* (not numerology). In my last column I described a mathematical art: the creation of mathematical structures, structures that dazzle, intrigue, excite, and/or amuse. There are many genres of this art, including algebras, geometries, games, puzzles, and tilings. This column is about one of the smaller genres: *numeration systems.*

Ancient History

I still remember the surprise I felt when my seventh-grade teacher (New Math, 1957) told us that a numeral was different from a number, that a numeral was a name for a number. What's in a name? How is that important? I knew about Roman numerals, of course. They were just annoying. Surprise turned to delight as we explored base-6, binary, and other numeration systems. It was, perhaps, my first aesthetic mathematical experience.

Are numeration systems *art*?

Numeration systems indeed fit my definition of mathematical structure—they can be described completely and unambiguously. But the systems that cultures evolved over millennia—Babylonian, Mayan, Hindu-Arabic ...—are no more or less art than the other tools they produced. The systems were intended to be useful but not, probably, to entertain me.

Most developers of modern systems are dealing with worldly problems—financial, mechanical, statistical.[1] But some were genuine artists. They created numerals that, as noted above, delight. That makes them artists.

It's difficult to know about the creators of systems centuries old. Binary is a good example of the problem. Binary was discovered independently by numerous thinkers starting at least 400 years ago. Gottfried Leibniz was one of those, not the first, and it thrilled him. The fact that all quantities could be described using only two characters, 0 and 1, seemed to him beautiful and philosophically significant. He wondered whether the power and symbolism of binary could be useful in uniting the religions of man.[2] Was he an artist?

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[1] In the epigraph to this article, Lord Churchill was complaining about decimal points.

[2] The mathematician, philosopher, physicist Leibniz was, professionally, a lawyer and diplomat. One project he longed to see accomplished was to heal the schisms among the Christian churches. See, for example, Frank J. Swetz, "Leibniz, the Yijing, and the Religious Conversion of the Chinese," *Mathematics Magazine* 76:4 (2003), 276–291.

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The reader is surely acquainted with binary, but let me describe it like this: the sequence of 0's and 1's (with a subscript of 2) 110101_2 names this sum:

$$\begin{array}{rcrcrcrcrcrcrcr} & & & & & & & & & & 1 & = & 1 \\ & & & & & & & & 2 & \cdot & 0 & = & 0 \\ & & & & & & 2 & \cdot & 2 & \cdot & 1 & = & 4 \\ & & & & 2 & \cdot & 2 & \cdot & 2 & \cdot & 0 & = & 0 \\ & & 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 1 & = & 16 \\ + & 2 \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 1 & = & 32 \\ \hline & & & & & & & & & & & & 53 \end{array}$$

Every natural number can be expressed uniquely in this way. Base 6 is the same, except the 2's are replaced by 6's and the digits used are 0, 1, 2, 3, 4, and 5.

Fourteen years after seventh grade, I read in the second volume of Donald Knuth's *The Art of Computer Programming* about a host of attractive variations.[3] For me, the loveliest of these were negabinary and balanced ternary. Negabinary is base -2. The numeral

$$110101_{-2}$$

represents

$$\begin{array}{rcrcrcrcrcrcrcr} & & & & & & & & & & 1 & = & 1 \\ & & & & & & & & -2 & \cdot & 0 & = & 0 \\ & & & & & & -2 & \cdot & -2 & \cdot & 1 & = & 4 \\ & & & & -2 & \cdot & -2 & \cdot & -2 & \cdot & 0 & = & 0 \\ & & -2 & \cdot & -2 & \cdot & -2 & \cdot & -2 & \cdot & 1 & = & 16 \\ + & -2 \cdot & -2 & \cdot & -2 & \cdot & -2 & \cdot & -2 & \cdot & 1 & = & -32 \\ \hline & & & & & & & & & & & & -11 \end{array}$$

Every integer has a unique numeral in negabinary. There are at least two ways for you to be convinced of this. One is to play with the system, to write and evaluate negabinary numerals. The second is to read the proof I've posted on the column website.[4]

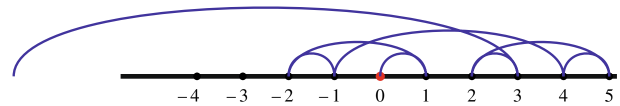
As an interesting experiment, we can take the natural numbers in binary,

$$0_2,\ 1_2,\ 10_2,\ 11_2,\ 100_2,\ \ldots,$$

and use them to "count" the integers in negabinary:

$$0_{-2},\ 1_{-2},\ 10_{-2},\ 11_{-2},\ 100_{-2},\ \ldots.$$

Graphically, we can watch as a point bounces along the line of integers, in the order dictated by the natural numbers in binary.



How this would have delighted Leibniz—naming every positive and negative whole number with just 0's and 1's!

Balanced ternary is like base 3 except that the digits are $+$, 0, and $-$, representing, respectively, 1, 0, and -1. The numeral

$$+-0-0+_{\mathrm{BT}}$$

represents

$$\begin{array}{rcrcrcrcrcrcrcr} & & & & & & & & & & 1 & = & 1 \\ & & & & & & & & 3 & \cdot & 0 & = & 0 \\ & & & & & & 3 & \cdot & 3 & \cdot & -1 & = & -9 \\ & & & & 3 & \cdot & 3 & \cdot & 3 & \cdot & 0 & = & 0 \\ & & 3 & \cdot & 3 & \cdot & 3 & \cdot & 3 & \cdot & -1 & = & -81 \\ + & 3 \cdot & 3 & \cdot & 3 & \cdot & 3 & \cdot & 3 & \cdot & 1 & = & 243 \\ \hline & & & & & & & & & & & & 154 \end{array}$$

Balanced ternary also can represent all integers (proof on the website).

Early Adventures

I found negabinary and balanced ternary so attractive that in the next two years, while doing a stint of middle-school teaching, I turned my hand to inventing numeration systems. I called my first system "pherocimals."[5,6] A pherocimal sequence

$$abcdef_{\mathrm{Ph}}$$

represents the following sum:

$$\begin{array}{rcccccccccccc} & & & & & & & & & & f \\ & & & & & & & & e & \cdot & f \\ & & & & & & d & \cdot & e & \cdot & f \\ & & & & c & \cdot & d & \cdot & e & \cdot & f \\ & & b & \cdot & c & \cdot & d & \cdot & e & \cdot & f \\ + & a \cdot & b & \cdot & c & \cdot & d & \cdot & e & \cdot & f \\ \hline \end{array}$$

Clearly, using 0's and 1's as digits doesn't work, but I found that using 1's and 2's works pretty well. 122121_{Ph}, for example, represents

[3] Addison-Wesley, 1969.

[4] www.math.smith.edu/~jhenle/pleasingmath.

[5] This was 40+ years ago. I imagined that my middle-school students would think ''ferocious decimals'' and get excited about them.

[6] If the reader is a middle-school student, I mean no disrespect.

$$1+2+2+4+8+8=25.$$

Every number can be represented in this way, but not uniquely. A simple example is

$$11_{\mathrm{Ph}} = 2_{\mathrm{Ph}}.$$

But duplication can be eliminated by forbidding two 1's in a row. A proof of this is on the column website.

I can't be sure of the motivations of Leibniz or other thinkers, but I am reasonably sure of my own. When I am constructing a numeration system, I'm out for a good time. I definitely want to tickle and intrigue.

Another system I invented was fracimals. Fracimals were inspired by the way the ancient Egyptians wrote fractions. With the exception of $\frac{2}{3}$, the Egyptians had symbols only for "unit" fractions: fractions with numerator 1. They had to express a quantity such as $\frac{3}{7}$ as a sum of distinct unit fractions, for example,

$$\frac{1}{3}+\frac{1}{11}+\frac{1}{231} \qquad \left(=\frac{3}{7}\right).$$

It's not clear why they had this restriction.[7]

The fracimal replaces the multiplication of pherocimals with division. For example, $.a.b.c.d.e.f.$ represents the sum[8]

$$\begin{array}{r} 1 \div a \\ 1 \div a \div b \\ 1 \div a \div b \div c \\ 1 \div a \div b \div c \div d \\ 1 \div a \div b \div c \div d \div e \\ + \quad 1 \div a \div b \div c \div d \div e \div f \\ \hline \end{array}$$

That is,

$$\frac{1}{a}+\frac{1}{ab}+\frac{1}{abc}+\frac{1}{abcd}+\frac{1}{abcde}+\frac{1}{abcdef}.$$

Every rational greater than 0 can be written in this way. The fraction $\frac{3}{7}$, for example, equals .3.6.2.3.4.7., while $\frac{7}{3}$ equals .1.1.3.

Once again, we don't have uniqueness; for example,

$$\frac{7}{13} = .2.13. = 3.2.5.7.13.$$

and

$$\frac{1}{2} = .2. = .3.2. = .3.3.2. = .3.3.3.2. = \cdots.$$

For fracimals of finite length, uniqueness can be enforced by requiring that the "digits" not decrease. Every rational number can still be expressed by such a fracimal. For instance, $\frac{3}{7}$ is .3.4.7.

Every real number in $(0,\infty]$ can be expressed by an infinite fracimal. The infinite fracimal, .1.1.1.1.1.1... represents ∞. Every other infinite fracimal converges to a positive real number. Proofs of everything are on the website.

And really, you have to like this fracimal:

$$.1.1.2.3.4.5\ldots = e.$$

Infinite fracimals avoid an annoying feature of decimal fractions. Among infinite decimals there are duplications:

$$.375000\overline{0}\ldots = .374999\overline{9}\ldots.$$

You must either live with the problem or else forbid either $\overline{9}\ldots$ or $\overline{0}\ldots$. Fracimals don't present this difficulty. You keep all infinite fracimals and discard the finite ones. Every finite fracimal can be reconfigured as an infinite one. For example, .3.7.12 is the infinite fracimal

$$\begin{aligned} .3.7.\overline{13}\ldots &= \frac{1}{3}+\frac{1}{3\cdot 7}\left(1+\frac{1}{13}+\left(\frac{1}{13}\right)^2+\cdots\right) \\ &= \frac{1}{3}+\frac{1}{3\cdot 7}\cdot\frac{1}{1-\frac{1}{13}} \\ &= \frac{1}{3}+\frac{1}{3\cdot 7}\left(1+\frac{1}{12}\right) \\ &= .3.7.12. \end{aligned}$$

In those early years I designed many systems—frictions, zerones, continued frictions, discontinued frictions ....[9]

Then I went back to serious stuff.

The State of the Art

I haven't mentioned continued fractions. That's a system of numeration, too, and it's a lot like the others. We can write the continued fraction

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\frac{1}{f}}}}}$$

like this:[10]

$$a+1\div b+1\div c+1\div d+1\div e+1\div f.$$

I haven't mentioned Zeckendorff's Fibonacci numeration, in which every natural number is uniquely named as a

[7]I've seen it suggested that the Egyptians thought that $\frac{1}{7}$ was unique, that is, there's really only one $\frac{1}{7}$, so you can't add up five of them—because there's really only one of them. This is a crazy explanation and I really like it.

[8]Important parentheses tastefully omitted.

[9]*Numerous Numerals*, The National Council of Teachers of Mathematics, 1975.

[10]Really important parentheses accidentally omitted.

sum of nonconsecutive Fibonacci numbers.[11] That's simply lovely.

And I haven't mentioned the systems discovered by Frank Gray: "Gray" codes.[12] Those are lovely too.[13]

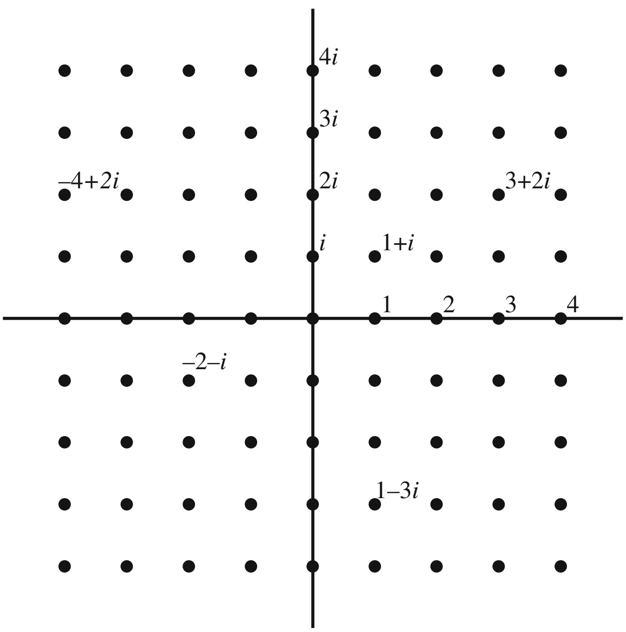
I assume there are many (mathematical) artists out there sculpting numeration systems. *I'd like to know about them.* I still fiddle with numeration on rainy days.[14]

Imaginary Names

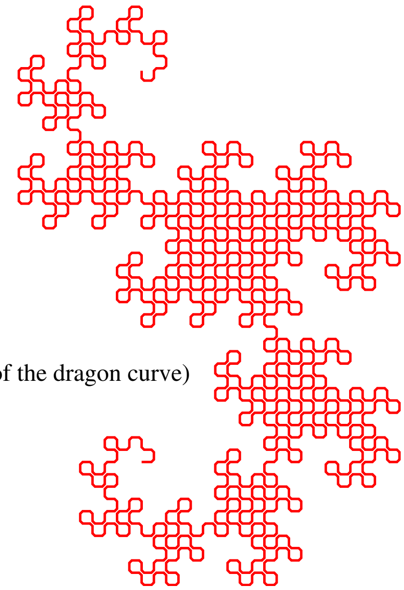
In 1970, Chandler Davis (editor emeritus of the *Intelligencer*) and Donald Knuth created a numeration system for the Gaussian integers.[15] This was tricky because the Gaussian integers form a two-dimensional set. They're represented by all expressions of the form,

$$m + ni,$$

where m and n are ordinary integers and i is an "imaginary" square root of -1. They arrange nicely in a plane:



Knuth and Davis were investigating the dragon curve, itself a marvelous work of mathematical art that had been discovered only a few years earlier:[16]



(Just part of the dragon curve)

To name the Gaussian integers, Knuth and Davis thought to use a base numeration system with a complex number as the base. Their choice was $1+i$. For digits, Knuth and Davis used 0, 1, -1, i, and $-i$. For neatness they wrote $\bar{1}$ for -1 and $\bar{\imath}$ for $-i$. The numeral

$$i\,\bar{\imath}\,1\,\bar{1}\,0\,i_{1+i},$$

for example, represents the sum

$$\begin{array}{rcccccccccccc}
 & & & & & & & & & & 1 & \cdot & i \\
 & & & & & & & & & & 1+i & \cdot & 0 \\
 & & & & & & & & 1+i & \cdot & 1+i & \cdot & \bar{1} \\
 & & & & & & 1+i & \cdot & 1+i & \cdot & 1+i & \cdot & 1 \\
 & & & & 1+i & \cdot & 1+i & \cdot & 1+i & \cdot & 1+i & \cdot & \bar{\imath} \\
+ & & 1+i & \cdot & 1+i & \cdot & 1+i & \cdot & 1+i & \cdot & 1+i & \cdot & i \\
\hline
\end{array}$$

which equals $2+i$.

Without further refinement, there are infinitely many duplications:

$$1 = \bar{\imath}\,i = \bar{\imath}\,0\,i\,i = \bar{\imath}\,0\,\bar{1} = 1\,\bar{\imath}\,\bar{\imath} = \cdots.$$

[11] A good source for the history of this is https://proofwiki.org/wiki/Zeckendorf's_Theorem.

[12] See https://en.wikipedia.org/wiki/Gray_code.

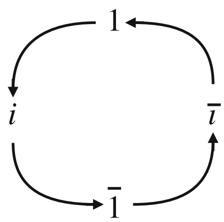
[13] I also haven't mentioned English and other natural languages. These have their fans. I have a small volume somewhere devoted to listing the first 1000 numbers alphabetically in English and alphabetically as Roman numerals.

[14] See *The Proof and the Pudding* (Princeton University Press, 2015) for unbalanced ternary, and "The Same, Only Different," *Mathematical Intelligencer* 39:2 (2017), 60–63 for the ring-a-ding system.

[15] "Number Representations and Dragon Curves," *J. Recreational Mathematics* 3 (1970), 66–81, 133–149. Reprinted in *Selected Papers on Fun & Games*, Donald Knuth, CSLI Publications 2011.

[16] There are many references to the dragon on the website. Martin Gardner's column on the subject is excellent: Martin Gardner. "Mathematical Games." *Scientific American*, March, April, July, 1967. Reprinted in his *Mathematical Magic Show*, pp. 207–209 and 215–220. Vintage, 1978.

For uniqueness you have to make restrictions. Knuth and Davis required that the nonzero "digits" appear in a particular order: in moving from right to left in the numeral, the next nonzero digit to the left of a 1 had to be an i and the next digit to the left of that had to be a $\bar{1}$, and to the left of that, an $\bar{\imath}$, and to the left of that, a 1, and so on:



Knuth and Davis called their numerals "revolving representations," and the diagram above shows why.

The restrictions mean that the numeral we gave for $2+i$, namely $i\,\bar{\imath}\,1\,\bar{1}\,0\,i_{1+i}$, is unlawful. But there is a lawful revolving representation for $2+i$:

$$\bar{\imath}\,\bar{1}\,i\,1_{RR}\,.$$

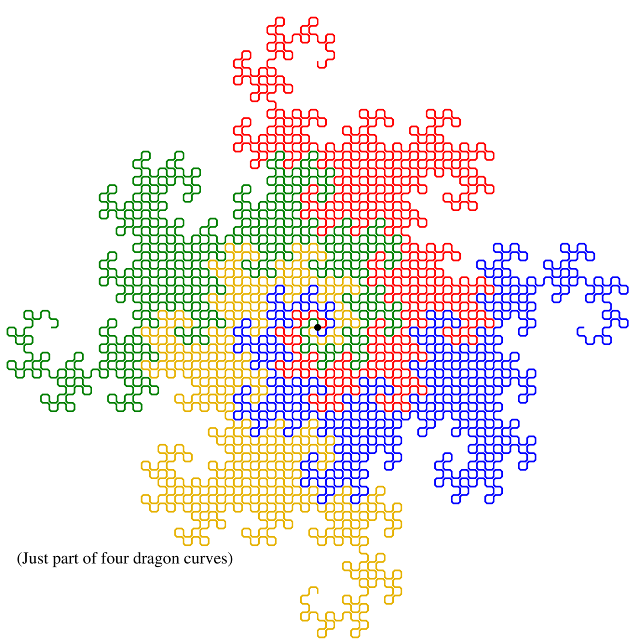
With Knuth and Davis's restriction there are still duplications, but they have a cute theorem: *Every Gaussian integer can be written in exactly four ways:*

- with a rightmost nonzero digit of 1,
- with a rightmost nonzero digit of i,
- with a rightmost nonzero digit of $\bar{1}$,
- with a rightmost nonzero digit of $\bar{\imath}$.

The other three ways of lawfully expressing $1+2i$ are

$$\bar{\imath}\,\bar{1}\,0\,i_{RR}, \quad \bar{1}\,i\,1\,0\,\bar{\imath}\,\bar{1}_{RR}, \quad \bar{1}\,i\,1\,0\,0\,\bar{\imath}_{RR}.$$

You may wonder why there are exactly four ways to describe each number. You might guess that it has something to do with the fact that exactly four copies of the dragon curve fill the plane:



(Just part of four dragon curves)

And indeed, the first known proof of this fact was Knuth and Davis's proof using revolving representations. All the details are in their paper.

Binary Revolving Representations

The cute theorem of Knuth and Davis suggests something nice: if we restrict ourselves to revolving representations with a rightmost nonzero digit of 1, then we *uniquely* name all the Gaussian integers.

That being so, we can simplify the numerals. We don't have to use all those digits; two are sufficient. We can take any sequence of 0's and 1's (without a leading 0), then moving from right to left, substitute for the 1's the digits in the circle starting with 1. The sequence

$$1\,0\,1\,1\,1\,0\,1\,0,$$

for example, results in

$$1\,0\,\bar{\imath}\,\bar{1}\,i\,0\,1\,0_{BRR}.$$

I call these binary revolving representations.

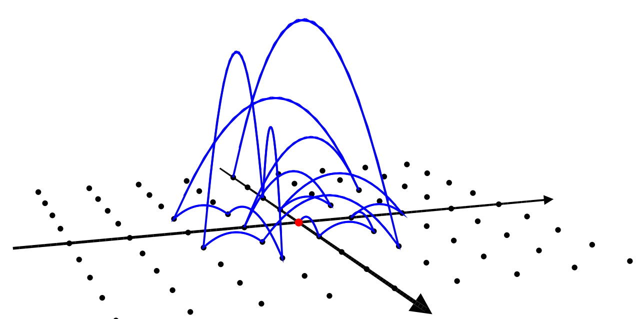
As with the integers, we can take the natural numbers in binary,

$$0_2, 1_2, 10_2, 11_2, 100_2, \ldots$$

and use them to count the Gaussian integers,

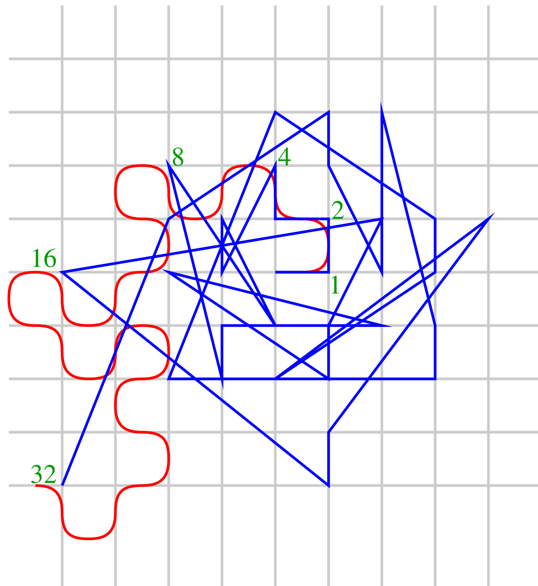
$$0_{BRR}, 1_{BRR}, 10_{BRR}, 11_{BRR}, 100_{BRR}, \ldots$$

bouncing in the complex plane:



This might not be the way you or I would go about counting the points in the plane. But the madness fits with the dragon in a nice way. The nth number in the bouncing sequence is exactly the nth step of the dragon curve—for all n's that are a power of 2.

Here are the two together, the dragon curve in red and the sequence of Gaussian integers in blue:



And here's a table of values (powers of 2 in green):

n	1	2	3	4	5	6	7	8
BRR	1	$1+i$	i	$2i$	-1	$-1+i$	$-i$	$-2+2i$
dragon	1	$1+i$	i	$2i$	$-1+2i$	$-1+i$	$-2+i$	$-2+2i$

n	9	10	11	12	13	14	15	16
BRR	$-1-2i$	$-1-i$	$2-i$	-2	$1-2i$	$1-i$	$2+i$	-4
dragon	$-3+2i$	$-3+i$	$-2+i$	-2	-3	$-3-i$	$-4-i$	-4

You can see that the numbers also match for 3, 6, 12, ..., each a sum of two consecutive powers of 2. This is an easy consequence of Knuth and Davis's work.

Complex numbers were unknown four hundred years ago. But pairs of integers were understood. Binary revolving representations effectively name all pairs, and they do it with only two symbols: 0 and 1.

That would have thrilled Gottfried Wilhelm Leibniz.

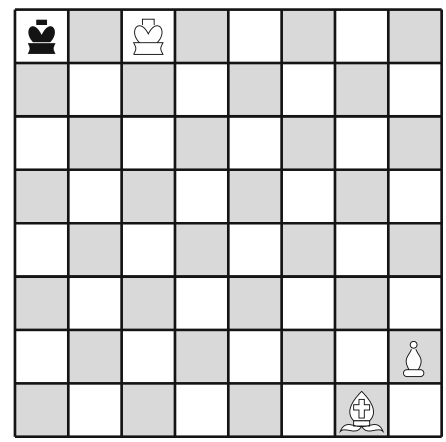
Looking Back

In my last column I left you with two structures invented by students of mine. One had been discovered much earlier and is the subject of many papers. The other is new. I didn't say which was which, but I'll tell you now. Halley Haruta's circles of numbers was noticed by Enrico Ducci in the 1930s. I learned about this when I read the paper by Achim Clausing in the (then latest) issue of the *Monthly*.[17] In the case in which there are four numbers in the circle, Clausing describes the action as the "well-known Four Number Game of E. Ducci." Clausing's paper has an extensive bibliography.

The other structure, by Amelia Austin, is, as far as I know, new and a source of interesting questions. Given a rectangle and a dot, where can you put an × to get a puzzle with a unique solution? Where can you put two ×'s?

A Retrograde Problem

A reader pointed out to me that the first retrograde analysis problem in the column on Raymond Smullyan[18]



has a second solution. One quickly sees that the previous move (we are told it was Black's) must have been the Black king moving up one square (from a7), and the puzzle then becomes figuring out how White had put him in check. Smullyan's solution is that the check was revealed by a white knight that moved to the upper left corner and was taken by the king. But my reader noticed that if you don't assume that White's side of the board is at the bottom, then another solution is that a White pawn moved to the bottom row and was promoted to a bishop.

The puzzle appears twice in *The Chess Mysteries of Sherlock Holmes*,[19] on the cover (as above and as it appeared in the column) and inside the book on page 34, where the bottom is labeled "White." Thus there are two solutions to the cover puzzle and one solution to the puzzle on page 34.

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[17]Achim Clausing, "Ducci Matrices," *American Mathematical Monthly* 125:10 (2018), 901–921.

[18]"The Entertainer," *Mathematical Intelligencer* 40:2 (2018), 76–80.

[19]Alfred A. Knopf, 1979.